Comparison of Traditional Design Nonlinear Programming Optimization and Stochastic Methods for Structural Design

Structural design generated by traditional method, optimization method and the stochastic design concept are compared. In the traditional method, the constraints are manipulated to obtain the design and weight is back calculated. In design optimization, the weight of a structure becomes the merit function with constraints imposed on failure modes and an optimization algorithm is used to generate the solution. Stochastic design concept accounts for uncertainties in loads, material properties, and other parameters and solution is obtained by solving a design optimization problem for a specified reliability. Acceptable solutions were produced by all the three methods. The variation in the weight calculated by the methods was modest. Some variation was noticed in designs calculated by the methods. The variation may be attributed to structural indeterminacy. It is prudent to develop design by all three methods prior to its fabrication. The traditional design method can be improved when the simplified sensitivities of the behavior constraint is used. Such sensitivity can reduce design calculations and may have a potential to unify the traditional and optimization methods. Weight versus reliability traced out an inverted-S-shaped graph. The center of the graph corresponded to mean valued design. A heavy design with weight approaching infinity could be produced for a near-zero rate of failure. Weight can be reduced to a small value for a most failure-prone design. Probabilistic modeling of load and material properties remained a challenge.

Comparison of Traditional Design Nonlinear Programming Optimization and Stochastic Methods for Structural Design

Surya N. Patnaik
Ohio Aerospace Institute
and
Shantaram S. Pai and Rula M. Coroneos
NASA Glenn Research Center

Presented by Rula M. Coroneos

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Objective

- Summarize work performed supported by the Subsonic Fixed Wing project under NASA's Fundamental Aeronautics Program.
- Focus:
- Comparison of Three Structural Design Methods
  - Traditional Design
  - Nonlinear Optimization Programming Techniques
  - Stochastic Design Optimization
    - Probabilistic Analysis – Four methods
      - Solution for a three bar truss
- Stochastic Design Optimization Code (SDOC)
  - CometBoards testbed as Optimizer
  - FPI of NESSUS code as Probabilistic estimator
  - MSC/Nastran as Structural Analyzer
- Results for Three Design Problems
  - Tapered Beam - Determinate
  - Clamped Beam - Indeterminate
  - Raked Wing Tip Structure of Boeing 767-ER airliner - Complex
- Conclusions

Methods for Structural Design

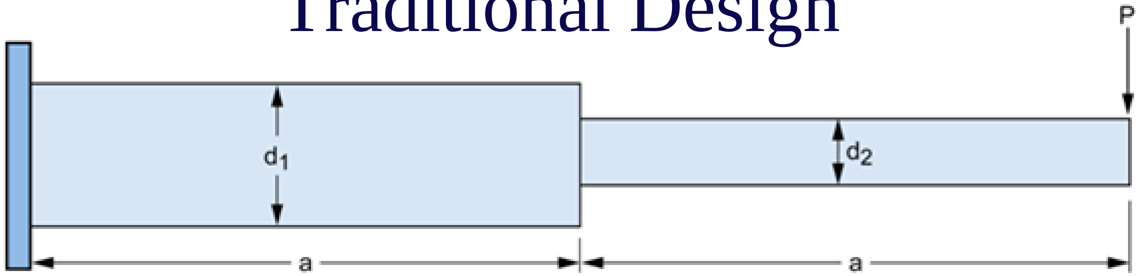
Traditional Design
Find n design variables {x} within bounds to satisfy (fully utilize) m constraints $g_j <= 0$
Constraints are manipulated to obtain the design.
A condition to minimize weight is *not* imposed

Deterministic Optimization
Find n design variables {x} within bounds
to minimize weight W(x)
subjected to m constraints $g_j <= 0$
Nonlinear mathematical programming algorithms

Stochastic Design
Find n mean values for n random variables {x} within bounds
to minimize mean value of weight W(x)
subjected to m constraint $g_j (p) <=0$
defined as: $P\{g_j <=0\} >= p_j$
Same as deterministic design but the parameters are random in nature

Probabilistic analysis
Primitive random variables are defined through a distribution function with a mean value and a standard deviation
Response is calculated via MPP of NESSUS code. The influence of each variable on the probability of constraint satisfaction can be obtained by calculating probabilistic sensitivities

Traditional Design



- Basic problem: Calculate design variables d_1 and d_2 to find the weight subject to strength and displacement constraints.
- Traditional design method is based on a fully utilized concept
- Design is first obtained for strength limitation:

$$\sigma_0 = \sigma_1 = \left(\frac{My}{I}\right)_1 \; and \; \sigma_0 = \sigma_2 = \left(\frac{My}{I}\right)_2$$

$$d_1 = \sqrt{6M_1 / b\sigma_o}; \; d_2 = \sqrt{6M_2 / b\sigma_o}$$

- Stiffness constraint can be written as:

$$g_j(d_1, d_2) = \left(\frac{Pa^3}{E}\right)\left(\frac{7}{d_1^3} + \frac{1}{d_2^3}\right) - \delta_0$$

- Design can be adjusted for violated stiffness constraints:

$$d_1^{new} = k\, d_1^s \qquad d_2^{new} = k\, d_2^s$$

$$k = a\left(\left(\frac{P}{\delta_{max} E}\right)\left(\frac{7}{(d_1^s)^3} + \frac{1}{(d_2^s)^3}\right)\right)^{\frac{1}{3}}; k \geq 1$$

- Constraint is satisfied by many sets of (d_1, d_2) and weight is not unique
- Weight is not used in traditional design but back calculated

Deterministic Optimization

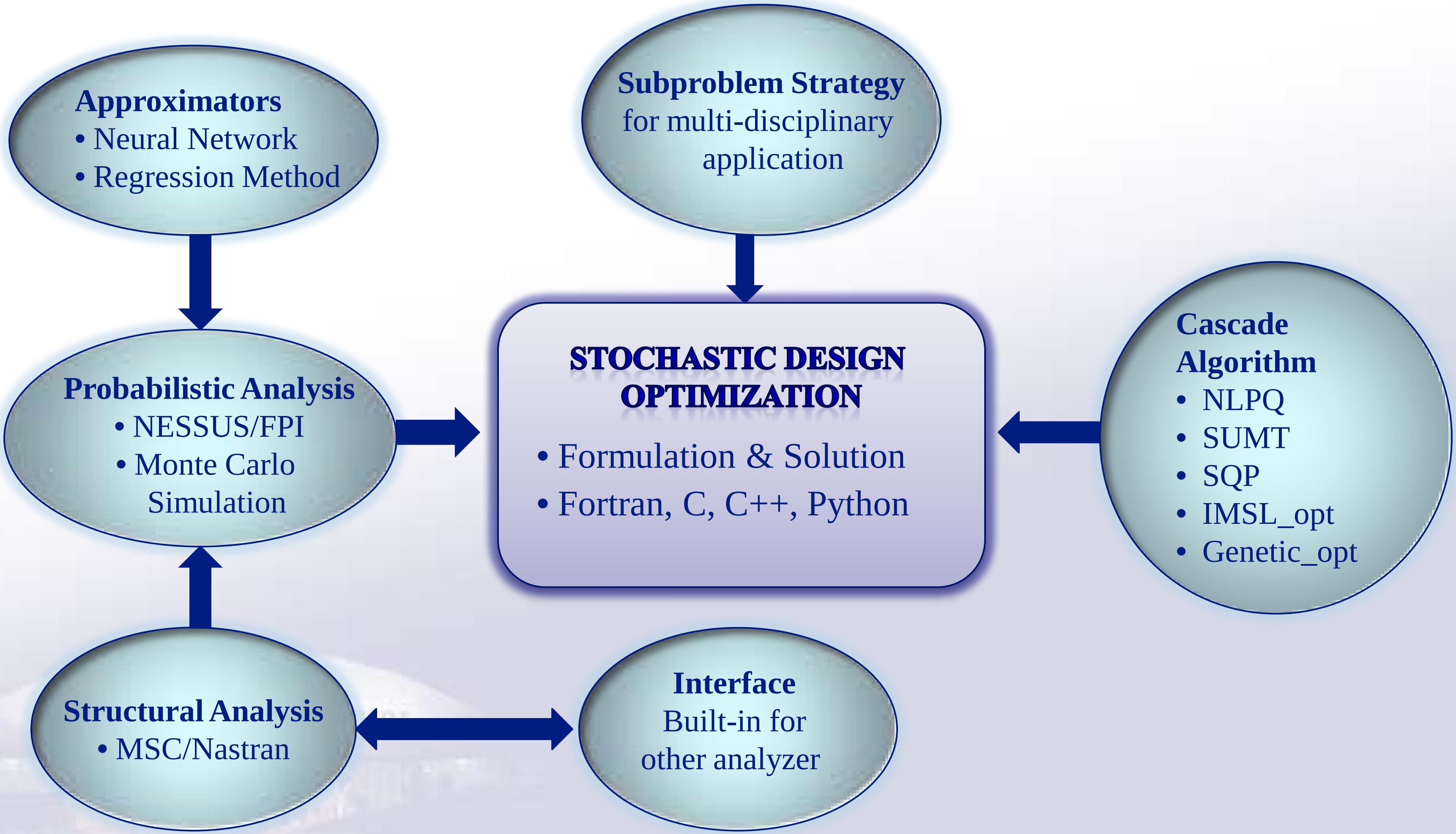
- Cast as a nonlinear mathematical programming problem
- Find **x** that minimize $\boldsymbol{W}(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$, $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x}_{lb} \leq \mathbf{x} \leq \mathbf{x}_{ub}$

  where $\boldsymbol{W}$ is an objective, **x** is a vector of design variables, **g** is a vector of inequality constraints, **h** is a vector of equality constraints, and $\mathbf{x}_{lb}$ and $\mathbf{x}_{ub}$ are vectors of lower and upper bounds on the design variables.

- Applications of nonlinear programming include: automobile design, naval architecture, electronics, computers, aerospace engineering, such as aircraft and spacecraft design, etc.

- Implemented several methods in Stochastic Design Software Code (SDOC)
  - MSC/Nastran was the analyzer
  - Optimizers were taken from CometBoards

Organization of Stochastic Design Optimization Code (SDOC) at NASA GRC



Can solve structures, airliner and jet engine design problems

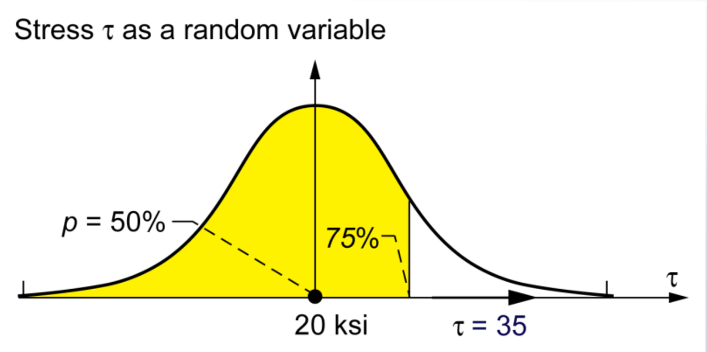
Being interfaced in OpenMDAO Framework at NASA GRC

Stochastic Design Optimization

- Similar to deterministic optimization but parameters are random in nature

$$g_j^{Deterministic} = \left(\left| \frac{\tau}{\tau_0} \right| \right) - 1 \le 0 \qquad \tau_0 = \frac{\tau_{yield}}{\textit{factor of safety}}.$$

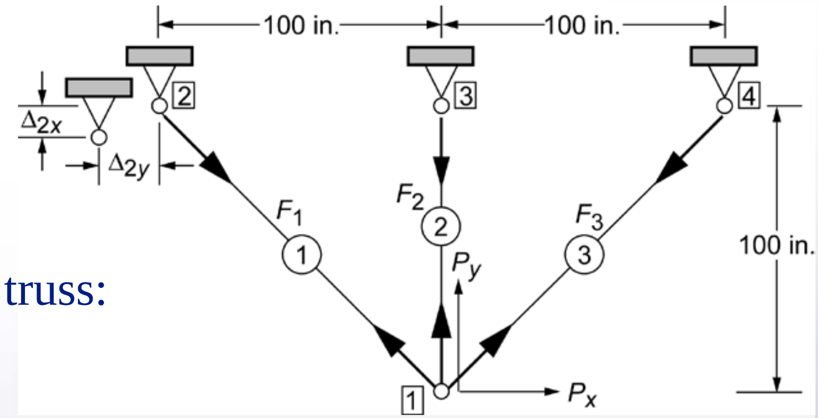
$$g_j^{Stochastic}(p) = \underbrace{\left[\left| \frac{\mu_{\tau_1}}{\mu_{\tau_{10}}} \right| - 1 \right]}_{\text{Mean value}} + \underbrace{\left[\Phi^*(p) \frac{\sqrt{\sigma_{\tau_1}^2 + \sigma_{\tau_{10}}^2}}{\mu_{\tau_{10}}} \right]}_{\text{Standard deviation}} \le 0$$



- For stress allowable we require three assumed parameters:
  - Distribution function (Normal) for strength (random variable)
  - Mean value
  - Standard deviation
- For stress response we assume a Normal distribution function
- Probability of success also needs to be chosen
- Accuracy of design depends on these parameters four of which were assumed

Summary of Probabilistic Analyses Methods

- Perturbation method (PM)
- Direct Monte Carlo simulation (DMCS)
- Latin hypercube simulation (LHS)
- Fast probabilistic integration (FPI)



Three bar truss

Ten primitive random variables (q_1-q_{10}) for 3-bar truss:

$$A_1 = \mu_{A_1}\left(1+q_{A_1}\right) = \mu_1\left(1+q_1\right)$$

$$A_2 = \mu_{A_2}\left(1+q_{A_2}\right) = \mu_2\left(1+q_2\right)$$

$$A_3 = \mu_{A_3}\left(1+q_{A_3}\right) = \mu_3\left(1+q_3\right)$$

$$E = \mu_E\left(1+q_E\right) = \mu_4\left(1+q_4\right)$$

$$\alpha = \mu_\alpha\left(1+q_\alpha\right) = \mu_5\left(1+q_5\right)$$

$$P_X = \mu_{P_X}\left(1+q_{P_X}\right) = \mu_6\left(1+q_6\right)$$

$$P_Y = \mu_{P_Y}\left(1+q_{P_Y}\right) = \mu_7\left(1+q_7\right)$$

$$T = \mu_T\left(1+q_T\right) = \mu_8\left(1+q_8\right)$$

$$\Delta_{2x} = \mu_{\Delta_1}\left(1+q_{\Delta_1}\right) = \mu_9\left(1+q_9\right)$$

$$\Delta_{2y} = \mu_{\Delta_2}\left(1+q_{\Delta_2}\right) = \mu_{10}\left(1+q_{10}\right)$$

- Covariance matrix included bar areas with

Mean value: $\begin{Bmatrix} \mu_{A_1} \\ \mu_{A_2} \\ \mu_{A_3} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}$ in.^{2}

Covariance matrix: $\operatorname{cov}\left(\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}\right) = \begin{bmatrix} 1.00 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.25 \\ 0.25 & 0.25 & 1.00 \end{bmatrix} \times 10^{-2}$

- Similar for other variables

Probabilistic Analyses Results

Parameter	Perturbation method (PM)		Direct Monte Carlo simulation (DMCS)		Latin hypercube simulation (LHS)		Fast probability integrator (FPI)	
	Mean value	Standard deviation	Mean value	Standard deviation	Mean value	Standard deviation	Mean value	Standard deviation
Force: $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}^{\text{Kip}}$	$\begin{Bmatrix} 62.77 \\ 61.24 \\ -7.95 \end{Bmatrix}$	$\begin{Bmatrix} 4.39 \\ 4.35 \\ 4.67 \end{Bmatrix}$	$\begin{Bmatrix} 62.77 \\ 61.27 \\ -7.93 \end{Bmatrix}$	$\begin{Bmatrix} 4.39 \\ 4.35 \\ 4.67 \end{Bmatrix}$	$\begin{Bmatrix} 62.76 \\ 61.25 \\ -7.96 \end{Bmatrix}$	$\begin{Bmatrix} 4.39 \\ 4.35 \\ 4.67 \end{Bmatrix}$	$\begin{Bmatrix} 62.78 \\ 61.21 \\ -7.93 \end{Bmatrix}$	$\begin{Bmatrix} 4.39 \\ 4.35 \\ 4.67 \end{Bmatrix}$
Stress: $\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}^{\text{Ksi}}$	$\begin{Bmatrix} 63.29 \\ 61.70 \\ -3.98 \end{Bmatrix}$	$\begin{Bmatrix} 6.71 \\ 6.20 \\ 2.34 \end{Bmatrix}$	$\begin{Bmatrix} 63.16 \\ 61.58 \\ -3.97 \end{Bmatrix}$	$\begin{Bmatrix} 5.89 \\ 5.42 \\ 2.79 \end{Bmatrix}$	$\begin{Bmatrix} 63.15 \\ 61.57 \\ -3.99 \end{Bmatrix}$	$\begin{Bmatrix} 5.77 \\ 5.38 \\ 2.75 \end{Bmatrix}$	$\begin{Bmatrix} 62.78 \\ 61.21 \\ -3.96 \end{Bmatrix}$	$\begin{Bmatrix} 6.34 \\ 6.03 \\ 2.69 \end{Bmatrix}$
Displacement: $\begin{Bmatrix} u \\ v \end{Bmatrix}^{\text{in.}}$	$\begin{Bmatrix} 0.20 \\ -0.24 \end{Bmatrix}$	$\begin{Bmatrix} 0.004 \\ 0.003 \end{Bmatrix}$	$\begin{Bmatrix} 0.20 \\ -0.24 \end{Bmatrix}$	$\begin{Bmatrix} 0.004 \\ 0.003 \end{Bmatrix}$	$\begin{Bmatrix} 0.20 \\ -0.24 \end{Bmatrix}$	$\begin{Bmatrix} 0.004 \\ 0.003 \end{Bmatrix}$	$\begin{Bmatrix} 0.20 \\ -0.24 \end{Bmatrix}$	$\begin{Bmatrix} 0.004 \\ 0.003 \end{Bmatrix}$
CPU seconds	7.0		4245.0		390.0		1.0	
Normalized time	1.0		606		56		1/7	

- Performance was satisfactory for all four methods
- Monte Carlo simulation is numerically expensive while FPI took the least cpu
- FPI is used for probabilistic analysis throughout this study

Comparison of Results for Three Structural Design Methods

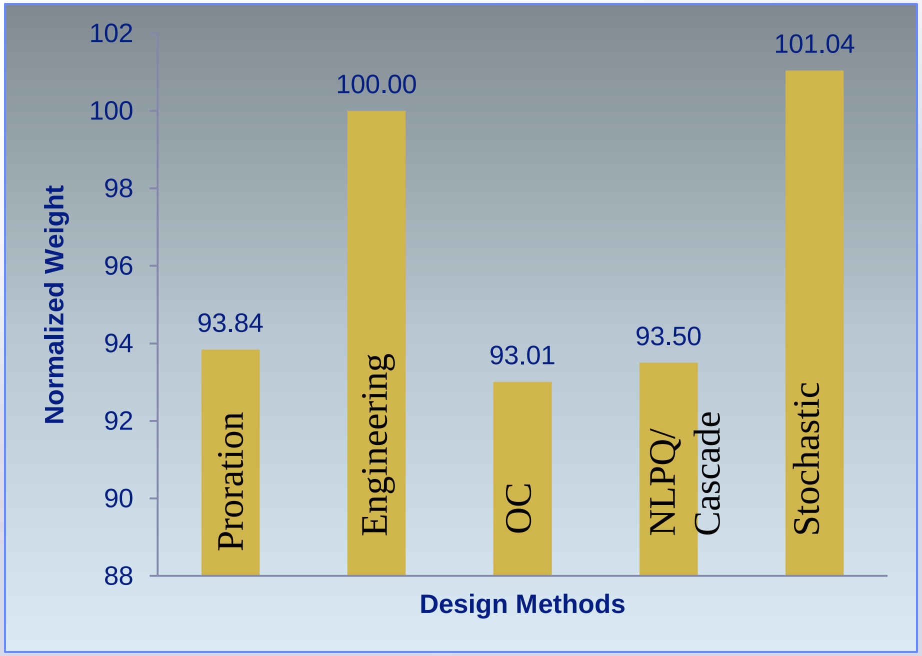
Design of a Tapered Beam

Calculate depths d_1 and d_2 for strength and displacement limitations

Variable	Deterministic value	Mean value	Standard deviation
Stress limit: σ_0	20 ksi	$\mu_{\sigma_0} = 25\ ksi$	$\bar{\sigma}_{\sigma_0} = 2.5\ ksi$
Disp. limit: δ	1.0 in.	$\mu_{\delta} = 1.25\ in$	$\bar{\sigma}_{\delta} = 0.125\ in.$
Modulus: E	10 ksi	$\mu_E = 10\ ksi$	$\bar{\sigma}_E = 1\ ksi$
Density: ρ	0.1 lb/in.	$\mu_{\rho} = 0.1$	$\bar{\sigma}_{\rho} = 0.005\ in.$
Load: P	20 kip	$\mu_P = 15\ kip$	$\bar{\sigma}_P = 3\ kip.$
Depth: d_1, d_2	Calculate	$\mu_{d_1} = 25\ in\ and\ \mu_{d2} = 10$	$\bar{\sigma}_{d_1} = 3\ in\ and\ \bar{\sigma}_{d2} = 2\ in$
Limitation		Assumed but not measured values	

Design Solutions for Tapered Beam by Different Methods

Method	Design variables in inch		Weight (lbf)	Max stress	Max disp
	d_1	d_2			
Proration Technique	19.4	13.8	955.4	11.4	1.0 (active)
Engineering Method	25.0	10.4	1018.1	20.0	1.0 (active)
Optimality Criteria	20.4	12.5	946.9	13.8	1.0 (active)
NLPQ Algorithm	20.4	12.6	951.9	13.6	1.0 (active)
Cascade Strategy	20.5	12.5	951.9	13.8	1.0 (active)
Stochastic Design for 1 failure in one million samples	25.1	10.2	1028.7	---	active

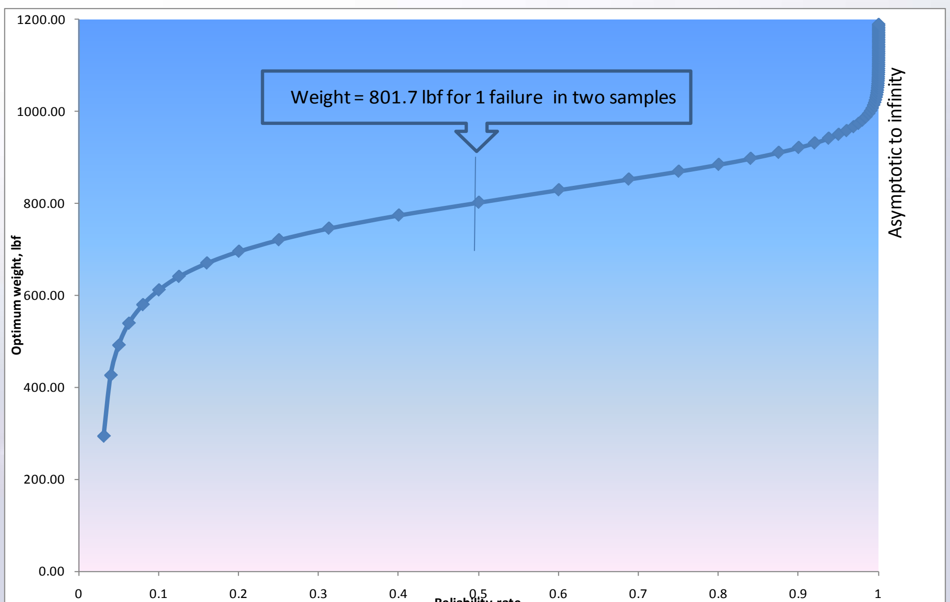


Summary

- No unique solution even for the determinate beam
- Variation in solutions
  - Weight: 8 percent
  - Design variables: 35 percent
- Performance
  - Acceptable by all methods

Stochastic Design Solution for Tapered Beam

N (One failure in N samples)	Design variable d_1 in inch.	Variable d_2	Weight in lbf (mean value)
2	17.2	10.6	801.7
10	19.8	12.2	921.0
100	21.5	13.2	1000.5
1000	22.7	14.0	1055.4
10,000	23.6	14.6	1100.6
100,000	24.5	15.1	1140.6
200,000	24.7	15.2	1152.1
500,000	25.1	15.4	1167.0
1 million	25.3	15.6	1178.0
1.25 million	25.4	15.6	1181.6
2 million	25.5	15.7	1189.0

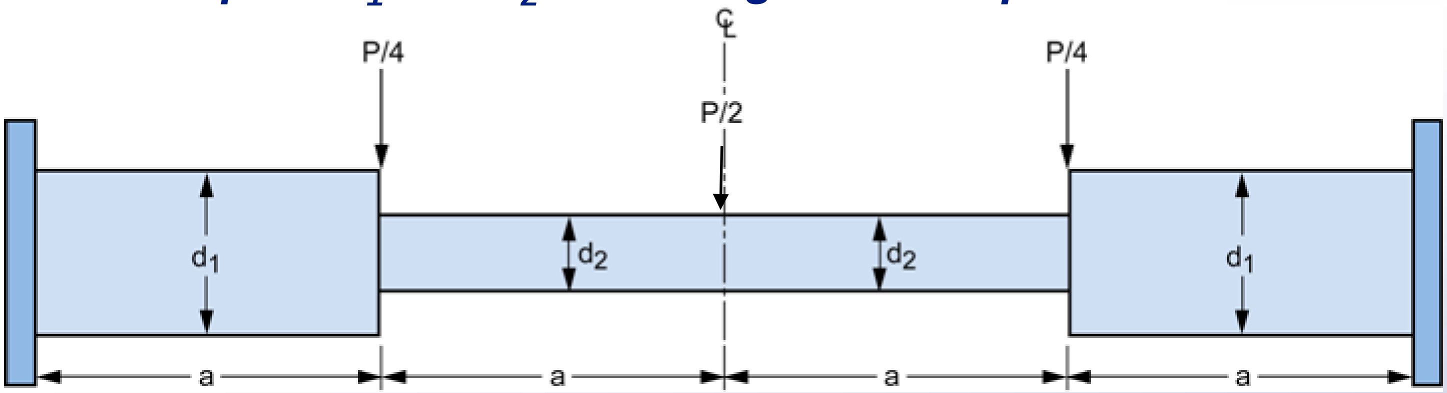


Summary

- Weight increased when risk was reduced & vice-versa
- For 50% rate of success weight = 802 lbf
- Weight $\rightarrow$ to ∞ for zero risk
- Weight $\rightarrow$ to 0 for most unreliable design
- Extreme values can not be captured

Design of a Clamped Beam by Different Methods

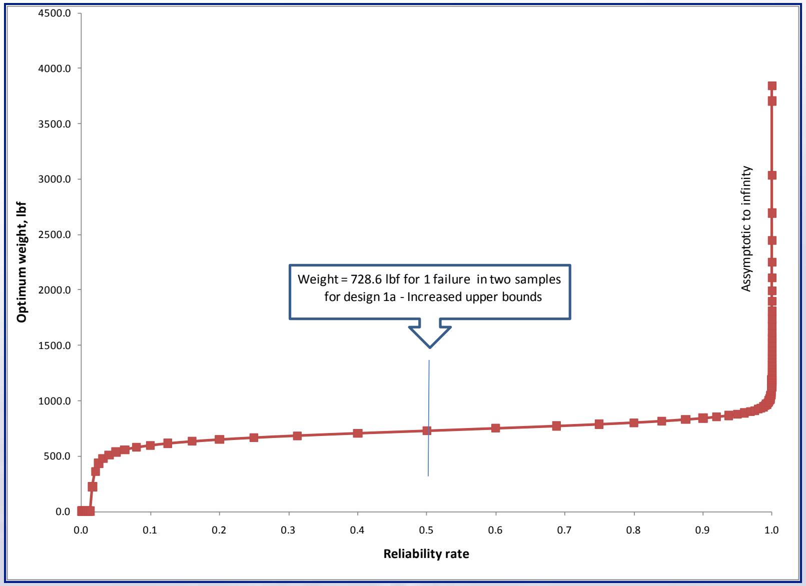
Calculate depths d_1 and d_2 for strength and displacement limitations

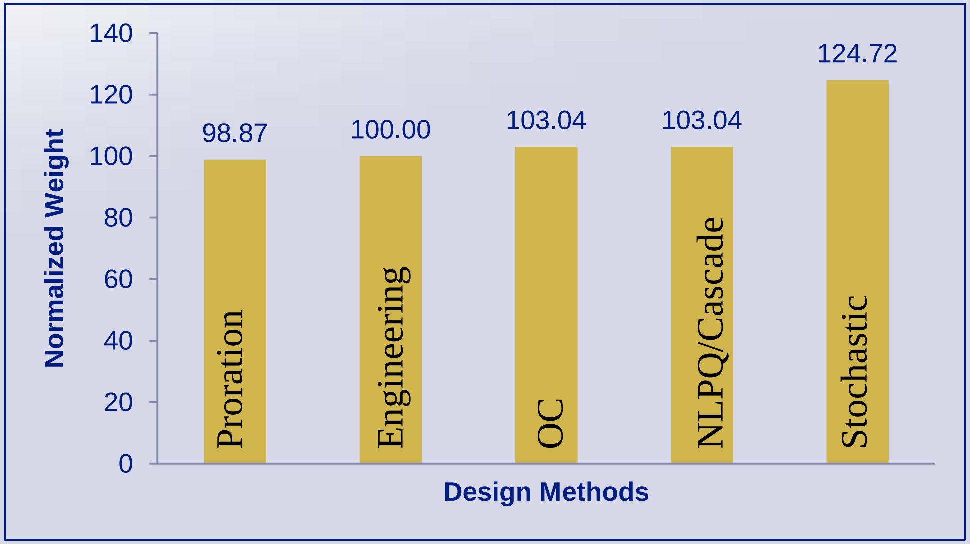


Variable	Deterministic value	Mean value	Standard deviation
Stress limit: σ_0	20 ksi	$\mu_{\sigma_0} = 25\ ksi$	$\bar{\sigma}_{\sigma_0} = 2.5\ ksi$
Disp. limit: δ	0.25 in.	$\mu_\delta = 0.251\ in$	$\bar{\sigma}_\delta = 0.005\ in.$
Modulus: E	10K ksi	$\mu_E = 10K\ ksi$	$\bar{\sigma}_E = 1K\ ksi$
Density: ρ	0.1 lb/in.	$\mu_\rho = 0.1$	$\bar{\sigma}_\rho = 0.005\ in.$
Load: P	25, 50, 25 kip	$\mu_P = 25.01, 49.98, 25.01$	$\bar{\sigma}_P = 0.01, 0.01, 0.01\ kip$
Depth: d_1,d_2	20.34,17.49	$\mu_{d_1} = 23.805\ in$ and $\mu_{d2} = 12.805\ in$	$\bar{\sigma}_{d_1} = 0.005\ in$ and $\bar{\sigma}_{d2} = 0.005\ in$
Limitation		Assumed but not measured values	

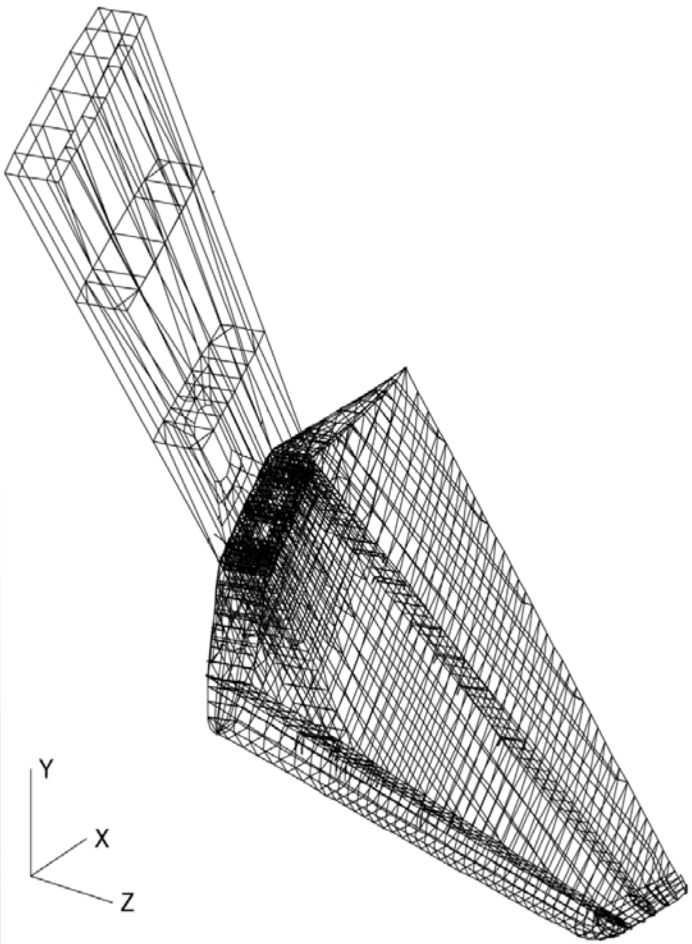
Design Solutions for the Clamped Beam

Method	Design variables in inch		Weight (lbf)	Max stress (ksi)	Max disp in inch
	d_1	d_2			
Proration Technique	23.5	12.9	699.1	15.13	0.25
Engineering Method	21.5	15.3	707.1	19.0	0.25
Optimality Criteria	23.8	12.8	702.7	15.4	0.25
NLPQ	24.5	13.5	728.6	14.2	0.25
Cascade Strategy	24.5	13.5	728.6	14.2	0.25
Stochastic Design for 1 failure in 10 samples	29.5	16.35	881.9	---	---



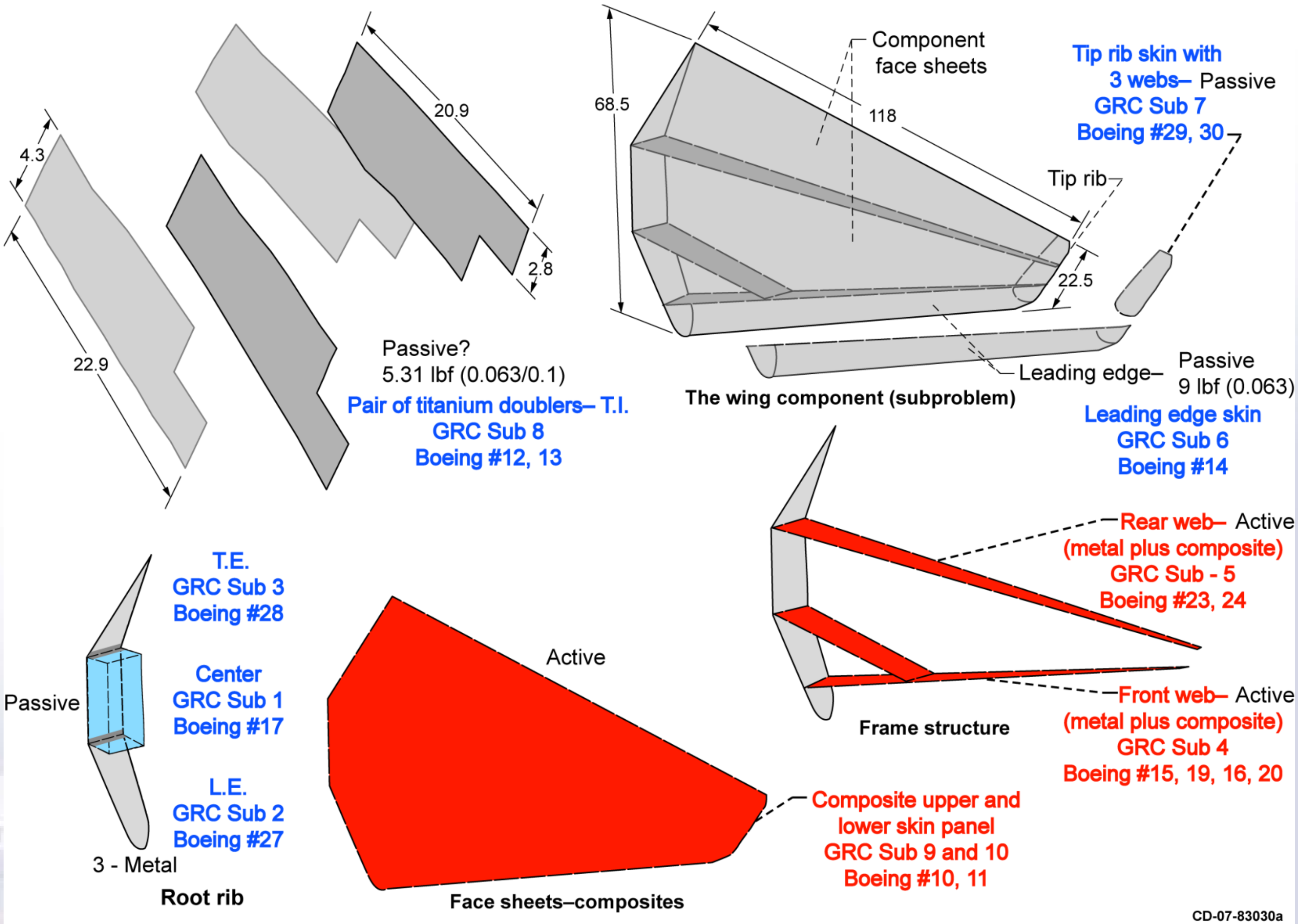


Design of a Raked Wing Tip of Boeing 767-400 ER Airliner



- The structure shown was made of composite and different types of metals
- Optimum design was obtained for 1 failure in N samples, Max N = 2 Million
- 13 sets of design variables
- 227 groups of strain & displacement constraints
- Solution required 5 days of execution in a Linux workstation.
- Limited information can be provided because data is Boeing proprietary

Ten Primary Components of Wing Tip



- Optimized four composite components (shown in red)
- The six metallic components were passive (shown in blue titles)

Orthotropic Material Model

Material properties	Deterministic values	Probabilistic values	
		Mean value	Standard deviation
Material w Fabric			
Young's modulus, E	8.1×10^6 psi	8.505×10^6 psi	0.638×10^6 psi
Shear modulus, G_{12}	7.1×10^5 psi	7.35×10^5 psi	0.551×10^5 psi
Honey comb			
Transverse shear modulus G_{1z}	4500	4725	334
G_{2z}	2500	2625	197
Strain allowable	4000 µs	5000	375

Limitation: Mean values & variations were assumed not measured

Failure theory: Maximum failure theory

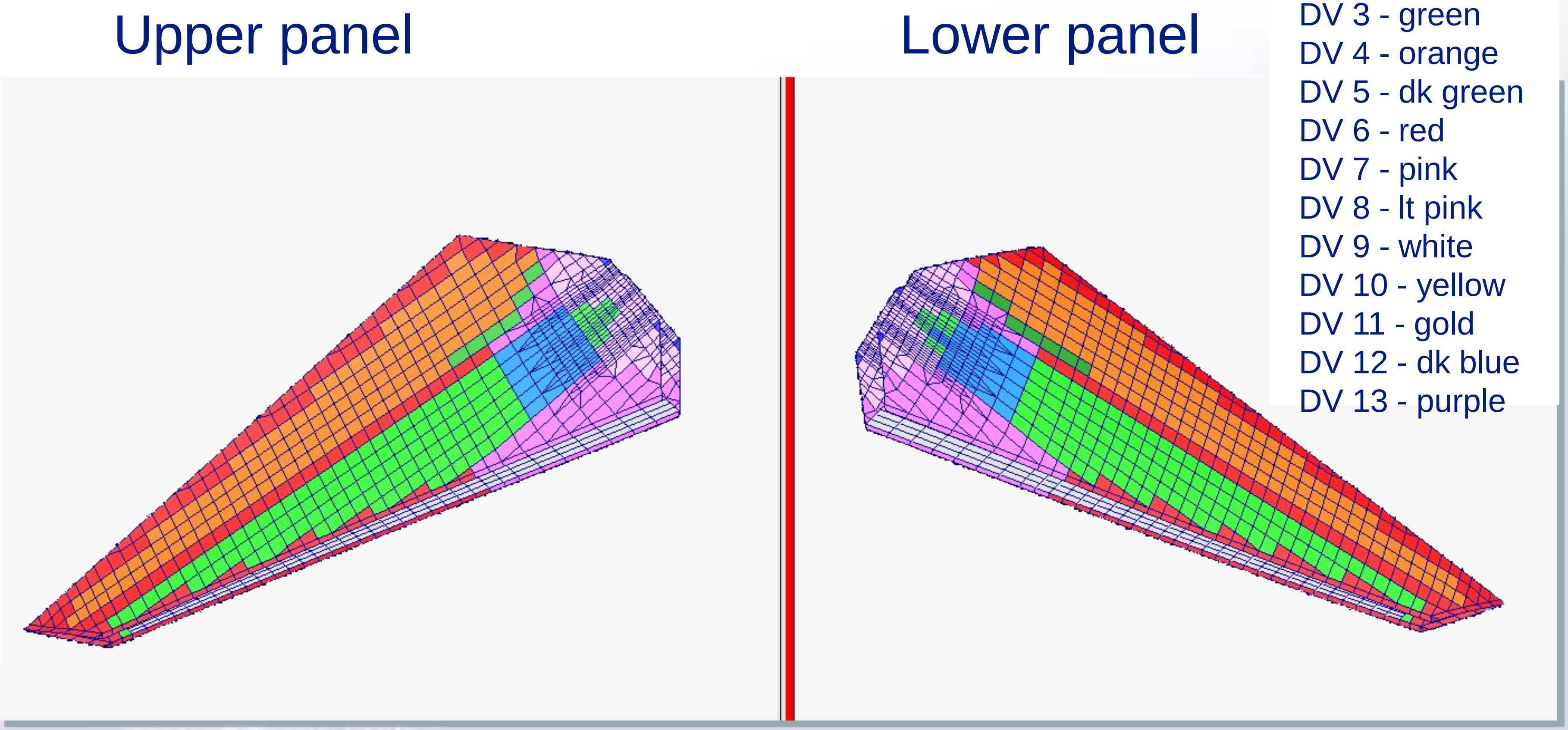
Design Variable and Constraint Definition

Design variable	Location	Honeycomb Thickness in inch	Number of plies*
1	Section 1: Upper & lower skin panel	0.65 - 0.75	7 - 16
2	Section 2: Upper & lower panel	0.65 - 0.75	18 - 27
3	Section 3: Upper & lower panel	0.65 - 0.75	28 - 38
4	Section 4: Upper & lower panel	0.5	7 - 14
5	Section 5: Upper & lower panel	0.5	15 - 21
6	Section 6: Upper & lower panel	None	9 - 17
7	Section 7: Upper & lower panel	None	18 - 27
8	Section 8: Upper & lower panel	None	28 - 38
9	Web of both spars	None	8 - 15
10	Front & aux spars	0.18 - 0.383 sq. in.	
11	Rear spar	0.27 - 0.343	
12	Upper -passive	36 plies	
13	Lower -passive	38 plies	

- Structure was made of (7-38) plies, honeycomb core and rod elements
- Behavior constraints: 227 (203 strain for upper & lower panels + 16 for rods)
- 3 translations (z-dir <= 13.68 in.; mean = 17.1, stdev = 1.71);
- 1 rotation (5^{o}; mean = 6.25, stdev = 0.625)

Design Variable Grouping for Panels

DV 1 - lt green
DV 2 - blue
DV 3 - green
DV 4 - orange
DV 5 - dk green
DV 6 - red
DV 7 - pink
DV 8 - lt pink
DV 9 - white
DV 10 - yellow
DV 11 - gold
DV 12 - dk blue
DV 13 - purple



- Design variable grouping was formulated by considering elements whose thickness range was similar.

Initial Analysis for Load Case Determination

Nominal Response				
Load Case	Max. Disp. in inch	Micro-Strain in C- panel		Strain Energy in in.-kip
		Lower	Upper	
1	-5.6	1042	3243	9
2	11.7	5744	1713	40
3	10.6	5648	1579	34
4	10.4	5689	1512	33
5	-5.5	985	2998	9
6	13.7	5897	3773	54
7	12.4	5189	3409	44
8	-10.0	1787	4737	28

- Frequencies: 1st mode =12.7 Hz; 2nd =16.4; 3rd =17.0
- Lower composite panel carried more load
- Weight of composite materials = 84.5 lbf

Deterministic Optimization Results

	Boeing Design	Design Model 1	Design Model 2
Weight, lbf	84.5	70.966	67.782
Design variables in inch			
1	0.0958	0.0680	0.0680
2	0.1791	0.1871	0.1788
3	0.2730	0.3012	0.3053
4	0.0899	0.0680	0.0692
5	0.1516	0.0680	0.0680
6	0.0998	0.1231	0.0808
7	0.1857	0.1563	0.1754
8	0.2727	0.1202	0.1224
9	0.0963	0.0786	0.0805
10	0.3065	0.2700	0.2700
11	0.2438	0.1800	0.1800
12	0.3060	0.3060	0.3060
13	0.3230	0.3230	0.3230
Active constraints		Six strains = 4000 μs	Six strains = 4000 μs
Displacement	13.68 in.	13.14	13.35
Rotation	4.24°	4.48	4.72
Frequency, Hz	16.36	16.45	20.33

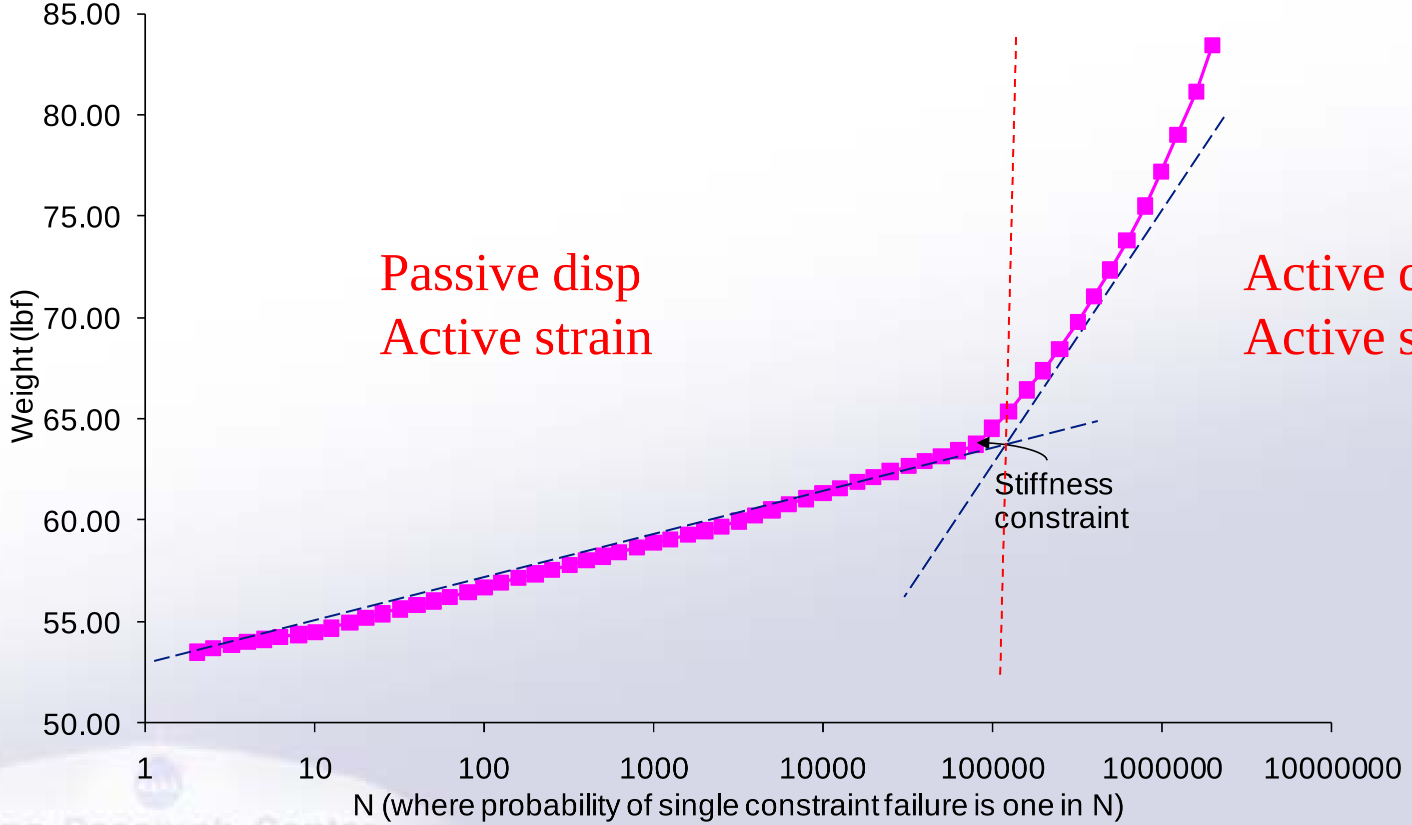
- Weight was lighter by 13.5 lbf or 16 percent of original design
- Strain was about 4000 μs for six of 13 groups of design variables
- Frequency was 16.45 Hz against 16.36 Hz for initial design

Stochastic Design Optimization Results

Probability of single constraint Failure	N One failure in N samples	Model-1 Load-A Strength + stiffness	Model-1 Load-B Strength or strength and stiffness	Model-2 Load-A Strength and stiffn
0.5	2	54.726	52.658	53.474
0.1	10	57.054	53.782	54.453
0.01	100	59.620	55.429	56.661
0.001	1,000	62.398	56.664	58.861
1E-4	10,000	64.844	57.727	61.322
1E-5	100,000	67.076	58.969	64.512
1E-6	1 million	78.626	60.553	77.170
8E-7	1,250,000	80.272	60.716	79.028
5E-7	2,000,000	84.607	61.057	83.427

- Weight varies depending on the failure rate – the less likely the failure the higher the weight
- Each row represents a different probability of single constraint failure

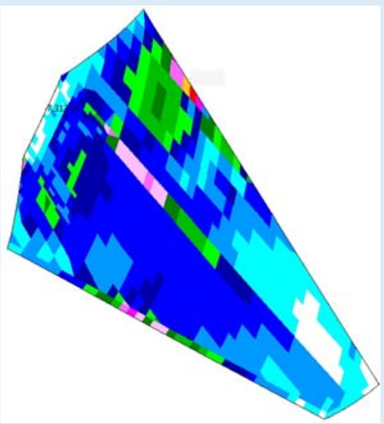
Inverted S-graph – for Model-2 and Load A x-axis in log scale, weight in y-axis



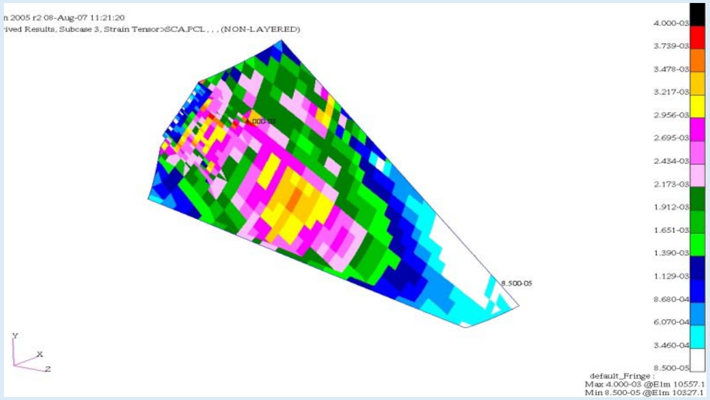
- Weight can be approximated by two linear segments
- First segment is for active strain constraints only
- Both strain and displacement are active for the second segment

Strain Distribution in Upper Panel

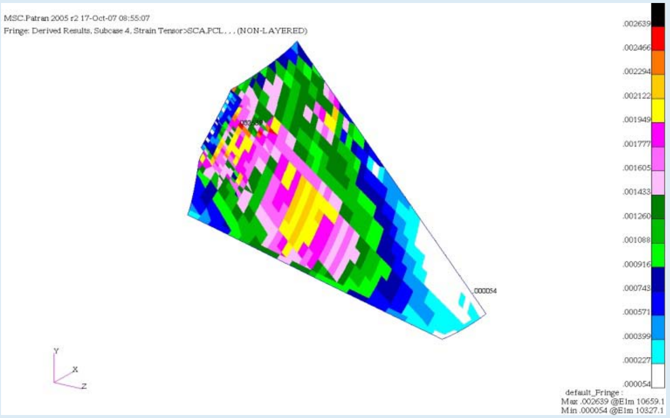
Boeing design
Max Strain = 5897 μs
Total Weight = 84.5 lbf



Deterministic design
Max Strain = 4000 μs
Total Weight = 70.97 lbf



Stochastic design for one failure in 2 Million samples:
Model-1, Load A
Max Strain = 4670 μs
Total Weight = 70.354 lbf



- In original design the strain field displayed few small areas of high strain (exceeding allowable) with the rest of the field having relatively low strain values
- Optimization process more evenly distributed the strain field

Run Time

- CometBoards was run on a Red Hat Linux 2.6.9-67.ELsmp O/S, with x86_64 architecture, 2600 MHz, 4 cpus, 8GB of memory, 32 bit numeric format
- MSC/Nastran version 2005.5.0 (2005R3)
- FPI of Nessus level 6.2, dated 29 Sept. 1995

Activity	Time
One analysis cycle – static	5 sec.
One analysis cycle – 20 frequencies	51 sec.
Deterministic optimization run (model-1)	39 min.
Stochastic analysis (model -1 +load-A):	
– (3031 number of random variables: 3025 ply strains + 6 displ)	47 min.
Stochastic optimization runs (GRC model + GRC load):	
– (61 number of p-levels) ≈ (5 days + 8 hrs)	128 hrs
Stochastic optimization runs (LaRC model + GRC load):	
– (61 number of p-levels) ≈ (5 days + 6 hrs)	126 hrs

Summary

- A single winner from the three methods can not be identified for design calculations
  - Traditional design is based on fully utilized concept. Weight is back calculated.
  - Deterministic optimization minimizes the weight with constraints on stresses and/or displacements – widely accepted method
  - Stochastic design similar to deterministic but parameters are random variables with specified reliability or 1 failure to N samples

- All three methods produced acceptable solutions with some variation which may be due to the indeterminacy that can influence stress or strain flow in the members

- Variation of weight was modest for all methods. In design variables up to one third variation was noticed

- In stochastic design weight versus reliability traced out an inverted S-shape graph. Weight increased when risk was reduced and vice-versa

- Stochastic design should be used with caution because the distribution function, mean value and standard deviation were assumed random variables