We provide a mathematical formalism for optimizing the mirror nodal positions along the optical axis and the tilt of a commonly employed detector configuration at the focus of a x-ray telescope consisting of nested mirror shells with known mirror surface prescriptions. We adopt the spatial resolution averaged over the field-of-view as the figure of merit $M$. A more complete description appears in our paper in these proceedings.

1. Variance in ray position on a focal surface $S$

$$M = \langle \sigma_{11}^2 \rangle = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} d\theta d\phi \, w_{\text{avg}}(\theta, \phi) \sigma_{11}^2(\theta, \phi)$$

where $w_{\text{avg}}(\theta, \phi)$ is a viewing angle weighting function and $\sigma_{11}(\theta, \phi)$ is the spatial variance for a point source on the sky at polar and azimuthal off-axis angles $(\theta, \phi)$ determined on the surface $S$ of the detectors. For $J$ mirror shells with $N$ multiply reflected rays through shell $j$, we have shown:

$$\sigma_{jj}^2(\theta, \phi) = \frac{N}{W} \sum_{j-1}^{N-1} \left( \frac{n_j - 1}{W} \right) \sigma_{jj}^2$$

$$\sigma_{ij}^2(\theta, \phi) = \frac{N}{W} \sum_{j-1}^{N-1} \left( \frac{n_j - 1}{W} \right) \sigma_{ij}^2$$

When $w_j = 1$, then $w_j = n_j$ and $W = N$. The ensemble average of a quantity, say the ray $x$ position on the surface $S$, in the $k$-th row and $j$-th column is given by:

$$\langle x_k \rangle_j = \frac{1}{W} \sum_{j-1}^{N-1} w_j \langle x_k \rangle_j$$

$$\langle y_k \rangle_j = \frac{1}{W} \sum_{j-1}^{N-1} w_j \langle y_k \rangle_j$$

Where $w_j$ is the weight assigned to the $k$-th ray from the $j$-th mirror shell. In order to account for dependence on energy $E$, say for optics with two segments per mirror shell, the natural weight to use is the product of the reflectivities from the primary, $R_1$, and secondary, $R_2$, mirror surfaces:

$$w_{jk} = R_1(\phi_j, \theta_j, E) \cdot R_2(\phi_j, \theta_j, E)$$

Here $\alpha_{jk}$ and $\beta_{jk}$ are the primary and secondary graze angles for the $k$-th ray from the $j$-th mirror shell.

Important result: $\sigma_{kk}^2$ is not a simple sum over the variances $\alpha_{kk}^2$ of the individual shells.

3. The merit function

Making use of additional definitions given in our paper to make the notation more compact, we write:

$$\left( \begin{array}{c} \langle x_k \rangle_j \\ \langle y_k \rangle_j \\ \langle z_k \rangle_j \end{array} \right) = (A_{jk}) \cdot \left( \begin{array}{c} \tan \phi_j \\ \tan \theta_j \end{array} \right) + \sum_{i=1}^{J} \left[ \left( B_{ij} x_i + C_{ij} y_i + D_{ij} z_i \right) \tan \phi_j + \left( E_{ij} x_i + F_{ij} y_i + G_{ij} z_i \right) \tan \theta_j \right]$$

The telescope/detector configuration is optimized by solving:

$$\frac{\partial \langle \sigma_{11}^2 \rangle_j}{\partial \tan \phi_j} = 2 \left( B_{ij} x_i + C_{ij} y_i + D_{ij} z_i \right) \tan \phi_j + \sum_{i=1}^{J} \left( E_{ij} x_i + F_{ij} y_i + G_{ij} z_i \right) \tan \phi_j = 0$$

$$\frac{\partial \langle \sigma_{11}^2 \rangle_j}{\partial \tan \theta_j} = 2 \left( D_{ij} x_i + E_{ij} y_i + F_{ij} z_i \right) \tan \theta_j = 0$$

Solve for $\tan \phi_j$ and $\tan \theta_j$:

$$\tan \phi_j = -\frac{ \sum_{i=1}^{J} \left( E_{ij} x_i + F_{ij} y_i + G_{ij} z_i \right) \tan \theta_j } { \sum_{i=1}^{J} \left( D_{ij} x_i + E_{ij} y_i + F_{ij} z_i \right) }$$

Substitute into equations for $\alpha_{jk}$ and $\beta_{jk}$ in the linear algebra (matrix) form:

$$\mathbf{T} = \mathbf{E} - \mathbf{F}$$

The column vectors $\mathbf{X}$ and $\mathbf{Y}$ are given by

$$\mathbf{X} = \left[ x_j \right]$$

$$\mathbf{Y} = \left[ y_j \right]$$

The element of the $J \times J$ matrix $\mathbf{B}$ in the $k$-th row and $j$-th column is given by:

$$b_{jk} = \delta_{jk} \left[ (C_{ij} - C_{ik}) - (E_{ij} - E_{ik}) \right] - \delta_{j-k} \left[ (C_{ij} - C_{ik}) - (E_{ij} - E_{ik}) \right]$$

Here $\delta_{jk}$ is the Kroenecker delta equal to 1 when $k = j$ and 0 otherwise.

2. Application to an inverted pyramid of detectors

Consider a pyramid of four tilted detectors, with apex pointing away from the nested shells, and each occupying a quadrant with one corner on the diagonal intersecting the optical axis (see Figure). In this case, the focal surface $S$ corresponds to the flat, but tilted, surfaces of the four detectors. We denote the tilt angle by $\gamma$. If shell $j$ is displaced along the optical axis so that the apex of the inverted pyramid is a distance $S_{j2}$ from the on-axis focus for that shell, then we have:

$$\sigma_{jj}^2 = a_j + 2 \beta_j \gamma \frac{2}{\gamma^2} + \gamma^2 \tan \phi_j \tan \theta_j + f_j \tan \gamma$$

In our coordinate system, $\gamma_j < 0$ if the apex is further from the shell mid-plane than its on-axis focus. The coefficients $a_j$, $\beta_j$, etc. are evaluated in the flat plane perpendicular to the optical axis and passing through the on-axis focus for shell $j$. Each coefficient is an ensemble average of the appropriate corresponding combination of the ray position and wave-vectors, and are given in our paper. To second order in $\tan \phi_j$ and $\tan \theta_j$, we also have:

$$\sigma_{jj}^2 = a_j + \beta_j \gamma \frac{2}{\gamma^2} + \gamma^2 \tan \phi_j \tan \theta_j + f_j \tan \gamma$$

Again our paper gives expressions for $a_j$, $\beta_j$, etc. in terms of appropriate ensemble averages.

4. Configuration solutions

For a single mirror shell with $j = 1$, the solutions for $\alpha_j$ and $\beta_j$ reduce to:

$$\alpha_j = \frac{\left( C_{ij} - C_{ik} \right) - \left( E_{ij} - E_{ik} \right) \tan \theta_j}{\left( D_{ij} - D_{ik} \right) \tan \theta_j}$$

$$\beta_j = \frac{\left( C_{ij} - C_{ik} \right) - \left( E_{ij} - E_{ik} \right) \tan \phi_j}{\left( D_{ij} - D_{ik} \right) \tan \phi_j}$$

For a set of $J$ nested mirror shells, we have $J$ linear equations for the $\alpha_j$ and an equation for each $\beta_j$ in terms of the $\beta_j$. We suggest that this system of linear equations ought to be linearly independent in the sense that the determinant $\det(B_{jk}) = 0$. However, even if this condition is satisfied, current wide field telescope designs approach 100 closely nested mirror shells, so numeric precision and convergence may be issues for any computer implementation of the solution of these equations.

When optimization of the prescriptions for the reflecting surfaces of the mirror shells is desired, the above procedure becomes more complex. For example consider the case of so-called polynomial x-ray optics, assuming two mirror segment surfaces and J mirror shells, for which M-1 higher order polynomial terms $\alpha_{jm}(z-w_{m})^n$ with $m = (2M)$, $j = (1, J)$ and $n = (1, 2)$ are added to a toleration prescription for the mirror segment radius squared. The merit function will now depend on the $\alpha_{jm}$ and $\beta_{jm}$, and derivatives with respect to these new parameters set to zero and simultaneously solved for in addition to the $\alpha_j$ and $\beta_j$. Assuming the $\alpha_{jm}$ are small enough to permit linearization of the new conditions, the linear system of equations now consists of $J \times (J+1)$ equations (including the equation for $\tan \phi_j$).