Dynamical Modeling and Control Simulation of a Large Flexible Launch Vehicle

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This paper presents dynamical models of a large flexible launch vehicle. A complete set of coupled dynamical models of propulsion, aerodynamics, guidance and control, structural dynamics, fuel sloshing, and thrust vector control dynamics are described. Such dynamical models are used to validate NASA’s SAVANT Simulink-based program which is being used for the preliminary flight control systems analysis and design of NASA’s Ares-1 Crew Launch Vehicle. SAVANT simulation results for validating the performance and stability of an ascent phase autopilot system of Ares-1 are also presented.

Nomenclature

\begin{align*}
R_E &= \text{Earth’s equatorial radius} = 20925646.325459 \text{ ft} \\
R_P &= \text{Earth’s polar radius} = 20855486.595144 \text{ ft} \\
J_2 &= \text{Earth’s second order zonal coefficient} = 1.082631 \times 10^{-3} \\
J_3 &= \text{Earth’s third order zonal coefficient} = -2.55 \times 10^{-6} \\
J_4 &= \text{Earth’s fourth order zonal coefficient} = -1.61 \times 10^{-6} \\
U &= \text{Earth’s gravitational potential} \\
\mu &= \text{Earth’s gravitational parameter} = 1.407644176 \times 10^{16} \text{ ft}^3 / \text{s}^2 \\
\phi &= \text{Earth’s geocentric latitude} \\
(g_x, g_y, g_z) &= \text{(x,y,z) components of the gravitational acceleration} \\
\vec{r} &= \text{vehicle’s position vector} \\
r &= \text{magnitude of vehicle’s position vector} \\
(x, y, z) &= \text{(x,y,z) components of vehicle’s position vector in an inertial reference frame} \\
(a_x, a_y, a_z) &= \text{(x,y,z) components of vehicle’s absolute acceleration in an inertial reference frame} \\
\vec{v} &= \text{vehicle’s absolute velocity vector}
\end{align*}

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\( (u, v, w) = (x, y, z) \) components of vehicle’s absolute velocity vector in an inertial reference frame

\( \vec{v}_{rel} = \) relative velocity vector of the vehicle (measured in a body-fixed reference frame)

\( \vec{\Omega}_e = \) angular velocity vector of the Earth

\( \omega_z = \) z component of the Earth’s angular velocity vector \( = 7.2921152087 \times 10^{-5} \) rad/s

\( \vec{\omega} = \) angular velocity vector of the vehicle

\( (p, q, r) = (x, y, z) \) components of vehicle’s angular velocity vector in a body-fixed reference frame

\( \vec{v}_w = \) velocity of the wind

\( \vec{V}_m = \) air-stream velocity vector

\( V_m = \) magnitude of air-stream velocity

\( (V_{m,xb}, V_{m,yb}, V_{m,zb}) = (x, y, z) \) components of air-stream velocity in an inertial reference frame

\( M = \) mach number

\( c = \) speed of sound

\( Q = \) dynamic pressure

\( \rho = \) density of the air

\( \alpha = \) angle of attack

\( \beta = \) angle of sideslip

\( F_{base} = \) base force

\( ARP = \) aerodynamics reference point \( = 275.6 \) ft from pin point

\( D = \) drag (axial) force

\( C = \) lateral (side) force

\( N = \) lift (normal) force

\( S = \) reference area \( = 116.2 \) ft²

\( h_{ref} = \) reference length \( = 12.16 \) ft

\( c_{ref} = \) reference chord \( = 12.16 \) ft

\( \zeta = \) damping ratio of the actuator dynamics \( = 1 \)

\( \omega_n = \) undamped natural frequency of the actuator dynamics \( = 32.6726 \) rad/sec

\( \delta_p = \) pitch gimbal angle

\( \delta_{pc} = \) pitch gimbal angle command

\( \ddot{\delta}_p = \) pitch gimbal angle acceleration

\( \delta_y = \) yaw gimbal angle

\( \delta_{yc} = \) yaw gimbal angle command

\( \ddot{\delta}_y = \) yaw gimbal angle acceleration of the actuator dynamics

\( T = \) total thrust inside the atmosphere

\( T_0 = \) total vacuum thrust

\( A_e = \) nozzle exit area \( = 122.137 \) ft²

\( p_0 = \) local atmospheric pressure
\((F_{\text{aero,}x}, F_{\text{aero,}y}, F_{\text{aero,}z})\) = (x,y,z) components of aerodynamic force in Body Frame

\((F_{\text{rkt,}x}, F_{\text{rkt,}y}, F_{\text{rkt,}z})\) = (x,y,z) components of rocket engine force in Body Frame

\((F_{\text{rcx,}x}, F_{\text{rcx,}y}, F_{\text{rcx,}z})\) = (x,y,z) components of reaction control force in Body Frame

\((F_{\text{slosh,}x}, F_{\text{slosh,}y}, F_{\text{slosh,}z})\) = (x,y,z) components of slosh force in Body Frame

\((F_{\text{total,}x}, F_{\text{total,}y}, F_{\text{total,}z})\) = (x,y,z) components of total force in Body Frame

\((F_{\text{total,}x}, F_{\text{total,}y}, F_{\text{total,}z})\) = (x,y,z) components of total force in Inertial Frame

\((T_{\text{aero,}x}, T_{\text{aero,}y}, T_{\text{aero,}z})\) = (x,y,z) components of aerodynamic torque in Body Frame

\((T_{\text{rkt,}x}, T_{\text{rkt,}y}, T_{\text{rkt,}z})\) = (x,y,z) components of rocket engine torque in Body Frame

\((T_{\text{rcx,}x}, T_{\text{rcx,}y}, T_{\text{rcx,}z})\) = (x,y,z) components of reaction control torque in Body Frame

\((T_{\text{slosh,}x}, T_{\text{slosh,}y}, T_{\text{slosh,}z})\) = (x,y,z) components of slosh torque in Body Frame

\(T_{\text{TWDp}}\) = pitch torque on the vehicle due to the TWD effect

\(T_{\text{TWDy}}\) = yaw torque on the vehicle due to the TWD effect

\(I_e\) = nozzle inertia in plane of movement = 19102.0833 lbf ft ⋅ s²

\(M_e\) = nozzle mass = 694.4 lb

\(l_{\text{sg}}\) = distance from vehicle’s center of gravity to nozzle pivot point

\(l_c\) = distance from nozzle pivot point to nozzle center of gravity = 1.2775 ft

\(\vec{r}_s\) = vector from vehicle’s center of mass to slosh fuel center of mass in Body Frame

\(\vec{i}_{\text{sg}}\) = center of mass position vector of vehicle in Body Frame

\(\vec{i}_{\text{tank}}\) = tank location vector in Body Frame

\(\vec{i}_s\) = slosh moment arm

\(M_s\) = slosh mass

\(\zeta_s\) = damping ratio of the slosh fuel dynamics

\(\omega_s\) = undamped natural frequency of the slosh fuel dynamics

\(\eta\) = flex mode state

\(\zeta_{\text{flex}}\) = damping ratio of flex modes

\(\omega_{\text{flex}}\) = undamped natural frequency of flex modes

\(m\) = vehicle mass

\((c_x, c_y, c_z)\) = (x,y,z) components of center of mass

\(T_s\) = sampling period = 0.02 s

\(K_p\) = proportional gain

\(K_i\) = integral gain

\(K_d\) = derivative gain

I. Introduction

NOTE to Session Organizer/Reviewers: This draft manuscript summarizes the preliminary results obtained during an early phase of a new project for the dynamical modeling and flight control design of large flexible
launch vehicles as applied to Ares-I Crew Launch Vehicle. During the next several months, a more detailed, rigorous study will be conducted in the areas of coupled dynamical modeling of propulsion, aerodynamics, guidance and control, and vehicle structure. A companion paper on flight control systems analysis and design for large flexible launch vehicles is also being submitted to the Space Exploration and Transportation GNC session.

II. Definition of Coordinate Frame

A. Geocentric Equatorial Inertial Frame or Inertial Frame

Geocentric equatorial inertial frame or simply inertial frame (Fig.1). Origin is at Earth Center. Axis $z_i$ is normal to equatorial plane, pointing to North Pole; Axes $x_i$ and $y_i$ are in equatorial plane, axis $x_i$ is along direction of vernal equinox, which is the direction of intersection of Earth equatorial plane and Sun ecliptic plane.

B. Geocentric Equatorial Rotating Frame or Central Earth Frame or Earth Frame

Geocentric equatorial rotating frame is fixed to the Earth, also called central Earth Frame (Fig.2). Origin is at Earth center. Axis $z_e$ is normal to equatorial plane, pointing to North Pole, hence coincides with $z_i$. Axes $x_e$ and $y_e$ are in equatorial plane, with axis $x_e$ in Greenwich meridian. This frame has angular velocity of Earth.
C. Body Frame or Body-fixed Frame

Body-fixed frame, simply body frame (Fig.3), is rigidly fixed to the vehicle body. Origin is at empty vehicle center of mass; Axis $x_b$ is along structural longitudinal axis, pointing forward; normal axis $z_b$ is in plane of symmetry, perpendicular to $x_b$ and pointing downward; axis $y_b$ is perpendicular to plane of symmetry and pointing rightward.

III. Aerodynamics Forces and Moments

From the state variables, we can determine the position and velocity of the vehicle in the inertial frame. And then, the geographical latitude, the height over the Earth surface, angle of attack, angle of sideslip and Mach number can be calculated. By looking up the data table, aerodynamic coefficients and base force can be found. The equations of aerodynamics forces and moments can be written as follows:

$$
\vec{V}_m = \vec{V}_{rel} - \vec{V}_w = \vec{V} - \vec{\Omega} \times \vec{r}_{rel} - \vec{V}_w
$$

$$
M = \frac{V_m}{c}
$$

$$
Q = \frac{1}{2} \rho V_m^2
$$

$$
\alpha = \arctan \frac{V_{m,zb}}{V_{m,xb}}
$$

$$
\beta = \arctan \frac{V_{m,yb}}{\sqrt{V_{m,xb}^2 + V_{m,zb}^2}}
$$

$$
\begin{cases}
D = C_A Q_S - F_{base} \\
C = C_{Y\beta} \beta Q_S \\
N = (C_{N0} + C_{Na} \alpha) Q_S
\end{cases}
$$

Where, $C_A$ is axial force coefficient; $C_{Y\beta}$ is side force curve slope; $C_{N0}$ is normal force coefficient at zero angle of attack; $C_{Na}$ is normal force curve slope.
\[
\begin{align*}
T_{\text{aero},xb} & = c_y (C_{N0} + C_{Na} \alpha)QS + c_z C_{Y \beta} QS + C_{Mr \beta} QS_{\text{ref}} \\
T_{\text{aero},yb} & = -c_z (F_{\text{base}} - C_{A} QS) + (-ARP - c_x)(C_{\alpha} N_0 + C_{N\alpha} \alpha)QS + (C_{Mr \theta} + C_{Mpa \alpha})QS_{\text{ref}} \\
T_{\text{aero},zb} & = (-ARP - c_x) C_{Y \beta} QS + c_y (F_{\text{base}} - C_{\alpha} QS) + C_{my \beta} QS_{\text{ref}}
\end{align*}
\]

Where, \(C_{Mr \beta}\) is rolling moment coefficient; \(C_{Mpa \alpha}\) is pitching moment coefficient at zero angle of attack; \(C_{my \beta}\) is pitching moment curve slope; \(C_{my \beta}\) is yawing moment curve slope.

In the SAVANT, \(C_{N0} = C_{Mpa \theta} = C_{Mr \beta} = 0\).

IV. TWD Model

Tail Wag Dog model

\[
T_{\text{TWD}p} = (I_e + M \ell_{cg} \ell_e) \dot{\delta}_p
\]

\[
T_{\text{TWD}y} = (I_e + M \ell_{cg} \ell_e) \dot{\delta}_y
\]

V. Flex Model

Flex state dynamics

\[
\ddot{\eta} + 2\zeta_{\text{flex}} \omega_{\text{flex}} \dot{\eta} + \omega_{\text{flex}}^2 \eta = F_{\text{rkt}}^T \Phi_{\text{rkt}}
\]

Where, \(\eta\) is the flex mode state column vector. \(F_{\text{rkt}}^T\) is the transpose of the rocket engine force column vector, and \(\Phi_{\text{rkt}}\) is a 3 by 6 flex mode parameter matrix.

Sensor error

\[
\begin{align*}
\mathbf{e}_{\text{angle flex}} & = \psi_{\text{nav1}} \mathbf{\hat{\eta}} \\
\mathbf{\dot{e}}_{\text{rate flex}} & = \begin{bmatrix} \psi_{\text{nav2}} \mathbf{\hat{\eta}} \\ \psi_{\text{nav1}} \mathbf{\hat{\eta}} \\ \psi_{\text{nav3}} \mathbf{\hat{\eta}} \end{bmatrix}
\end{align*}
\]

LOX

\[
\begin{align*}
\Phi_{\text{lox flex}} & = \Phi_{\text{slm1}} \mathbf{\hat{\eta}} \\
\Phi_{\text{lox flex}} & = \Phi_{\text{slm1}} \mathbf{\hat{\eta}} \\
\Phi_{\text{lox flex}} & = \Phi_{\text{slm1}} \mathbf{\hat{\eta}}
\end{align*}
\]

LH2

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092407
\[ \Phi_{lh2\_flex} = \Phi_{slm2}\eta \]
\[ \Phi_{lh2\_flex} = \Phi_{slm2}\eta \]
\[ \Phi_{lh2\_flex} = \Phi_{slm2}\eta \]

Gimbal compliance

\[ \Psi_{rkt\_flex} = \Psi_{rkt}\eta \]

Where, \[ \Psi_{nav1} \cdot \Psi_{nav2} \cdot \Psi_{nav3} \cdot \Phi_{slm1} \cdot \Phi_{slm2} \cdot \Psi_{rkt} \] are 3 by 6 parameter matrix respectively.

VI. Slosh Model

The sloshing will be modeled as a spring-mass-damper system in y-z plane; we do not consider the x component. For this program, it does not include the flex model effect to the slosh model.

\[ \ddot{\mathbf{r}}_s = -2\zeta_s \omega_s \dot{\mathbf{r}}_s - \omega^2_s \mathbf{r}_s - \{\ddot{\mathbf{r}}_{rel} + \dot{\omega} \times (\dot{\mathbf{r}}_s - \dot{l}_{eg} - \dot{l}_{tank}) + 2\dot{\omega} \times \dot{\mathbf{r}}_s + \ddot{\omega} \times [\dot{\omega} \times (\dot{\mathbf{r}}_s - \dot{l}_{eg} - \dot{l}_{tank})]\} \]

Matrix Form in Body Frame:

\[ \begin{pmatrix} \ddot{x}_s \\ \ddot{y}_s \\ \ddot{z}_s \end{pmatrix} = -2\zeta_s \omega_s \begin{pmatrix} 0 \\ \ddot{y}_s \\ \ddot{z}_s \end{pmatrix} - \omega^2_s \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} - \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \begin{pmatrix} 0 & -\dot{r} & \dot{\omega} \\ \dot{r} & 0 & -\dot{p} \\ -\dot{\omega} & \dot{p} & 0 \end{pmatrix} \begin{pmatrix} x_{loc} - c_s - l_{tank} \\ y_{loc} - c_y \\ z_{loc} - c_z \end{pmatrix} \]

\[ \begin{pmatrix} F_{slosh,xb} \\ F_{slosh,yb} \\ F_{slosh,zb} \end{pmatrix} = \begin{pmatrix} m_{slosh,xb} & m_{slosh,yb} & m_{slosh,zb} \end{pmatrix} \begin{pmatrix} \ddot{x}_s \\ \ddot{y}_s \\ \ddot{z}_s \end{pmatrix} \]

\[ \begin{pmatrix} T_{slosh,xb} \\ T_{slosh,yb} \\ T_{slosh,zb} \end{pmatrix} = \begin{pmatrix} 0 & c_z & -c_y \\ -c_z & 0 & x_{loc} - c_s - l_{tank} \\ c_y & -x_{loc} + c_s + l_{tank} & 0 \end{pmatrix} \begin{pmatrix} F_{slosh,xb} \\ F_{slosh,yb} \\ F_{slosh,zb} \end{pmatrix} \]

VII. Rocket Model

Rocket Propulsion:

\[ T = T_0 - p_0 A_s \]

The x, y and z components of the thrust in the Body Frame:
\[
\begin{pmatrix}
F_{rkt,xb} \\
F_{rkt,yb} \\
F_{rkt,zb}
\end{pmatrix} =
\begin{pmatrix}
\cos \delta_p & 0 & -\sin \delta_p \\
0 & 1 & 0 \\
\sin \delta_p & 0 & \cos \delta_p
\end{pmatrix} \begin{pmatrix}
\cos \delta_y & \sin \delta_y & 0 \\
-\sin \delta_y & \cos \delta_y & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
F \\
F \\
F
\end{pmatrix}
\]

The x, y and z components of the torque due to the thrust in the Body Frame:

\[
\begin{pmatrix}
T_{rkt,xb} \\
T_{rkt,yb} \\
T_{rkt,zb}
\end{pmatrix} =
\begin{pmatrix}
0 & c_z & -c_y \\
-c_z & 0 & c_x - l_{rkt} \\
c_y & l_{rkt} - c_z & 0
\end{pmatrix} \begin{pmatrix}
F_{rkt,xb} \\
F_{rkt,yb} \\
F_{rkt,zb}
\end{pmatrix}
\]

VIII. Gravity model

\[r = \sqrt{x^2 + y^2 + z^2}\]

\[\sin \phi = \frac{z}{r}\]

\[
\begin{align*}
g_x &= \frac{\mu x}{r^3} \\
&= \left[-1 + \frac{3J_x R^2_e}{2r^3} (3 \sin^2 \phi - 1) + \frac{4J_y R^2_e}{2r^3} (5 \sin^3 \phi - 3 \sin \phi) + \frac{5J_z R^4_e}{8r^4} (35 \sin^4 \phi - 30 \sin^2 \phi + 3)\right] \\
&+ \sin \phi \left[\frac{3J_x R^2_e}{2r^3} \sin \phi + \frac{J_y R^2_e}{2r^3} (15 \sin^2 \phi - 3) + \frac{J_z R^4_e}{2r^4} (35 \sin^3 \phi - 15 \sin \phi)\right]
\end{align*}
\]

\[
\begin{align*}
g_y &= \frac{\mu y}{r^3} \\
&= \left[-1 + \frac{3J_y R^2_e}{2r^3} (3 \sin^2 \phi - 1) + \frac{4J_x R^2_e}{2r^3} (5 \sin^3 \phi - 3 \sin \phi) + \frac{5J_z R^4_e}{8r^4} (35 \sin^4 \phi - 30 \sin^2 \phi + 3)\right] \\
&+ \sin \phi \left[\frac{3J_y R^2_e}{2r^3} \sin \phi + \frac{J_x R^2_e}{2r^3} (15 \sin^2 \phi - 3) + \frac{J_z R^4_e}{2r^4} (35 \sin^3 \phi - 15 \sin \phi)\right]
\end{align*}
\]

\[
\begin{align*}
g_z &= \frac{\mu z}{r^3} \\
&= \left[-1 + \frac{3J_z R^2_e}{2r^3} (3 \sin^2 \phi - 1) + \frac{4J_y R^2_e}{2r^3} (5 \sin^3 \phi - 3 \sin \phi) + \frac{5J_x R^4_e}{8r^4} (35 \sin^4 \phi - 30 \sin^2 \phi + 3)\right] \\
&+ \cos^2 \phi \left[\frac{3J_z R^2_e}{2r^3} \sin \phi + \frac{J_y R^2_e}{2r^3} (15 \sin^2 \phi - 3) + \frac{J_x R^4_e}{2r^4} (35 \sin^3 \phi - 15 \sin \phi)\right]
\end{align*}
\]

IX. Actuator Model

The actuator model is a second order system.

\[\ddot{\delta}_p + 2 \zeta \omega_n \dot{\delta}_p + \omega_n^2 \delta_p = \omega^2_p \delta_{pc}\]
\[
\begin{align*}
\ddot{\delta}_p &= \left[ \begin{array}{c}
0 \\
-\omega^2_n
\end{array} \right] \dot{\delta}_p + \left[ \begin{array}{c}
1 \\
-2\zeta \omega_n \delta_p
\end{array} \right] \delta_p + \left[ \begin{array}{c}
0 \\
\omega^2_n
\end{array} \right] \delta_{pc} \\
\delta_p + 2\zeta \omega_n \delta_p + \omega^2_n \delta_p &= \omega^2_n \delta_{pc} \\
\ddot{\delta}_y &= \left[ \begin{array}{c}
0 \\
-\omega^2_n
\end{array} \right] \dot{\delta}_y + \left[ \begin{array}{c}
1 \\
-2\zeta \omega_n \delta_y
\end{array} \right] \delta_y + \left[ \begin{array}{c}
0 \\
\omega^2_n
\end{array} \right] \delta_{yc}
\end{align*}
\]

**X. Force and Momentum**

In Body Frame:

\[
\begin{pmatrix}
F_{\text{total},xb} \\
F_{\text{total},yb} \\
F_{\text{total},zb}
\end{pmatrix} = \begin{pmatrix}
F_{\text{aero},xb} \\
F_{\text{aero},yb} \\
F_{\text{aero},zb}
\end{pmatrix} + \begin{pmatrix}
F_{\text{rkt},xb} \\
F_{\text{rkt},yb} \\
F_{\text{rkt},zb}
\end{pmatrix} + \begin{pmatrix}
F_{\text{rcs},xb} \\
F_{\text{rcs},yb} \\
F_{\text{rcs},zb}
\end{pmatrix} + \begin{pmatrix}
F_{\text{slosh},xb} \\
F_{\text{slosh},yb} \\
F_{\text{slosh},zb}
\end{pmatrix}
\]

To transfer the (x,y,z) components of total force from Body Frame to Inertial Frame by the quaternion \( \mathbf{q} = (q_1, q_2, q_3, q_4)^T \).

\[
\begin{pmatrix}
F_{\text{total},xi} \\
F_{\text{total},yi} \\
F_{\text{total},zi}
\end{pmatrix} = \begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix} \begin{pmatrix}
q_1 & q_2 & q_3 \\
q_4 & -q_3 & q_2 \\
-q_2 & q_1 & q_4
\end{pmatrix} \begin{pmatrix}
F_{\text{total},xb} \\
F_{\text{total},yb} \\
F_{\text{total},zb}
\end{pmatrix}
\]

In Inertial Frame:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix}
a_x \\
a_y \\
a_z
\end{pmatrix} = \frac{1}{m} \begin{pmatrix}
F_{\text{total},xi} \\
F_{\text{total},yi} \\
F_{\text{total},zi}
\end{pmatrix} + \begin{pmatrix}
g_x \\
g_y \\
g_z
\end{pmatrix}
\]

Kinematical Equation of Rotation:

\[
\begin{pmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix} = \frac{1}{\cos \theta} \begin{pmatrix}
1 & \sin \phi \sin \theta & \cos \phi \sin \theta \\
0 & \cos \phi \cos \theta & -\sin \phi \cos \theta \\
0 & \sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
\]

Angular acceleration:
\[
\begin{pmatrix}
I_x & I_{xy} & I_{xz} \\
I_{xy} & I_y & I_{yz} \\
I_{xz} & I_{yz} & I_z
\end{pmatrix}
\begin{pmatrix}
p \\
\dot{q} \\
\dot{r}
\end{pmatrix} =
\begin{pmatrix}
0 & -r & q \\
r & 0 & -p \\
-q & p & 0
\end{pmatrix}
\begin{pmatrix}
I_x & I_{xy} & I_{xz} \\
I_{xy} & I_y & I_{yz} \\
I_{xz} & I_{yz} & I_z
\end{pmatrix}
\begin{pmatrix}
p \\
q \\
r
\end{pmatrix}
+ \begin{pmatrix}
T_{aero,xb} \\
T_{aero,yb} \\
T_{aero,zb}
\end{pmatrix} + \begin{pmatrix}
T_{rkt,xb} \\
T_{rkt,yb} \\
T_{rkt,zb}
\end{pmatrix} + \begin{pmatrix}
T_{rcs,xb} \\
T_{rcs,yb} \\
T_{rcs,zb}
\end{pmatrix} + \begin{pmatrix}
T_{slosh,xb} \\
T_{slosh,yb} \\
T_{slosh,zb}
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[\text{XI. Quaternion}\]

To renormalize quaternion:
\[
\begin{pmatrix}
q_{1n} \\
q_{2n} \\
q_{3n} \\
q_{4n}
\end{pmatrix} = (1.5 - 0.5\sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2})
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix}
\]

Where, \(\begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T\) is the quaternion column vector and \(\begin{pmatrix} q_{1n} & q_{2n} & q_{3n} & q_{4n} \end{pmatrix}^T\) is the quaternion column vector after renormalization.

Quaternion derivatives:
\[
\begin{pmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
0 & r & -q & p \\
r & 0 & p & q \\
-q & p & 0 & r \\
-p & -q & -r & 0
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix}
\]

Quaternion conjugate:
\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix}^* =
\begin{pmatrix}
-q_1 \\
-q_2 \\
-q_3 \\
q_4
\end{pmatrix}
\]

\[\text{XII. Guidance Commands}\]

\[\text{XIII. Control System}\]

Discrete PID controller for roll, pitch and yaw channels
\[
u(t) = K_p e(t) + K_i \int e(t) + K_d \dot{e}(t)
\]
Discrete integration

\[ Y_i(t) = Y_i(t-1) + \frac{T}{2} [e(t) + e(t-1)] \]

XIV. System Analysis

1. Center of Pressure

\[ M_{LE} = -c_s L + M_{cg} = -x_{cp} L \]

\[ x_{cp} = c_s - \frac{M_{cg}}{L} \]

Where, \( M_{LE} \) is the pitching moment relative to the pin point of the vehicle; \( M_{cg} \) is the pitching moment with respect to the center of mass, \( x_{cp} \) is the location of center of pressure.

According to the definition (force-and-moment system) of center of pressure, the plot of center of pressure can be seen in the following figure.
Figure 6. Moments of inertia

Figure 7. Relative velocity

Figure 8. Mach number

Figure 9. Dynamic pressure
Figure 10. Pitch gimbal angle command

Figure 11. Yaw gimbal angle command

Figure 12. Angle of attack and sideslip angle

Figure 13. Total thrust
Figure 14  Solid rocket booster weight flow rate

Figure 15  Solid rocket boost specific impulse

Figure 16  Euler angle

Figure 17  Axial force coefficient
Figure 18. Normal force curve slop

Figure 19. Side force curve slop

Figure 20. Pitching moment curve slop

Figure 21. Yawing moment curve slop
2. Stability Analysis

XV. Sample Case

\[
\Phi_{\text{rh}} = \begin{pmatrix}
0.00000272367963 & 0.00000174392026 & -0.00000347086527 \\
-0.000364943105155 & 0.006281028219530 & 0.000491932740239 \\
0.006281175443849 & 0.000364891432306 & -0.006260333099131 \\
-0.00000266173427 & 0.000000329288262 & -0.000000369058169 \\
-0.006259406451949 & -0.000542750533582 & -0.007673360355205 \\
-0.000491798506301 & 0.007676195145027 & -0.000542218216634
\end{pmatrix}
\]

Figure 22. Total Mass
Table 1. Initial States’ Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial Value (unit)</th>
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<tbody>
<tr>
<td>U</td>
<td>1340.649569297657 (ft/sec)</td>
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<tr>
<td>V</td>
<td>6.542058392976782 (ft/sec)</td>
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<tr>
<td>W</td>
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<tr>
<td>P</td>
<td>0.000034916172322 (rad/sec)</td>
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<tr>
<td>Q</td>
<td>0.000064018398388 (rad/sec)</td>
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<tr>
<td>R</td>
<td>1.01643953670516e-20 (rad/sec)</td>
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<td>X</td>
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<tr>
<td>Y</td>
<td>1838483.09303243 (ft)</td>
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<tr>
<td>Z</td>
<td>996046.310921878 (ft)</td>
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<td>q1</td>
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<tr>
<td>q2</td>
<td>0.608859075914155</td>
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<tr>
<td>q3</td>
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<tr>
<td>q4</td>
<td>0.60720847366793</td>
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</table>

XVI. Conclusion

Appendix

Acknowledgments

References