Reduced Order Modeling of Incompressible Flows

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Develop low-dimensional model that can represent the dynamics of a higher dimensional system

Uses:
- Models for real-time control
- Sub-model generation

Difficulties:
- Round-off sensitive
- Unstable / poorly conditioned
- Units inconsistency for systems
Particle Modeling

- Incompressible, flow over a sphere with variable inlet velocity

\[ u = u_\infty (1 + A \sin(2\pi t/T)) \]

- Axial Velocity

- Centerline pressure

- Force
Generate Modes

- Find decomposition $\sum_j a_j(t)\phi_j(x)$ to represent solution $u(x, t)$
- Choose optimal functions by maximizing mean square projection:
  \[ \Pi(\phi) = \frac{\langle (\phi, u)^2 \rangle}{(\phi, \phi)} \]
- Some trouble:
  \[ \vec{u} = \begin{bmatrix} u_r \\ u_z \\ p \end{bmatrix} \]
  \[ \Pi(\phi) = \frac{\langle (\vec{\phi}, \vec{u})^2 \rangle}{(\vec{\phi}, \vec{\phi})} \]

Dimensionally inconsistent?
Possible Choices for $\vec{u}$

- Depends on non-dimensionalization:

  $$\vec{u} = (u_r(\vec{x}, t), u_z(\vec{x}, t), p(\vec{x}, t)/(\rho u_\infty))$$

- Could be imaginary:

  $$\vec{u} = \left( u_r(\vec{x}, t), u_z(\vec{x}, t), \sqrt{2(p(\vec{x}, t))}/\rho \right)$$

- Also has arbitrary constant:

  $$\vec{u} = \left( u_r(\vec{x}, t), u_z(\vec{x}, t), \sqrt{2(p(\vec{x}, t) + p_0)}/\rho \right)$$
31 steady snapshots from $Re = 0.1$ to 100

- Eigenvalues exponentially decay
- Small number of modes can capture most of the energy
- Last five modes of Lapack DGESVD are negative
Failure of Galerkin Projection

\[
\int_{\Omega} \left[ \phi_r, \phi_z, \phi_p \right] \left[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\nabla p + \mu \nabla^2 \vec{V} \right] d\Omega = 0
\]

- Modes are all incompressible
- Continuity is always zero
- \( \nabla p \) term integrated by parts is zero
- Pressure modes may be undetermined
Streamwise-upwind-Petrov-Galerkin variational approach

Allows us to seek $u_r, u_z, p$ with no pressure decoupling

Upwinds all the terms in the governing equations consistently

Results in a system of $M$ ODE’s (Solved using DIRK & Newton-Rhapson)
Accuracy of drag prediction versus $Re$ for 5 and 10 modes

Significant improvement over DNS (10 vs. 21,000 degrees of freedom)

Close to empirical correlations in performance

All three sets of modes perform similarly
Unsteady Data

- $Re = 0.1, 1, 10, 100 \times St = 0.1, 0.5, 1, 2, 10$
- 15 time-steps/period, 20x15 snapshots

All eigenvalues after $\approx 60$ are probably round-off dominated.

Using Lapack DGESVD, all modes beyond 180 have negative eigenvalues
Unsteady Results

Re = 0.1, St = 0.1

Re = 0.1, St = 10

Re = 100, St = 0.1

Re = 100, St = 10
Comparison of Modes

- $Re = 1$ and $St = 1$.
- SVD modes seem ok in spite of eigenvalue distribution.
- Low Mach $\neq$ modes give similar results with 20 modes, but for 60 modes Newton diverges.
Conclusions

▶ The devil is in the details!!!
  ▶ Need stable numerical methods
  ▶ Round off error can be considerable
  ▶ Not convinced modes are correct for incompressible flow
▶ Nonetheless, can derive compact and accurate reduced-order models.
▶ Can be used to generate actuator models or full flow-field models