Reduced Order Modeling of Incompressible Flows

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POD-based Reduced Order Modeling

- Develop low-dimensional model that can represent the dynamics of a higher dimensional system
- Uses:
  - Models for real-time control
  - Sub-model generation
- Difficulties:
  - Round-off sensitive
  - Unstable / poorly conditioned
  - Units inconsistency for systems
Particle Modeling

- Incompressible, flow over a sphere with variable inlet velocity

\[ u = u_\infty (1 + A \sin(2\pi t/T)) \]

- Axial Velocity

- Centerline pressure

- Force
Generate Modes

- Find decomposition $\sum_j a_j(t)\phi_j(x)$ to represent solution $u(x, t)$
- Choose optimal functions by maximizing mean square projection:
  \[ \Pi(\phi) = \frac{\langle (\phi, u)^2 \rangle}{\langle \phi, \phi \rangle} \]
- Some trouble:
  \[ \vec{u} = \begin{bmatrix} u_r \\ u_z \\ p \end{bmatrix} \]
  \[ \Pi(\phi) = \frac{\langle \vec{\phi}, \vec{u} \rangle^2}{\langle \vec{\phi}, \vec{\phi} \rangle} \]
  Dimensionally inconsistent?
Possible Choices for $\vec{u}$

- Depends on non-dimensionalization:
  \[
  \vec{u} = (u_r(\vec{x}, t), u_z(\vec{x}, t), p(\vec{x}, t)/(\rho u_\infty))
  \]

- Could be imaginary:
  \[
  \vec{u} = \left( u_r(\vec{x}, t), u_z(\vec{x}, t), \sqrt{2(p(\vec{x}, t))}/\rho \right)
  \]

- Also has arbitrary constant:
  \[
  \vec{u} = \left( u_r(\vec{x}, t), u_z(\vec{x}, t), \sqrt{2(p(\vec{x}, t) + p_0)}/\rho \right)
  \]
Eigenvalues of Steady Problem

- 31 steady snapshots from $Re = 0.1$ to 100
- Eigenvalues exponentially decay
- Small number of modes can capture most of the energy
- Last five modes of Lapack DGESVD are negative
Failure of Galerkin Projection

\[
\int_{\Omega} \left[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\nabla p + \mu \nabla^2 \vec{v} \right] \, d\Omega = 0
\]

- Modes are all incompressible
- Continuity is always zero
- \( \nabla p \) term integrated by parts is zero
- Pressure modes may be undetermined
SUPG Projection

\[
\sum_{e=1}^{n_{el}} \left\{ \int \int_{\Omega} \left[ \left( -\frac{\partial \phi^T}{\partial \xi} \bar{e} - \frac{\partial \phi^T}{\partial \eta} \bar{f} \right) \right] d\Omega + \int_{\Gamma} \phi^T \left( \bar{e}, \bar{f} \right) \cdot \vec{n}_\Gamma \ d\Gamma \\
+ \int \int_{\Omega} \left[ \frac{\partial \phi^T}{\partial \xi} \frac{\partial \bar{e}}{\partial \bar{u}} + \frac{\partial \phi^T}{\partial \eta} \frac{\partial \bar{f}}{\partial \bar{u}} \right] \ T \left[ \frac{\partial}{\partial \xi} \bar{e} + \frac{\partial}{\partial \eta} \bar{f} \right] d\Omega \right\} = 0 \ \forall \phi
\]

- Streamwise-upwind-Petrov-Galerkin variational approach
- Allows us to seek \( u_r, u_z, p \) with no pressure decoupling
- Upwinds all the terms in the governing equations consistently
- Results in a system of \( M \) ODE’s (Solved using DIRK & Newton-Rhapson)
Accuracy of drag prediction versus $Re$ for 5 and 10 modes

Significant improvement over DNS (10 vs. 21,000 degrees of freedom)

Close to empirical correlations in performance

All three sets of modes perform similarly
Unsteady Data

- $Re = 0.1, 1, 10, 100 \times St = 0.1, 0.5, 1, 2, 10$
- 15 time-steps/period, 20x15 snapshots

All eigenvalues after $\approx 60$ are probably round-off dominated.

Using Lapack DGESVD, all modes beyond 180 have negative eigenvalues.
Unsteady Results
Comparison of Modes

- \( \text{Re} = 1 \) and \( \text{St} = 1 \).
- SVD modes seem ok in spite of eigenvalue distribution.
- Low Mach \# modes give similar results with 20 modes, but for 60 modes Newton diverges.
Conclusions

- The devil is in the details!!!
  - Need stable numerical methods
  - Round off error can be considerable
  - Not convinced modes are correct for incompressible flow
- Nonetheless, can derive compact and accurate reduced-order models.
- Can be used to generate actuator models or full flow-field models