High Speed Solution of Spacecraft Trajectory Problems Using Taylor Series Integration

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Taylor series integration is implemented in a spacecraft trajectory analysis code – the Spacecraft N-body Analysis Program (SNAP) – and compared with the code’s existing eighth-order Runge-Kutta Fehlberg time integration scheme. Nine trajectory problems, including near Earth, lunar, Mars and Europa missions, are analyzed. Head-to-head comparison at five different error tolerances shows that, on average, Taylor series is faster than Runge-Kutta Fehlberg by a factor of 15.8. Results further show that Taylor series has superior convergence properties. Taylor series integration proves that it can provide rapid, highly accurate solutions to spacecraft trajectory problems.

Nomenclature

\[ x_1 = \text{x component of spacecraft position relative to inertial frame centered at the central body} \]
\[ x_2 = \text{y component of spacecraft position relative to inertial frame centered at the central body} \]
\[ x_3 = \text{z component of spacecraft position relative to inertial frame centered at the central body} \]
\[ x_4 = \text{x component of spacecraft velocity relative to inertial frame centered at the central body} \]
\[ x_5 = \text{y component of spacecraft velocity relative to inertial frame centered at the central body} \]
\[ x_6 = \text{z component of spacecraft velocity relative to inertial frame centered at the central body} \]
\[ x_7 = \text{spacecraft mass} \]
\[ \vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \text{spacecraft state vector} \]
\[ \vec{x} = (x_1, x_2, x_3) = \text{spacecraft position vector} \]
\[ \vec{v} = (x_4, x_5, x_6) = (v_1, v_2, v_3) = \text{spacecraft velocity vector} \]
\[ t = \text{time} \]
\[ \dot{t} = \frac{dt}{dt} = \text{derivative with respect to time} \]
\[ \ddot{t} = \frac{d^2t}{dt^2} = \text{derivative with respect to time} \]
\[ m = \text{mass flow rate} \]
\[ T = \text{thrust magnitude} \]
\[ a_1 = \text{x component of spacecraft acceleration relative to inertial frame centered at the central body} \]
\[ a_2 = \text{y component of spacecraft acceleration relative to inertial frame centered at the central body} \]
\[ a_3 = \text{z component of spacecraft acceleration relative to inertial frame centered at the central body} \]
\[ \ddot{x} = (a_1, a_2, a_3) = \text{acceleration vector} \]
\[ \dot{x}_j = \text{position vector of the } j^{\text{th}} \text{ other body relative to the central body} \]
\[ G = \text{gravitational constant} \]
\[ M = \text{body mass} \]
\[ h = \text{step size} \]
\[ \tau = \text{local error tolerance} \]
\[ \eta = \text{step multiplication factor} \]

I. Introduction

The advantages of Taylor series integration in solving ordinary differential equations have been known for some time\(^1\)\(^-\)\(^\text{25}\). Foremost among these is the ability to maintain very high computational efficiency while achieving high accuracy. In fact, comparisons with other methods have shown that Taylor series integration can be faster by a factor of twenty or more\(^20\).

The success of the method depends on recasting the governing differential system into a canonical form whereby system derivatives can be obtained to arbitrary order through recursion. This obviates the need to directly calculate derivatives, and makes it possible to obtain derivative information cheaply. The system variables can thus be expanded in a highly accurate series at each time level at minimal cost.

It turns out that most differential systems can be recast into the required canonical form in a straightforward manner. This has led a number of authors to develop general purpose software which can recast an arbitrary system into canonical form and solve the resulting equations automatically\(^10, 11, 15\)\(^-\)\(^\text{18, 22-24}\). Taylor series integration can thus be used as both a general purpose solver and also for specific applications.

The focus here is on calculating spacecraft trajectories. Previous work using Taylor series to calculate trajectories includes that in Refs. 21 and 22. Unlike Refs. 21 and 22, which used an automated Taylor series package\(^16, 22\) to propagate trajectories in Earth orbit, the present work focuses on using Taylor series integration in an existing trajectory analysis code, SNAP (Spacecraft N-body Analysis Program)\(^26\). Developed at NASA’s Glenn Research Center, SNAP is a high fidelity trajectory propagation program that can propagate the trajectory of a spacecraft about virtually any body in the solar system. The equations of motion include the effects of central body gravitation with N x N harmonics, other body gravitation with N x N harmonics, solar radiation pressure, atmospheric drag (for Earth orbits) and spacecraft thrusting (including shadowing). The equations are solved using an eighth-order Runge-Kutta Fehlberg (RKF)\(^27\) single step method with variable step size control.

The purpose of this paper is to demonstrate the use of Taylor series (TS) integration in a high fidelity trajectory analysis code, SNAP, and to provide a detailed comparison of TS performance to eighth-order RKF\(^27\). Section II presents the equations of motion, Section III describes the TS formulation and Section IV discusses the numerical implementation. Section V compares TS and RKF on a representative set of spacecraft trajectory problems, including near Earth, lunar, Mars and Europa missions. It is shown that TS is faster than RKF by an average factor of 15.8, while simultaneously improving accuracy.

II. Equations of Motion

Let \( \vec{X} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \) denote the spacecraft state vector, where \( (x_1, x_2, x_3) = \vec{x} \) is the spacecraft position in Cartesian coordinates relative to an inertial frame centered at the central body, \( (x_4, x_5, x_6) = \vec{v} \) is the spacecraft velocity relative to an inertial frame centered at the central body, and \( x_7 \) is the spacecraft mass. The equations of motion are

\[
\begin{align*}
x'_1 &= x_4 \\
x'_2 &= x_5 \\
x'_3 &= x_6
\end{align*}
\]
\[ x_4' = a_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, t) \]
\[ x_5' = a_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, t) \]
\[ x_6' = a_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, t) \]
\[ x_7' = -\dot{m}(t) \]

where \( a_i \) is the acceleration in the \( i^{th} \) coordinate direction and \( \dot{m} \) is the mass flow rate. The acceleration is a function of the forces acting on the spacecraft. Forces included in SNAP are central body, other body, thrust, atmospheric drag (for low Earth orbits), solar radiation pressure, oblateness effects of Earth, and oblateness effects of other bodies, so that

\[
\vec{a} = \vec{a}_{cb} + \vec{a}_{ob} + \vec{a}_{th} + \vec{a}_{d} + \vec{a}_{srp} + \vec{a}_{obe} + \vec{a}_{obo}
\]

This paper considers only the first four acceleration terms, which are given below.

\[
a_{cb,i} = -G M_{cb} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} := -G M_{cb} \frac{x_i}{|\mathbf{x}|^3}
\]

where \( G \) is the gravitational constant, \( M_{cb} \) is the central body mass, \( i = 1,2,3 \) denotes the coordinate direction, and the spacecraft mass is much less than the central body mass.

\[
a_{ob,i} = \sum_j -G M_j \left( \frac{x_i - x_{i,j}}{|\mathbf{x} - \mathbf{x}_j|^3} + \frac{x_{i,j}}{|\mathbf{x}_j|^3} \right)
\]

where \( j \) denotes the \( j^{th} \) body and \( |\mathbf{x}_j| := \left( x_{1,j}^2 + x_{2,j}^2 + x_{3,j}^2 \right)^{1/2} \).

\[
a_{th,i} = T \frac{V_i}{|\mathbf{v}|} \frac{1}{x_7}
\]

where \( T \) is the constant thrust magnitude and the thrust direction is parallel to the velocity vector.

\[
a_{d,1} = c_1 \left( x_4 + c_2 x_2 \right)^2 + \left( x_5 - c_2 x_1 \right)^2 + x_6^2 \left( x_4 + c_2 x_2 \right) / x_7
\]
\[
a_{d,2} = c_1 \left( x_4 + c_2 x_2 \right)^2 + \left( x_5 - c_2 x_1 \right)^2 + x_6^2 \left( x_5 - c_2 x_1 \right) / x_7
\]
\[
a_{d,3} = c_1 \left( x_4 + c_2 x_2 \right)^2 + \left( x_5 - c_2 x_1 \right)^2 + x_6^2 \left( x_6 \right) / x_7
\]

where \( c_1 = -\frac{1}{2} C_D \cdot (\text{spacecraft area}) \cdot (\text{atmospheric density}) \), \( c_2 = \text{rotation rate of Earth} \) and \( C_D \) is the drag coefficient.
III. Taylor Series Formulation

Let the state vector $\mathbf{X}$ have initial condition $\mathbf{X}_0$. Within the radius of convergence, the system variables $x_n(t)$ can be expanded in a Taylor series,

$$x_n(t) = \sum_{k=0}^{\infty} \frac{x_n^{(k)}(t_0)}{k!}(t - t_0)^k, \quad n = 1, \ldots, 7$$  

(9)

where the derivatives $x_n^{(k)}$ are obtained by successively differentiating the right hand side of Eqs. (1). This can be efficiently accomplished using recurrence relations, as follows. Consider only the central body acceleration term, so that

$$x_1' = x_4$$
$$x_2' = x_5$$
$$x_3' = x_6$$

$$x_4' = -G M_{cb} \frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$
$$x_5' = -G M_{cb} \frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$
$$x_6' = -G M_{cb} \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

$$x_7' = 0$$

Introduce $x_8$ and $x_9$,

$$x_8 = x_1^2 + x_2^2 + x_3^2$$

(11)

$$x_9 = x_8^{3/2}$$

(12)

Eqs. (10) become

$$x_1' = x_4$$
$$x_2' = x_5$$
$$x_3' = x_6$$

$$x_4' = -G M_{cb} \frac{x_1}{x_9}$$

(13)

$$x_5' = -G M_{cb} \frac{x_2}{x_9}$$
\[ x_6' = -G M_{cb} \frac{x_3}{x_9} \]
\[ x_9' = 0 \]

and two auxiliary equations are added to the system:
\[ x_8' = 2x_1x_4 + 2x_2x_5 + 2x_3x_6 \]  
\[ x_9' = \frac{3}{2} \frac{x_9x_8'}{x_9} \]  
\[ \text{(14)} \]

\[ \text{The right hand side of the new system, Eqs. (13) – (15), can now be differentiated using recurrence relations for products and quotients.} \]

For a function \( w(t) = f(t) g(t) \), the Leibnitz rule for differentiating products gives \(^{16}\)
\[ W(k) = \sum_{j=0}^{k} F(j) G(k-j) \]  
\[ \text{(16)} \]

where \( W(k) := \frac{w^{(k)}(t_0)}{k!} \), \( F(j) := \frac{f^{(j)}(t_0)}{j!} \) and \( G(k-j) := \frac{g^{(k-j)}(t_0)}{(k-j)!} \) are reduced derivatives.

For a function \( w(t) = \frac{f(t)}{g(t)} \), the recurrence relation for quotients gives \(^{11}\)
\[ W(k) = \frac{1}{g} \left[ F(k) - \sum_{j=1}^{k} G(j)W(k-j) \right] \]  
\[ \text{(17)} \]

where \( W, F \) and \( G \) are reduced derivatives as above.

The recurrence relations are derived as follows. Let
\[ u_n = x_n', \quad n = 1, \ldots, 9 \]  
\[ \text{(18)} \]

Eqs. (13) – (15) become
\[ u_1 = x_4 \]  
\[ u_2 = x_5 \]  
\[ u_3 = x_6 \]  
\[ u_4 = -G M_{cb} \frac{x_1}{x_9} \]  
\[ u_5 = -G M_{cb} \frac{x_2}{x_9} \]  
\[ \text{(19)-(23)} \]
\[ u_6 = -G M_{cb} \frac{x_3}{x_9} \]  
\[ u_7 = 0 \]  
\[ u_8 = 2x_1x_4 + 2x_2x_5 + 2x_3x_6 \]  
\[ u_9 = \frac{3}{2} x_9 u_8 \]  

Introduce auxiliary functions \( w_4 = \frac{x_1}{x_9}, \quad w_5 = \frac{x_2}{x_9}, \quad w_6 = \frac{x_3}{x_9}, \quad w_{8,1} = x_1 x_4, \quad w_{8,2} = x_2 x_5, \quad w_{8,3} = x_3 x_6, \) and \( w_9 = \frac{x_9 u_8}{x_9}. \)

Using Eq. (18) one obtains

\[ u_n^{(k-1)} = x_n^{(k)}, \quad k \geq 1 \]  
\[ \frac{u_n^{(k-1)}}{(k-1)!} = \frac{x_n^{(k)}}{(k-1)!}, \quad k \geq 1 \]  
\[ \Rightarrow U_n(k-1) = k X_n(k) \Rightarrow X_n(k) = \frac{U_n(k-1)}{k}, \quad k \geq 1 \]  

The recurrence relations are then, for all \( k \geq 1, \)

\[ U_1(k) = X_4(k) = \frac{U_4(k-1)}{k} \]  
\[ U_2(k) = X_5(k) = \frac{U_5(k-1)}{k} \]  
\[ U_3(k) = X_6(k) = \frac{U_6(k-1)}{k} \]  
\[ U_4(k) = -G M_{cb} W_4(k) \]  
\[ U_5(k) = -G M_{cb} W_5(k) \]  
\[ U_6(k) = -G M_{cb} W_6(k) \]
\[ U_7(k) = 0 \] (37)
\[ U_8(k) = 2 W_{8,1}(k) + 2 W_{8,2}(k) + 2 W_{8,3}(k) \] (38)
\[ U_9(k) = \frac{3}{2} W_9(k) \] (39)

where
\[ W_4(k) = \frac{1}{x_9} \left[ X_1(k) - \sum_{j=1}^{k} X_9(j) W_4(k - j) \right] = \frac{1}{x_9} \left[ \frac{U_1(k-1)}{k} - \sum_{j=1}^{k} \frac{U_9(j-1)}{j} W_4(k - j) \right] \] (40)
\[ W_{8,1}(k) = \sum_{j=0}^{k} X_1(j) X_4(k - j) = x_1 \frac{U_4(k-1)}{k} + \frac{U_1(k-1)}{k} x_4 + \sum_{j=1}^{k} \frac{U_1(j-1)}{j} \frac{U_4(k - j - 1)}{k - j} \] (41)

etc., and a similar expression can be derived for \( W_9 \).

The Taylor series coefficients are then
\[ \frac{x_n^{(k)}}{k!} := X_n(k) = \frac{U_n(k-1)}{k}, \quad 1 \leq k \leq K \] (42)

and the local series solution is
\[ x_n(t) = \sum_{k=0}^{K} \frac{x_n^{(k)}(t_0)}{k!} (t - t_0)^k + T_{n,K} \] (43)

where \( U_n(k) \) is defined by Eqs. (31) – (39), \( U_n(0) \) is defined by Eqs. (19) – (27), \( K \) is the number of terms in the series and \( T_{n,K} \) is the truncation error.

The other acceleration terms can be handled similarly. Only other body acceleration, Eq. (4), requires special consideration, due to the need for the motion of other bodies. This can generally be obtained from ephemeris files. However, integration by Taylor series requires derivatives not available from ephemeris files. It is thus necessary to integrate the other body motion as part of the governing differential system. This leads to a substantially larger system of equations, but fortunately can still be integrated efficiently.

Once the recurrence relations are derived for all acceleration terms and the state vector specified, Eqs. (42) – (43) are used to expand the system variables in a series from \( t_0 \) to \( t_1 \), where the step size \( h_t := t_1 - t_0 \) is determined to meet the local error tolerance. From \( t_1 \), the variables are expanded in a new series to \( t_2 \), and so forth. Thus, by a process of “analytic continuation,” one obtains a set of overlapping series solutions that cover the integration domain.

**IV. Numerical Implementation**

Taylor series integration was implemented in SNAP by making some minor modifications to existing source code and adding three additional subroutines - a driver routine which automatically introduces auxiliary variables, sets up initial conditions and integrates; a routine which calculates system reduced derivatives using recurrence
relations for the following quotients and products: \( \frac{x_m}{x_n}, \frac{x_m x_n}{x_n}, \frac{x_m}{x_n}, \frac{x_m}{x_n}, \frac{x_m x_n}{x_n}, \) and \( x_m x_n \), and a routine which determines the step size and sums the series. The number of series terms is variable up to a maximum of 30, but remains constant throughout the integration. Positive and negative terms are summed separately to avoid cancellation of significant digits.

The step size can be determined from the standard formula

\[ h_{\text{next}} = \eta h \left( \frac{\tau}{e_{\text{max}}} \right)^{1/M} \]  

where \( h \) denotes the current step, \( \tau \) the local error tolerance, \( e_{\text{max}} \) the estimate of maximum truncation error, \( M \) the order of the maximum truncation error estimate and \( \eta < 1 \) the step multiplication factor. Eq. (44) is more or less restrictive depending on \( \eta \) and the truncation error estimate \( e_{\text{max}} \). Generally, \( e_{\text{max}} \) should not be calculated from the next series term, due to the extra computation required and the fact that it is not a reliable error estimate. A conservative approach which takes advantage of the series terms already computed leads to

\[ e_{\text{max}} = \text{Max}_n \left[ \left| X_n(K-1) \right| h^{K-1} + \left| X_n(K) \right| h^K \right] \]  

where the expression in brackets is derived by subtracting the Taylor series solution of degree \( K - 2 \) from the solution of degree \( K \) and taking absolute values of individual terms. Eq. (45) can be viewed as a truncation error estimate for the series of degree \( K - 2 \) which is then applied to the more accurate series of degree \( K \).

An alternative to Eqs. (44) – (45) is to simply require \( h \) to be small enough that the system variables directly satisfy the absolute error tolerance requirement

\[ \left| X_n(K-1) \right| h^{K-1} + \left| X_n(K) \right| h^K \leq \tau \]  

for all \( n \). Eq. (46) can be solved by fixed point iteration,

\[ h_{t+1} = \exp \left( \frac{1}{K-1} \ln \left( \frac{\tau}{\left| X_n(K-1) \right| + h_t \left| X_n(K) \right|} \right) \right) \]  

The smallest \( h \) is chosen over all \( n \), and multiplied by the step multiplication factor \( \eta \). This approach offers the advantage of directly calculating the step size without the need for a previous step, and guarantees that the error tolerance is met. Eq. (44), on the other hand, requires a previous step and will require a repeat step whenever \( e_{\text{max}} > \tau \).

The step selection methods above performed very similarly in the current study. Both methods provided stable, accurate solutions and used approximately the same number of time steps in head-to-head calculations.

V. Results

We compare RKF and TS performance on the trajectory problems in Table 1. All calculations were run on a Dell PowerEdge 2600 with two 3.066 GHz processors and four GB of RAM. Source code was compiled using the Absoft Fortran 90 compiler without optimization. SNAP was run with all intermediate print and stop options turned off. All TS calculations used a series with 20 terms and a variable step size determined by Eq. (47) with a step multiplication factor that ranged from 0.75 to 0.9.
Table 1

<table>
<thead>
<tr>
<th>Problem</th>
<th>Title</th>
<th>Description</th>
<th>Central Body</th>
<th>Other Bodies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Satellite in low Earth orbit (LEO)</td>
<td>A 10,000 kg satellite orbits Earth for 10 days at an inclination of 28.45 degrees.</td>
<td>Earth</td>
<td>Moon</td>
</tr>
<tr>
<td>2</td>
<td>Satellite in LEO with drag</td>
<td>A 10,000 kg satellite orbits Earth for 10 days with constant drag at an inclination of 28.45 degrees.</td>
<td>Earth</td>
<td>Moon</td>
</tr>
<tr>
<td>3</td>
<td>Spacecraft spiraling out of Earth’s gravity well</td>
<td>A 10,000 kg spacecraft spirals out of Earth’s gravity well in a low thrust trajectory. Calculation stops when the semi-major axis of trajectory equals 40,000 km.</td>
<td>Earth</td>
<td>Sun, Moon</td>
</tr>
<tr>
<td>4</td>
<td>Spacecraft from near Earth to lunar orbit</td>
<td>A 3580 kg spacecraft 400 km above Earth has been propelled with sufficient energy to reach the Moon. Spacecraft coasts to Moon, performs insertion burn, propagates to apolune, and performs final burn to achieve 500 km by 10,000 km polar lunar orbit with an argument of perilune equal to 90 degrees. See Fig. 1.</td>
<td>Moon</td>
<td>Earth, Sun</td>
</tr>
<tr>
<td>5</td>
<td>Spacecraft in lunar orbit</td>
<td>Spacecraft with 2848.56 kg mass coasts for 10 days in 500 km by 10,000 km polar lunar orbit with an argument of perilune equal to 90 degrees. See Fig. 1.</td>
<td>Moon</td>
<td>Earth, Sun</td>
</tr>
<tr>
<td>6</td>
<td>Spacecraft thrusting from near Earth to Mars coast</td>
<td>A 585 kg spacecraft near Earth thrusts for 38.45 days to achieve sufficient energy to coast to Mars. See Fig. 2.</td>
<td>Sun</td>
<td>Earth, Moon, Venus, Mars, Jupiter barycenter, Saturn barycenter</td>
</tr>
<tr>
<td>7</td>
<td>Spacecraft coast to Mars flyby</td>
<td>A 555.66 kg spacecraft coasts to Mars flyby for 161.55 days. See Fig. 2.</td>
<td>Sun</td>
<td>Earth, Moon, Venus, Mars, Jupiter barycenter, Saturn barycenter</td>
</tr>
<tr>
<td>8</td>
<td>Spacecraft thrusting tangentially out of Europa orbit</td>
<td>A 10,000 kg spacecraft in Europa orbit thrusts tangentially to spiral out until the semi-major axis equals 10,000 km.</td>
<td>Europa</td>
<td>Jupiter, Sun, Ganymede, Io Callisto</td>
</tr>
<tr>
<td>9</td>
<td>Spacecraft coast near Europa</td>
<td>A 9800.49 kg spacecraft coasts for one day after spiraling out of Europa orbit.</td>
<td>Europa</td>
<td>Jupiter, Sun, Ganymede, Io Callisto</td>
</tr>
</tbody>
</table>

Figure 1. Earth to Moon Trajectory

Figure 2. Earth to Mars Flyby
Each trajectory was integrated at five error tolerances from $10^{-10}$ to $10^{-14}$. Tables 2 - 10 summarize the results. RKF results are shown on top and TS on bottom. Spacecraft positions are in kilometers and CPU ratio is RKF/TS. RKF and TS velocities agreed equally well as spacecraft position, but are omitted here for brevity.

Table 2      Results for Problem 1  
<table>
<thead>
<tr>
<th>τ</th>
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<th>y</th>
<th>z</th>
<th>(sec)</th>
<th>ratio</th>
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</thead>
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<td>2416.2243648200</td>
<td>6.91</td>
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<td>2416.2234952381</td>
<td>9.10</td>
<td></td>
</tr>
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11
Table 11 summarizes the CPU ratios. TS is faster than RKF by more than an order of magnitude in 18 of 45 cases. The average speedup is 15.8. For the interplanetary trajectory problems (4-9) the average speedup is 4.53. The gain here is smaller due to the additional equations that TS must solve to account for other body motion. As noted previously, TS must integrate other body motion as part of the differential system, whereas RKF obtains other body motion from ephemeris files. This difference in the integration methods explains the small differences in spacecraft positions observed in Tables 2-10.
Another important property to consider is convergence. Tables 12 - 20 present the number of converged digits obtained for each spacecraft coordinate at each error tolerance, where the $\tau = 10^{-14}$ case was used as the fully converged solution. RKF results are on top and TS on bottom. TS has more converged digits than RKF in 103 out of 108 cases, while RKF has more converged digits in one case. On average, TS has 2.63 more converged digits per case. The results also indicate that TS solutions are nearly fully converged at all error tolerances, suggesting that the step selection method may be too conservative. Finally, it should be noted that convergence itself does not necessarily imply accuracy. However, it does indicate that a necessary condition for accuracy is satisfied.

<table>
<thead>
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VI. Conclusion

Taylor series integration was implemented in a high fidelity trajectory analysis code (SNAP) and compared with 8th order Runge-Kutta Fehlberg on a representative set of trajectory problems. On average, TS was more than an order of magnitude faster than RKF. TS also showed superior convergence properties, having more converged digits than RKF in 103 out of 108 cases. Taylor series integration thus proved that it can provide rapid, highly accurate solutions to spacecraft trajectory problems. This is consistent with other reports which have found Taylor series integration to be superior to conventional methods in both speed and accuracy 11,16,20.

Table 20 Number of Converged Digits for Problem 9

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References

Taylor series integration is implemented in a spacecraft trajectory analysis code – the Spacecraft N-body Analysis Program (SNAP) – and compared with the code’s existing eighth-order Runge-Kutta Fehlberg time integration scheme. Nine trajectory problems, including near Earth, lunar, Mars and Europa missions, are analyzed. Head-to-head comparison at five different error tolerances shows that, on average, Taylor series is faster than Runge-Kutta Fehlberg by a factor of 15.8. Results further show that Taylor series has superior convergence properties. Taylor series integration proves that it can provide rapid, highly accurate solutions to spacecraft trajectory problems.