Tiltle: Closed-form and numerically-stable solutions to problems related to the optimal two-impulse transfer between specified terminal states of Keplerian orbits

The first part of the paper presents some closed-form solutions to the optimal two-impulse transfer between fixed position and velocity vectors on Keplerian orbits when some constraints are imposed on the magnitude of the initial and final impulses. Additionally, a numerically-stable gradient-free algorithm with guaranteed convergence is presented for the minimum delta-v two-impulse transfer. In the second part of the paper, cooperative bargaining theory is used to solve some two-impulse transfer problems when the initial and final impulses are carried by different vehicles or when the goal is to minimize the delta-v and the time-of-flight at the same time.

Closed-form and numerically-stable solutions to problems related to the optimal two-impulse transfer between specified terminal states of Keplerian orbits

July 6, 2011

1 Introduction

The optimal two-impulse transfer between fixed position and velocity vectors on Keplerian orbits problem has been studied intensively in the past. Numerical and closed-form solutions to various formulations of the the minimum delta-v problem have been presented in the literature (see for example [1],[?], [6], [7] or [8]). In this paper the optimization problem is formulated in a different way. If we considered the case where the 2 impulse maneuvers are implemented by two different vehicles (e.g. Earth-departure stage (EDS) and a crew exploration vehicle (CEV)) and considering that the unused Δv_{EDS} cannot be re-utilized, minimizing $J = \Delta v_{EDS} + \Delta v_{CEV}$ might not lead to the optimal performance. If we take into account that the Earth-departure stage can provide a maximum Δv (in interplanetary trajectories this would be a maximum v_∞), we can formulate the problem in a different way. Given the initial and final states: $\mathbf{r}_0, \mathbf{v}_0$ and $\mathbf{r}_f, \mathbf{v}_f$, calculate two impulse maneuvers such that:

$$\begin{aligned} J = \min \Delta v_{CEV} & \\ \textit{subject to} \quad & \Delta v_{EDS} \leq \Delta v_{max} \end{aligned}$$

In this paper, we show that a closed-form solution of the above problem can be obtained by solving a 4th order polynomial. A similar solution can be also obtained for the cases when a constraint in Δv_{CEV} is added to the previous problem or when the goal is to minimize the first maneuver subject to a constraint in the magnitude of the second maneuver.

When the optimization index is

$$J = \Delta v_1 + \Delta v_2 \tag{1}$$

no closed-form solution can be obtained according to ([?] and [4]). In this paper we present a numerically-stable gradient-free algorithm with guaranteed convergence. The solution consists of solving two 4th order polynomials and then using the bisection method developed in[5].

The final sections of the paper are devoted to solve the previous problems using Cooperative Bargaining theory (see [3]). If the two impulsive maneuvers are implemented by two different vehicles, we can formulate the problem as two agents trying to obtain their maximum benefit (minimizing its Δv) while cooperating with a second agent. In Cooperative Bargaining theory Eq. 1 is known as the Utilitarian solution, (see [3]). In this paper, we examine some properties of the Utilitarian solution (the most common one used in trajectory optimization problems) and also the properties of some other solutions known in the Cooperative Bargaining literature that could be used to solve the two-impulse transfer problem. Once the properties of the Cooperative bargaining solutions have been explained, an example of the application of this theory to the problem of minimizing the time of flight of the transfer while minimizing the total Δv is presented.

2 Problem setup

The problem we are trying to solve consists on calculating a two-impulse transfer between two known states: $\mathbf{r}_i, \mathbf{v}_{i,0}$, with $i = 1, 2$ defining the departure and arrival states respectively. In order to solve the optimization problem, the following the formulation will be used (see also Battin):

$$\mathbf{v}_1 = \mathbf{v}_c + \mathbf{v}_{\rho 1} \quad \mathbf{v}_2 = \mathbf{v}_c - \mathbf{v}_{\rho 2}$$

where:

$$\mathbf{v}_c = \bar{v}_c x \mathbf{i}_c \quad \mathbf{v}_{\rho i} = \frac{\bar{v}_\rho}{x} \mathbf{i}_{ri}$$

$$\mathbf{i}_{ri} = \frac{\mathbf{r}_i}{r_i} \quad \mathbf{i}_c = \frac{\mathbf{r}_2 - \mathbf{r}_1}{c} \quad c = \|\mathbf{r}_2 - \mathbf{r}_1\|$$

$$\bar{v}_c = \frac{c\sqrt{\mu}}{r_1 r_2 \sin\theta} \quad \bar{v}_\rho = \frac{\sqrt{\mu}\,(1 - \cos\theta)}{\sin\theta}$$

$\theta \in [0, \pi]$ angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, $\quad x = \pm\sqrt{p}$, $\quad p:$ parameter of transfer conic

The sign of x will determine the type of transfer: short ($< \pi$) or long (π). No multi-rev solutions will be considered in this paper.

It is well known that the velocity locus for $\mathbf{v}_i$ will correspond to two hyperbolas (see Sun, Battin) for the short and long transfers (see Figure 1). A complete description of the parameters of the hyperbola can be found in Sun "on optimun...". The parameters used in this paper are described in Table 1.

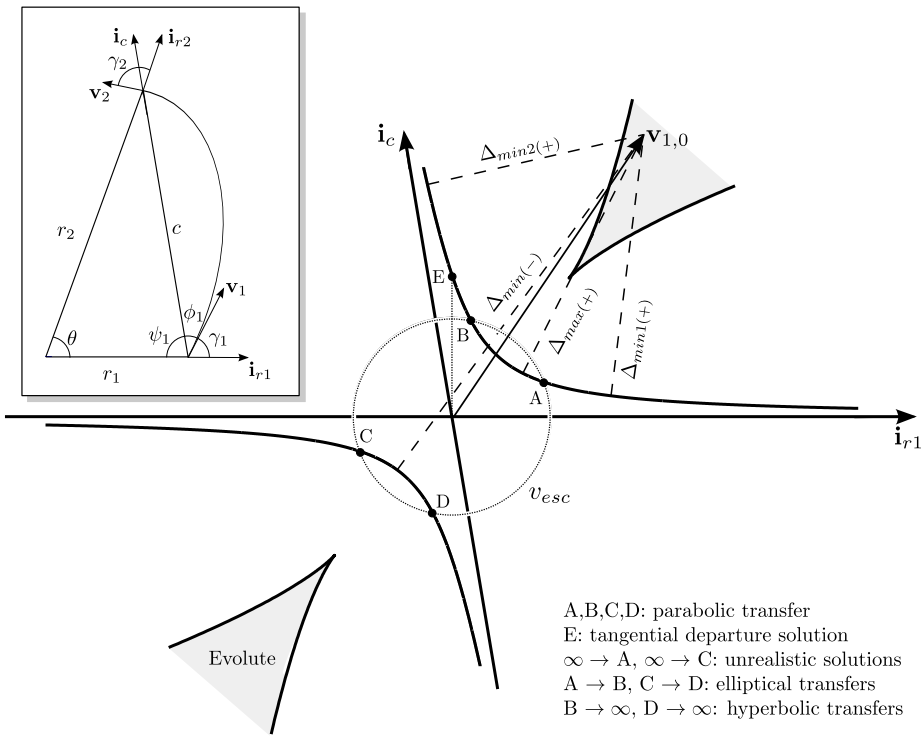

Figure 1: Geometry of the hyperbolic velocity locus for terminal 1. Since the initial velocity $\mathbf{v}_{1,0}$ is inside the evolute, four extrema are found. Three of them when $x > 0$: $\Delta_{min1(+)}, \Delta_{max(+)}, \Delta_{min2(+)}$ and one for $x < 0$: $\Delta_{min(-)}$.

Table 1: Geometrical Elements of the hyperbolic locus of velocity

Asymptotes	$\pm\mathbf{i}_{r1}$, $\mathbf{i}_c$
	$\pm\mathbf{i}_{r2}$, $\mathbf{i}_c$
Included between asymptotes	ϕ
Semi-transversal axis	$A_h = 2\sqrt{\overline{v}_c\overline{v}_\rho}\cos\frac{\phi}{2} = \sqrt{2\tan\frac{\theta}{2}\cot\frac{\phi_i}{2}}\sqrt{\frac{\mu}{r_i}}$
Semi-conjugate axis	$B_h = 2\sqrt{\overline{v}_c\overline{v}_\rho}\sin\frac{\phi}{2}$

Table 2: Characterization of the extrema of the solutions to the minimum Δv_i transfer problem.

1 local minimum for $x > 0$	1 local minimum for $x < 0$
2 local minima, 1 local maximum for $x > 0$	1 local minimum for $x < 0$
1 local minimum for $x > 0$	2 local minima, 1 local maximum for $x < 0$

3 Computing the optimal $\Delta v_{1,2}$

A tangent vector to the velocity locus is defined by (see Battin):

$$\mathbf{v}_{t1} := \mathbf{v}_c - \mathbf{v}_{\rho 1} \quad \mathbf{v}_{t2} := \mathbf{v}_c + \mathbf{v}_{\rho 2}$$

Given the departure or arrival velocities $\mathbf{v}_{i,0}$ we can calculate the parameter x that minimizes the departure or arrival Δv_i by simply:

$$\left(\mathbf{v}_c \pm \mathbf{v}_{\rho i}\right)^{\mathsf{T}} \boldsymbol{\Delta}\mathbf{v}_i = 0 \tag{2}$$

Alternatively, we can obtain the same result by

$$\min \Delta v_i = \min \left\|\mathbf{v}_i - \mathbf{v}_{i,0}\right\|$$

and solving for x in

$$\frac{d\left\|\mathbf{v}_i - \mathbf{v}_{i,0}\right\|}{dx} = 0$$

The solution to eq. 2 will require the solution of the following quartic polynomials on x for terminals 1 and 2 respectively:

$$\bar{v}_c x^4 - \bar{v}_c \left(\mathbf{i}_c \cdot \mathbf{v}_{1,0}\right) x^3 + \bar{v}_\rho \left(\mathbf{i}_{r1} \cdot \mathbf{v}_{1,0}\right) x - \bar{v}_\rho^2 = 0 \tag{3}$$

$$\bar{v}_c x^4 - \bar{v}_c \left(\mathbf{i}_c \cdot \mathbf{v}_{2,0}\right) x^3 - \bar{v}_\rho \left(\mathbf{i}_{r2} \cdot \mathbf{v}_{2,0}\right) x - \bar{v}_\rho^2 = 0 \tag{4}$$

Given the signs of the coefficients of the polynomials in eqs. 3 and 4, the number of solutions that we can find for x are described in table **. A similar approach is described in Sun and other. The solutions provide by both equations will create an envelope(s) of trajectories with very interesting properties. For a complete description of the trajectory envelopes see Sun.

We can now compute the global optimal by simply computing the Δv associated to the real solutions of the quartic polynomials. Let $x_j$be the real solutions to Eq 3 or 4. Let $\Delta v(x_j)$ be the associated Δv, the global optimal solution

$$\Delta v^* = \min_{x_j} \Delta v(x_j) \tag{5}$$

3.1 Existence of multiple extrema

It is worth studying the case when multiple extrema exist. In Figure ** we can see an example of such a case. In this example, the initial velocity vector $\mathbf{v}_{1,0}$ is inside the evolute associated with the velocity locus of $\mathbf{v}_1$. Therefore, there are three normals to the hyperbola that intersect at $\mathbf{v}_{1,0}$, that is why three extrema appear in the solution eq. 3. The normal in the middle will correspond to a local maxima while the two other extreme will correspond with two local minima.

Given a specific problem, we can calculate a necessary condition for the magnitude of the velocity vectors $v_{i,0}$ such that $\mathbf{v}_{i,0}$ is inside the hyperbola evolute. For that purpose we will compute the magnitude of the velocity at the cusp of the evolute

$$v_{cusp} = \frac{A_h^2 + B_h^2}{A_h} = \frac{2\sqrt{\bar{v}_c \bar{v}_\rho}}{\cos\frac{\phi_i}{2}} = \frac{A_h}{\cos^2\frac{\phi_i}{2}}$$

A_h is also the initial velocity of the minimum-energy transfer orbit v_{m1}, and it always corresponds with an elliptical orbit:

$$\frac{v_{mi}}{\sqrt{\frac{\mu}{r_i}}} = \frac{A_h}{\sqrt{\frac{\mu}{r_i}}} = \sqrt{2\tan\frac{\theta}{2}\cot\frac{\phi_i}{2}} = \sqrt{2}\sqrt{2\tan\frac{\theta}{2}\cot\frac{\phi_i}{2}} < \sqrt{2}$$

Therefore, even initial (or final) elliptical parking orbits could be such that $\mathbf{v}_{i,0}$ is inside the hyperbola evolute.

Finally, a necessary condition for $\mathbf{v}_{i,0}$ to be inside the evolute is:

$$v_{i,0} \geq \frac{v_{mi}}{\cos^2\frac{\phi_i}{2}} \tag{6}$$

4 Computing the optimal $\Delta v_{1,2}$ when constraints in x are present

In the previous section we have seen that in order to compute the optimal departure or arrival Δv_i it is necessary to solve a a quartic polynomial in x. We have also seen that given the geometry of the problem several local optima can appear. If no constraints are present, we can use Eq. 5 to compute the global solution. But if constraints on p (or x) are present a different approach is necessary.

Let $x \in C_x = [x_{c1}, x_{c2}]$ be the constraint set for x, with $0 < x_{c1} < x_{c2}$. An algorithm to compute the global optimal Δv_1 can be found (a similar one Δv_2 can be obtained).

1. Solve Eq. 3.

2a. For the real positive solutions of 3. If only one extrema is found, x_{min}, get Δv^* from Table 3. If three extrema are found $x_{min1} < x_{max} < x_{min2}$, $S_x = [x_{min1}, x_{min2}]$, get Δv^* from Table 4

2b. Repeat 2a for the real negative solutions of 3

3. The global Δv^* will the minimum between the steps 2a and 2b.

Table 3: Algorithm for the one-extrema case
x_{min}, $C_x = [x_{c1}, x_{c2}]$

$x_{min} \in C_x$	Δv^*
F	if $C_x < x_{min} \rightarrow \Delta v^* = \Delta v(x_{c2})$ if $C_x > x_{min} \rightarrow \Delta v^* = \Delta v(x_{c1})$
T	$\Delta v^* = \Delta v(x_{min})$

Table 4: Algorithm for the three-extrema case
$x_{min1} < x_{max} < x_{min2}$, $S_x = [x_{min1}, x_{min2}]$, $C_x = [x_{c1}, x_{c2}]$

$x_{min1} \in C_x$	$x_{max} \in C_x$	$x_{min2} \in C_x$	Δv^*
F	F	F	if $C_x < S_x \rightarrow \Delta v^* = \Delta v(x_{c2})$ if $C_x > S_x \rightarrow \Delta v^* = \Delta v(x_{c1})$
F	F	T	$\Delta v^* = \Delta v(x_{min2})$
F	T	F	$\Delta v^* = \min[\Delta v(x_{c1}), \Delta v(x_{c2})]$
F	T	T	$\Delta v^* = \Delta v(x_{min2})$
T	F	F	$\Delta v^* = \Delta v(x_{min1})$
T	F	T	Not possible
T	T	F	$\Delta v^* = \Delta v(x_{min1})$
T	T	T	From Eq. 5

If there are several constraint sets, the previous algorithms will be called several times for each set. The global minimum Δv will be the minimum of all of the solutions for each set. Additionaly, if the constraint set C_x contains negative and positive numbers, C_x will be subdivided into two different sets containing only negative and only positive numbers and the optimization algorithm will be called twice.

Once we have an algorithm to compute the global optimal when constraints in x are present, all we need to do is to express different constraints of the problem (e.g. minimum radius, minimum flight-path-angle or maximum Δv) in terms of x. The next sections describe several methods to achieve this.

5 Constraints in position magnitude $r \geq r_{min}$

In this section, we are going to describe how to impose a constraint in the position magnitude during the transfer:

$$t \in [t_1, t_2],\ r(t) \geq r_{min}$$

We can translate the above equation into a set(s) of constraints for p. First, we are going to calculate the sets of p that make the eccentricy vector to be contained between $\mathbf{r}_1$ and $\mathbf{r}_2$ during the transfer and then we are going to compute the sets of p that satisfy $r_p \geq r_{min}$. The combination of these two sets will provide the final constraint set(s).

Given that the eccentricy vector is defined by (see Battin ref.)

$$\mathbf{e} = A\mathbf{r}_1 + B\mathbf{r}_2$$

where

$$A = \frac{1}{\sin^2\theta}\left[\left(\frac{p}{r_1} - 1\right) - \left(\frac{p}{r_2} - 1\right)\cos\theta\right]$$

$$B = \frac{1}{\sin^2\theta}\left[\left(\frac{p}{r_2} - 1\right) - \left(\frac{p}{r_1} - 1\right)\cos\theta\right]$$

Therefore we need to calculate the values of p that make $A > 0$ and $B > 0$,

$$p \geq p_A = \frac{r_1 r_2 (1 - \cos\theta)}{r_2 - r_1\cos\theta},\ p \geq p_B = \frac{r_1 r_2 (1 - \cos\theta)}{r_1 - r_2\cos\theta}$$

$$p \in [p_A, +\infty) \cap [p_B, +\infty) = [\max(p_A, p_B), +\infty) = S_{p1} \rightarrow \mathbf{e} \text{ between } \mathbf{r}_1, \mathbf{r}_2$$

Since $p > 0$, there are solutions only if $\min\left(\frac{r_1}{r_2}, \frac{r_2}{r_1}\right) > \cos\theta$.

For $x > 0$ (short transfer) a periapse passage occurs during the transfer if $p \in S_{p1}$, for $x < 0$ (long transfer) a periapse passage occurs during the transfer if $p \notin S_{p1}$:

$$x > 0 \quad \rightarrow \quad x \in \left[\sqrt{\max(p_A, p_B)}, +\infty\right) = S_{n1} \tag{7}$$

$$x < 0 \quad \rightarrow \quad x \in \left[-\sqrt{\max(p_A, p_B)}, 0\right] = S_{n2} \tag{8}$$

The next step will be to determine the values of p (and therefore x) that make $r_p \geq r_{pmin}$. First, we will calculate $e^2 = A^2 + B^2 + 2AB\cos\theta$. After some algebraic manipulations we can obtain:

$$e^2 = \frac{c^2}{\sin^2\theta r_1^2 r_2^2}p^2 - \frac{2(r_1 + r_2)}{(1 + \cos\theta) r_1 r_2}p + \frac{2}{1 + \cos\theta}$$

or

$$e^2 = \frac{1}{d^2}p^2 - 2\frac{p_{min_e}}{d^2}p + 2\frac{p_{min_e}}{d^2}\frac{r_1 r_2}{r_1 + r_2}$$

Since $\frac{1}{d^2} > 0$, e, p define a hyperbola, we can rewrite the above equation to show this:

$$\frac{1}{d^2}(p - p_{min_e})^2 - e^2 = \frac{p_{min_e}}{d^2}\left(p_{min_e} - 2\frac{r_1 r_2}{r_1 + r_2}\right) \tag{9}$$

we can also proof by contradiction that

$$p_{min_e} - 2\frac{r_1 r_2}{r_1 + r_2} < 0$$

Therefore we can say that e and p define a vertical hyperbola centered at $p = p_{min_e}, e = 0$ where p is along the x-axis and e is along the y-axis (only the positive branch is considered). The asymptotes are given by $e_{asy} = \pm d\left(p - p_{min_e}\right)$

We can calculate the minimum eccentricy transfer by:

$$\frac{de^2}{dp} = 0,\ 2\frac{de}{dp} = \frac{2c^2}{\sin^2\theta r_1^2 r_2^2}p - \frac{2\left(r_1 + r_2\right)}{\left(1 + \cos\theta\right) r_1 r_2} = 0$$

$$p_{min_e} = \frac{r_1 r_2 \left(r_1 + r_2\right)\left(1 - \cos\theta\right)}{c^2}$$

If we define $r_{pmin} := r_{min}$, we can calculate the constraint for p as follows:

$$\frac{p}{1 + e} > r_{pmin}$$

$$\frac{p}{r_{pmin}} - 1 > e$$

the left-hand side of the previous equation defines a line with slope $\frac{1}{r_{pmin}}$ and the right-hand side of the equation is just the positive branch of the hyperbola defined above. The set of values of p make satisfy the constraint are the ones within the intersection of the line defined by $\frac{p}{r_{pmin}} - 1$ and the hyperbola defined by e. Therefore, we need to first find the values of p (if any) at the intersection points. For that we'll solve the following equation:

$$\left(\frac{p}{r_{pmin}} - 1\right)^2 = e^2$$

$$e^2 = \frac{1}{d^2}p^2 - 2\frac{p_{min_e}}{d^2}p + 2\frac{p_{min_e}}{d^2}\frac{r_1 r_2}{r_1 + r_2}$$

$$p^2\left(\frac{1}{r_{pmin}^2} - \frac{1}{d^2}\right) + p\left(-\frac{2}{r_{pmin}} + 2\frac{p_{min_e}}{d^2}\right) + \left(1 - 2\frac{p_{min_e}}{d^2}\frac{r_1 r_2}{r_1 + r_2}\right) = 0 \quad (10)$$

We have three possible solution sets:

$$\begin{aligned} p &\in [p_{rp1}, p_{rp2}] \\ p &\in [p_{rp1}, +\infty] \\ p &\in \emptyset \end{aligned}$$

Since we are solving Eq. 10 instead of $\frac{p}{r_{pmin}} - 1 = e$. It is necessary to check for extraneous solutions. We will assume that p_{rpi} are the positive real solutions to Eq. 10 that also satisfy the original equation. P_{rpi} is a valid solution if $\frac{P_{rpi}}{r_{pmin}} - 1 > 0$. In terms of x we obtain:

$$\begin{aligned} \text{if two solutions exist} \quad x &\in \left[\sqrt{p_{rp1}}, \sqrt{p_{rp2}}\right] \cup \left[-\sqrt{p_{rp2}}, -\sqrt{p_{rp1}}\right] = Sr_{p1} \cup Sr_{p2} && (11) \\ \text{if only one solutions exists} \quad x &\in \left[\sqrt{p_{rp1}}, +\infty\right] \cup \left[-\infty, -\sqrt{p_{rp1}}\right] = Sr_{p1} \cup Sr_{p2} && (12) \\ \text{if no solutions exist} \quad x &\in \emptyset && (13) \end{aligned}$$

If we combine the conditions in 7 and 11, x should satisfy:

$$x \in Sr_{p1} \cup Sr_{p2} \Longleftrightarrow r_p \geq r_{pmin} \text{ or}$$

$$x \notin (Sr_{p1} \cup Sr_{p2}) \text{ and } x \notin (S_{n1} \cup S_{n2}) \Longleftrightarrow r_p < r_{pmin} \text{ and no periapse passage}$$

Finally, if the initial or final terminal is inside the forbiden area, there are no solutions

$$\text{if } r_1 < r_{min} \text{ or } r_2 < r_{min} \rightarrow \text{no solutions}$$

6 Unrealistic solutions

Not all the initial velocities in the hyperbola will generate a trajectory that connects the terminals. If $v_1 > \sqrt{2\frac{\mu}{r_1}}$ and $\gamma_1 > 0$ the transfer trajectory will result on a hyperbola that goes to infinity. We need a criteria to avoid calculating trajectories in that region. First, we need to compute the values of p for $e = 1$. Using eq. 9 we obtain

$$p_{par1,2} = P_{min_e} \pm \frac{4\left(r_1 r_2\right)^{3/2}}{c^2} \sin^2\frac{\theta}{2}\cos\frac{\theta}{2} = P_{min_e} \pm \frac{4\left(r_1 r_2\right)^{3/2}}{2c^2} \sin\frac{\theta}{2}\sin\theta$$

In this way the realistic set of transfers is composed of elliptical transfers

$$p \in [p_{par1}, p_{par2}] \Longleftrightarrow x \in \left[-\sqrt{p_{par2}}, -\sqrt{p_{par1}}\right] \cup \left[\sqrt{p_{par1}}, \sqrt{p_{par2}}\right] = S_{el_1} \cup S_{el_2}$$

and hyperbolic transfers with $\gamma_1 \leq 0$. We need to find a criteria to determine the flight-path-angle at $\mathbf{r}_1$. We can use the condition:

$$\mathbf{v}_1 \cdot \hat{\mathbf{i}}_{r_1} = \left(\bar{v}_c x \hat{\mathbf{i}}_c + \frac{\bar{v}_\rho}{x}\hat{\mathbf{i}}_{r_1}\right) \cdot \hat{\mathbf{i}}_{r_1} \leq 0$$

We have two possible cases when $x > 0$:

$$\text{if } \hat{\mathbf{i}}_c \cdot \hat{\mathbf{i}}_{r1} = \cos\psi_1 \geq 0 \rightarrow x^2 \leq -\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1} \text{ has no solution } \rightarrow \gamma_1 > 0$$

$$S_{\gamma b} = \emptyset$$

$$\text{if } \hat{\mathbf{i}}_c \cdot \hat{\mathbf{i}}_{r1} = \cos\psi_1 < 0 \rightarrow x^2 \geq -\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1}$$

$$S_{\gamma b} = \left[\sqrt{-\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1}}, +\infty\right)$$

We have two possible cases when $x < 0$:

$$\text{if } \hat{\mathbf{i}}_c \cdot \hat{\mathbf{i}}_{r1} = \cos\psi_1 \geq 0 \rightarrow x^2 \geq -\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1} \text{ has always a solution}$$

$$S_{\gamma a} = (-\infty, 0]$$

$$\text{if } \hat{\mathbf{i}}_c \cdot \hat{\mathbf{i}}_{r1} = \cos\psi_1 < 0 \rightarrow x^2 \leq -\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1}$$

$$S_{\gamma a} = \left[-\sqrt{-\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1}}, 0\right]$$

$$x \in S_{\gamma a} \cup S_{\gamma b}$$

Therefore, the set of x that generates realistic trajectories is determined by:

$$x \in= S_{el_1} \cup S_{el_2} \Longleftrightarrow e < 1, \quad \text{and}$$

$$x \notin (S_{el_1} \cup S_{el_2}) \text{ and } x \in (S_{\gamma a} \cup S_{\gamma b}) \Longleftrightarrow e \geq 1, \gamma_1 \leq 0$$

We can obtain the same result using the equation that relates γ_1 and p from Battin for when $\gamma_1 = 0$

$$p = p_m \frac{c}{r_1 - r_2 \cos\theta} = \frac{r_1 r_2 (1 - \cos\theta)}{r_1 - r_2 \cos\theta}$$

Using,

$$\cos\psi_1 = -\cos\psi_1' = \frac{r_2^2 - r_1^2 - c^2}{2r_1 c} = \frac{r_2^2 - r_1^2 - \left(r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta\right)}{2r_1 c} = \frac{-r_1^2 + r_1 r_2 \cos\theta}{r_1 c}$$

where ψ_1' is the interior angle between $\mathbf{r}_1$ and $\mathbf{c}$.

$$p = -\frac{\bar{v}_p}{\bar{v}_c \cos\psi_1} = \frac{-r_1 r_2 (1-\cos\theta)}{c\cos\psi_1} = \frac{r_1 r_2 (1-\cos\theta)}{r_1 - r_2\cos\theta}$$

We can conclude that only tangential departure trasfers will exist if

$$r_1 - r_2\cos\theta > 0 \Longleftrightarrow \frac{r_1}{r_2} > \cos\theta$$

Therefore, the set of x that generates realistic trajectories is determined by:

$$x \in= S_{el_1} \cup S_{el_2} \Longleftrightarrow e < 1, \quad \text{and}$$

$$x \notin (S_{el_1} \cup S_{el_2}) \text{ and } x \in (S_{\gamma a} \cup S_{\gamma b}) \Longleftrightarrow e \geq 1, \gamma_1 \leq 0$$

After combining both conditions we will obtain a maximum of 4 sets (2 for $x < 0$, and 2 for $x > 0$): $S_{r1}, S_{r2}, S_{r3}, S_{r4}$.

7 Constraints on the departure and arrival flight-path-angles

We can impose limits on the terminal flight-path-angles γ_1 and γ_2 by traslating those limtis into constraints on the parameter p (or x). First, we can relate γ_i to the parameter by

$$p = p_m c \frac{\cos\gamma_1}{r_1\cos\gamma_1 - r_2\cos(\gamma_1+\theta)} \text{ or } p = p_m c \frac{\cos\gamma_2}{r_2\cos\gamma_2 - r_1\cos(\gamma_2-\theta)}$$

If $\gamma_{min} \leq \gamma_i < \gamma_{max}$,

$$p \in [p(\gamma_{max}), p(\gamma_{min})] \text{for terminal 1, since } \frac{dp}{d\gamma_1} < 0$$

$$p \in [p(\gamma_{min}), p(\gamma_{max})] \text{for terminal 2, since } \frac{dp}{d\gamma_2} > 0$$

8 Constraints on the arrival and departure Δv

We can impose constraints in the magnitude of the maneuvers $\Delta v_{i,min}$ by solving a quartic polynomial in x. First, by using the definitions,

$$\Delta v_i = \|\mathbf{v}_i - \mathbf{v}_{i,0}\| \leq \Delta v_{i,min} \Longleftrightarrow (\mathbf{v}_i - \mathbf{v}_{i,0}) \cdot (\mathbf{v}_i - \mathbf{v}_{i,0}) \leq \Delta v_{i,min}^2$$

This will lead to the solution of the following quatic polynomial for terminal 1:

Table 5: Characterization of the solutions to the maximum Δv_i contraint problem.

real solutions to Eqs. 14 and 15:$x_1 < x_2 < x_3 < x_4$

$x > 0$	$x < 0$
0 solutions $\rightarrow C_{x1} = \emptyset$	2 solutions $\rightarrow C_{x2} = [x_1, x_2]$
2 solutions $\rightarrow C_{x1} = [x_1, x_2]$	0 solutions $\rightarrow C_{x2} = \emptyset$
2 solutions $\rightarrow C_{x1} = [x_3, x_4]$	2 solutions $\rightarrow C_{x2} = [x_1, x_2]$
4 solutions $\rightarrow C_{x1} = [x_3, x_4]$, $C_{x2} = [x_3, x_4]$	0 solutions
0 solutions	4 solutions $\rightarrow C_{x1} = [x_3, x_4]$, $C_{x2} = [x_3, x_4]$
0 solutions$\rightarrow C_{x1} = \emptyset$	0 solutions $\rightarrow C_{x2} = \emptyset$

$$\bar{v}_c x^4 - \bar{v}_c \left(\mathbf{i}_c \cdot \mathbf{v}_{1,0}\right) x^3 + \left[2\bar{v}_c \bar{v}_\rho \left(\mathbf{i}_c \cdot \mathbf{i}_{r1}\right) + v_{1,0}^2 - \Delta v_{i,min}^2\right] x^2 - 2\bar{v}_\rho \left(\mathbf{i}_{r1} \cdot \mathbf{v}_{1,0}\right) x + \bar{v}_\rho^2 = 0 \tag{14}$$

and for terminal 2:

$$\bar{v}_c x^4 - \bar{v}_c \left(\mathbf{i}_c \cdot \mathbf{v}_{2,0}\right) x^3 + \left[-2\bar{v}_c \bar{v}_\rho \left(\mathbf{i}_c \cdot \mathbf{i}_{r2}\right) + v_{2,0}^2 - \Delta v_{i,min}^2\right] x^2 + 2\bar{v}_\rho \left(\mathbf{i}_{r2} \cdot \mathbf{v}_{2,0}\right) x + \bar{v}_\rho^2 = 0 \tag{15}$$

Given the signs of the polynomials in Eqs. 14 and 15, we can establish the nature of the solutions (see Table 5) and from them we can create the constraint sets.

References

[1] M.Avendano, D.Mortari: A closed-form solution to the minimum Δv^2 Lambert problem. Journal of Celestial Mechanics and Dynamical Astronomy 106(1) pp. 25-37 (2010)

[2] D.L. Jezewski, Optimal Two-impulse Transfer Between Specified Terminal States of Keplerian Orbits. Optimal Control Applications & Methods, vol. 3, 257-267 (1982)

[3] William L. Thomson Cooperative Models of Bargaining Chapter 35 in Handbook of game theory (R. Aumann and S. Hart, eds), North-Holland. (1995).

[4] Escobal, P.R. Methods of astrodynamics by Escobal, New York, Wiley, 1968.

[5] R. G. Gottlieb, B. F. Thompson. Bisected Direct Quadratic Regula Falsi. Applied Mathematical Sciences, Vol. 4, 2010, no. 15, 709 - 718

[6] Prussing, J.E., A Class of Optimal Two-Impulse Rendezvous Using Multiple-Revolution Lambert Solutions, The Journal of the Astronautical Sciences., Vol. 48, No. 2 and 3, Apr-Sep 2000, pp. 131-148.

[7] Shen, H. and Tsiotras, P., Optimal Two-Impulse Rendezvous Between Two Circular Orbits Using Multiple-Revolution Lambert's Solutions, Journal of Guidance, Control, and Dynamics, Vol. 26, No. 1, pp. 50–61, 2003.

[8] Battin, R.H. An Introduction to the Mathematics and Methods of Astrodynamics, AIAA, New York (1987).