High-Tc Superconducting Bolometer Noise Measurement using Low Noise Transformers - Theory and Optimization

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Abstract. Care must always be taken when performing noise measurements on high-Tc superconducting materials to ensure that the results are not from the measurement system itself. One situation likely to occur is with low noise transformers. One of the least understood devices, it provides voltage gain for low impedance inputs (< 100Ω), e.g., YBaCuO and GdBaCuO thin films, with comparatively lower noise levels than other devices for instance field effect and bipolar junction transistors. An essential point made in this paper is that because of the complex relationships between the transformer ports, input impedance variance alters the transformer’s transfer function—in particular, the low frequency cutoff shift. The transfer of external and intrinsic transformer noise to the output along with optimization and precautions are treated; all the while, we will cohesively connect the transfer function shift, the load impedance, and the actual noise at the transformer output.

1. Introduction

Complete transformer theory and analysis is widely covered in literature, e.g., Ref. [1]; yet, there is little that also treats noise. This tutorial paper explores the noise generated within the transformer and its relationship to the input and output ports. We examine the transformer as a passive, noise-free network described by an impedance matrix and establish voltage gain relationships between the input and output ports, considering the input source impedance and output load, as well as other relationships [2]. The noise spectral density of the output is then calculated and its connection to other parts of the transformer investigated. Referring the noise to the input by creating equivalent noise sources and the understanding of a simplified noise-generating circuit will illuminate the limits of the noise measurements in terms of source impedance and frequency [3]. From this same circuitry, a commonly used figure of merit—the noise factor—is derived and its advantages are discussed [8]. Finally, we look at the transformer output impedance and its relationship to a subsequent gain device(s), such as a low noise amplifier or a spectrum analyzer, in order to examine the total system noise and the signal-to-noise ratio [13].

2. Transformer Network Analysis

In this work, the transformer is treated as a network in which the internal elements are described mathematically at the input/output ports [2]. The transformer shown in figure 1(a) consists of two windings of resistances \( R_p \) and \( R_s \) with inductances \( L_p \) and \( L_s \) mutually connected by \( M \)—subscripts ‘p’ and ‘s’ indicate primary and secondary. Mutual inductance is defined as 
\[
M = k \sqrt{L_p \cdot L_s},
\]
where \( 0 \leq k \leq 1 \) is the coefficient of coupling between the windings. In this work, \( k \rightarrow 1 \) and stray capacitances are neglected. The turns-ratio of the p-s windings is 
\[
n = \sqrt[\frac{L_s}{L_p}} \frac{V_s}{V_p},
\]
where \( V_p \) and \( V_s \) are the voltages across \( L_p \) and \( L_s \).

An active or passive device connected to signal source \( V_s \) of internal impedance \( Z_s \) and to load \( Z_L \) at the output can be analyzed as a two-port, \( \rho \)-parameter network as depicted in figure 1(b). The central larger box represents the transformer circuit of figure 1(a) and, for now, is considered noise free. The impedance matrix \( Z \) represents elements inside the box such that the voltages and currents at the input and output ports are described by,
\[
V = Z \cdot I = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}
\]  

(1)

The immittance parameters \( z_{11} \) and \( z_{22} \) in (1) are the *self-impedances* and \( z_{12} \) and \( z_{21} \) are the *transfer-or*—for the transformer—*mutual-impedances*. The \( z \)-parameters are determined by a combination of ratio and open-circuit configurations of the voltages and currents at and into the ports:

\[
(a) \quad z_{11} = \frac{V_1}{I_1} \bigg|_{I_2 = 0} \quad (b) \quad z_{12} = \frac{V_1}{I_2} \bigg|_{I_1 = 0} \quad (c) \quad z_{21} = \frac{V_2}{I_1} \bigg|_{I_2 = 0} \quad \text{and} \quad (d) \quad z_{22} = \frac{V_2}{I_2} \bigg|_{I_1 = 0}
\]  

(2)

Applying (2a)-(2d) to the transformer network gives the following \( z \)-parameters as functions of \( s = j \omega \),

\[
(a) \quad z_{11}(s) = R_p + s \cdot L_p \quad (b) \quad z_{12}(s) = z_{21}(s) = s \cdot M \quad (c) \quad z_{22}(s) = R_s + s \cdot L_s
\]  

(3)

Looking into the \( k \)-th port with the \( f \)-th port open \((I_f = 0)\) implies \( M = 0 \); thus impedances \( z_{ik} \) are separate series combinations of winding resistances and inductances—\( z_{12} = z_{21} \) is found through the Laplace transform of Faraday’s Law, \( v_g = M \frac{di}{dt} \).

Referring to figure 1(b), the input and output impedances are determined by looking into the primary winding with signal source \( V_g \) and source and load impedances, \( Z_g \) and \( Z_s \), respectively. The primary impedance, \( Z_p \), is found by inserting \( V_2 = -I_2 \cdot Z_L \) into (1) and solving for \( I_2 \). The secondary impedance, \( Z_s \), is found by setting \( V_g = 0 \), substituting \( V_1 = -I_1 \cdot Z_g \) in (1) and solving for \( I_1 \). The results are,

\[
(a) \quad Z_p(s) = z_{11}(s) - \frac{z_{12}^2(s)}{z_{22}(s) + Z_L(s)} \quad \text{and} \quad (b) \quad Z_s(s) = z_{22}(s) - \frac{z_{12}^2(s)}{z_{11}(s) + Z_g(s)}
\]  

(4)

Relating this to the transformer by substituting (3a)-(3c) into (4a) and (4b) gives,

\[
(a) \quad Z_p(s) = \left( R_p + s \cdot L_p \right) - \frac{s^2 \cdot M^2}{(R_s + s \cdot L_s) + Z_L(s)} \quad \text{and} \quad (b) \quad Z_s(s) = \left( R_s + s \cdot L_s \right) - \frac{s^2 \cdot M^2}{(R_p + s \cdot L_p) + Z_g(s)}
\]  

(5)

The last terms on the right hand sides of (4a) and (4b)—or (5a) and (5b)—are the transformer’s *secondary* and primary reflection impedances, \( Z_s \) and \( Z_p \), respectively. The impedance looking into a transformer port equals its self-impedance, \( z_{ik} \), minus the reflected impedance of the other port(s); thus, the primary and secondary impedances are \( Z_p = z_{11} - Z_s \) and \( Z_s = z_{22} - Z_p \). The reflected impedances have a phase relationship to \( z_{ik} \) that increases the magnitude and, depending on termination impedances and frequency, can make up a good portion of the total port impedance.

The transformer network of figure 1(b) has two types of voltage gain: the *system voltage gain*, \( H(s) = V_2(s)/V_g(s) \), and the *network voltage gain*, \( T(s) = V_2(s)/V_1(s) \). Since \( Z_g \) is the internal impedance of the voltage source \( V_g \), \( H(s) \) cannot be directly measured. \( H(s) = T(s) \) can be a good approximation, but when dealing with comparable input impedances, a distinction must be made—references to \( V_g \) must involve \( V_1 \) in figure 1(b). Also, as a passive device, the transformer’s impedances and correspondingly the system and network gains are dependent on termination impedances \( Z_s \) and \( Z_L \).

Considering the two voltage gains under an unloaded condition (i.e., removing \( Z_t \)) sets \( I_2 = 0 \) in (1) yielding \( V_1 = z_{11} I_1 \) and \( V_2 = z_{21} I_1 \). These give open-circuit expressions for \( T(s) \) and \( H(s) \),

\[
(a) \quad T_0(s) = \left. \frac{V_2(s)}{V_1(s)} \right|_{I_2 = 0} = \frac{z_{12}(s)}{z_{11}(s)} \quad \text{and} \quad (b) \quad H_0(s) = \left. \frac{V_2(s)}{V_g(s)} \right|_{I_2 = 0} = \frac{z_{12}(s)}{z_{11}(s) + Z_g(s)}
\]  

(6)

Reinserting \( Z_L \) and replacing both \( V_g \) and \( Z_s \) with voltage source \( V_1 \), the network voltage gain from (1) is,
The substitution of \( V_1 = V_0 \cdot z_{12} \) into (7) gives the system transfer function,

\[
T(s) = \frac{V_2(s)}{V_1(s)} = \frac{z_{12}(s) \cdot Z_L(s)}{z_{11}(s) \cdot (z_{22}(s) + Z_L(s)) - z_{12}^2(s)} = T_0(s) \cdot \frac{Z_L(s)}{(z_{22}(s) + Z_L(s)) - z_{12}(s) \cdot T_0(s)}
\]  

(7)

As \( |Z_L| \to \infty \), (7) and (8) approach the open-circuit equations of (6a) and (6b), respectively. Equation (8) merely adds \( Z_g \) to \( z_{11} \) in (7) making it the special case \( T(s) = H(Z_g = 0, s) \).

From here on, to simplify matters, \( Z_g \) is replaced with source resistance \( R_g \) and \( Z_L \) is replaced with load resistance \( R_L \). Substituting the \( z \)-parameters of (3a)-(3c) into (8) results in the system transfer function in terms of the transformer elements,

\[
H(R_g, s) = \frac{s \cdot M \cdot R_s}{s^2 \cdot M^2 - (R_g + R_p + s \cdot L_p) (R_L + R_s + s \cdot L_s)}
\]  

(9)

Examination of (9) reveals that the shape of the transfer function shifts with \( R_g \)—that is, there is sensitivity to \( R_g \). There is sensitivity to \( R_L \) as well; however, it is usually constant and typically \( R_L >> R_s \).

In fact to simplify matters again, occurrences of \( R_L + R_s \) terms will be replaced with \( R_L \) in this work.

Note that (9) can be rearranged into the familiar second-order bandpass statement,

\[
H(s) = \frac{a_1 \cdot s}{s^2 + \left( \frac{\omega_m}{Q} \right) \cdot s + \omega_m^2}
\]  

(10)

where the natural frequency, \( \omega_m \), and quality factor, \( Q \), (hence bandwidth, \( \omega_B = \frac{\omega_m}{Q} \)) are functions of \( R_g \).

The coefficients are, \( a_1 = \frac{R_L \cdot M}{L_p \cdot L_s - M^2}, \omega_m(R_g) = \frac{(R_g + R_p) L_s + R_L \cdot L_p}{L_p \cdot L_s - M^2}, \) and \( \omega_m(R_g) = \sqrt{(R_g + R_p) \cdot R_L} \).

The magnitude density, \( A(R_g, \omega) = |H(R_g, s)|_{\omega = \omega_m} \), and the phase angle, \( \theta(R_g, \omega) = \arg[H(R_g, s)]_{\omega = \omega_m} \),

are found to be,

\[
A(R_g, \omega) = \frac{R_L \cdot M \cdot \omega}{\left[ (M^2 - L_p \cdot L_s) \cdot \omega^4 + \left( \frac{R_g + R_p}{L_p} \right) R_L \cdot M^2 + L_p^2 \cdot R_L^2 + \left( \frac{R_g + R_p}{L_s} \right) \cdot L_s^2 \right]^{1/2}}
\]  

(11)

and

\[
\theta(R_g, \omega) = \arctan \left[ \frac{(M^2 - L_p \cdot L_s) \cdot \omega^2 + \left( \frac{R_g + R_p}{L_p} \right) R_L}{L_p \cdot R_L + L_s \cdot \left( \frac{R_g + R_p}{L_s} \right) R_L} \right]
\]  

(12)

The relationship for maximum gain, \( A_{\text{max}} = \frac{a_1}{\omega_B} \), and the lower 3dB cutoff frequency,

\[
\omega_L = \frac{1}{2} \left( \sqrt{4 \cdot \omega_m^2 + \omega_B^2} - \omega_B \right)
\]  

(13)

and,

\[
A_{\text{max}}(R_g) = \frac{R_L \cdot M}{(R_g + R_p) L_s + R_L \cdot L_p}
\]  

(13)

and,
The high frequency cutoff is
\[ \omega_H = \omega_B - \omega_L; \]
typically, however, \( \omega_H \approx \omega_B \).

A low noise transformer model is developed here based on the PARC 1900 low-noise transformer using the 1:1000 ports [4]. The model element values are listed in inset of figure 2—from here on these will be referred to as the “model parameters.” The parameters, along with \( R_g \) and \( R_L \) values, are inserted into (11) and (12) to give the responses seen in figures 2(a) and 2(b). The gain is displayed in dB, \( A_{db} = 20 \cdot \log(A) \), and the phase in degrees. Figure 2(a) depicts the shift of the frequency response as \( R_g \) varies. The shift of the upper cutoff frequency is negligible compared to the lower cutoff frequency.

Utilizing (14), a linear relationship between \( R_g \) and lower cutoff frequency, \( f_L \), is displayed in figure 3(a). The absolute sensitivity of \( f_L \) to \( R_g \), (15a), is shown in figure 3(b). Extrapolation shows a near-unity sensitivity peak at \( R_g = \, 6.4 \Omega \); afterwards, it gradually descends to zero (not shown).

For later discussion, three forms of power gains are defined: i) power gain, \( G_p = \frac{P_2}{P_1} \), ii) available power gain, \( G_a = \frac{P_{2a}}{P_{1a}} \), and iii) transducer power gain, \( G_t = \frac{P_2}{P_{1a}} [2] \). \( P_1 \) is the primary average power and \( P_2 \) is the secondary average power delivered to the load. \( P_{1a} \) is the primary maximum available average power—the power extracted through power matching with \( V_g \) and \( Z_g \)—and \( P_{2a} \) is the load maximum available average power. Conjugate-matched terminations are assumed for \( P_{1a} \) and \( P_{2a} \). The powers \( P_1, P_2 \) and \( P_{1a} \) are defined as,

\[ (a) \ P_1 = |I_1|^2 \cdot \text{Re}(Z_p) \quad \text{and} \quad (b) \ P_2 = |I_2|^2 \cdot \text{Re}(Z_L), \quad \text{and} \quad (c) \ P_{1a} = \frac{|V_g|^2}{4 \cdot \text{Re}(Z_g)} \]  

A Thévenin equivalent network of figure 1(b) with \( Z_g \) terminated into \( V_g \) defines \( P_{2a} \). Looking back into the open-circuit output, the equivalent voltage is determined by (6b) and the equivalent secondary impedance by (4b). For series-connected \( V_{eq} \) and \( Z_{eq} \),

\[ (a) \ V_{eq} = \frac{z_{12}}{z_{11} + Z_g} \cdot V_g \quad \text{and} \quad (b) \ Z_{eq} = z_{22} - \frac{z_{12}^2}{z_{11} + Z_g} \]  

The secondary’s maximum available average power occurs at \( Z_L = Z_{eq}^* \). Applying \( Z_{eq} + Z_{eq}^* = 2 \cdot \text{Re}(Z_{eq}) \) to \( |V_{eq}|^2 / (Z_{eq} + Z_{eq}^*) \) in (16a) and (16b), the load’s maximum available power is,

\[ P_{2a} = \frac{|z_{12}|^2 \cdot |V_g|^2}{4 \cdot |z_{11} + Z_g|^2 \cdot \text{Re}(Z_{eq})} \]  

To facilitate calculating power gains, current gain \( I_2/I_1 \) is derived by substituting \( V_2 = -I_2 \cdot Z_L \) in (1),

\[ \frac{I_2}{I_1} = \frac{z_{12}}{z_{22} + Z_L} \]  

which when worked into ratio \( P_2/P_1 \) gives the simple power gain. The ratio of (17) and (15c) yields the available power gain. Both power gains are written respectively as,
\[(a) \quad G_p = \frac{P_2}{P_1} = \frac{z_{12}^2 \cdot \text{Re}(Z_L)}{|z_{22} + Z_L|^2 \cdot \text{Re}(Z_p)} \quad \text{and} \quad (b) \quad G_a = \frac{P_{2a}}{P_{1a}} = \frac{|z_{12}|^2 \cdot \text{Re}(Z_g)}{|z_{11} + Z_g|^2 \cdot \text{Re}(Z_s)} \]  

in which \(Z_{eq}\) is replaced with \(Z_s\). Finally, the transducer power gain from the ratio of (15b) and (15c), the application of (18), and the substitution of \(V_g = I_1(Z_s + Z_p)\) is given by,

\[G_t = \frac{P_2}{P_{1a}} = \frac{4|z_{21}|^2 \cdot \text{Re}(Z_g) \text{Re}(Z_L)}{(Z_g + Z_p)(z_{22} + Z_L)^2} = \frac{4|z_{21}|^2 \cdot \text{Re}(Z_g) \text{Re}(Z_L)}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}^2} \]  

After substituting (3a)-(3c) and (5a)-(5b) into (19a) and (19b), the power gains as functions of \(\omega\) and \(R_g\) are,

\[G_p(\omega) = \frac{R_l \cdot M^2 \cdot \omega^2}{(R_l \cdot M^2 + R_p \cdot I_{s}^2) \omega^2 + R_p \cdot R_l^2} \]  

and

\[G_a(R_g, \omega) = \frac{R_g \cdot M^2 \cdot \omega^2}{(R_g + R_p)^2 \omega^2 + R_s \cdot (R_g + R_p)^2} \]  

followed by substitution of (3a)-(3c) into the right-hand side of (20) to derive the transducer power gain,

\[G_t(R_g, \omega) = \frac{4 \cdot R_g \cdot R_l \cdot M^2 \cdot \omega^2}{(M^2 - L_p \cdot I_s) \omega^4 + 2 \left(\frac{R_g + R_p}{R_l}\right) R_l \cdot M^2 + R_p^2 \cdot I_{s}^2 + \left(\frac{R_g + R_p}{R_l}\right)^2 \cdot I_{s}^2 \omega^2 + \left(\frac{R_g + R_p}{R_l}\right)^2 \cdot R_l^2} \]  

Comparison of (22) to (11) reveals the relationship, \(G_t = 4 \cdot \left(\frac{R_g}{R_l}\right) \cdot A^2\). Because the transformer is a passive device, i.e., no external power is extracted, power gain equations (21a), (21b), and (22) are always less than unity.

3. Noise Generation in the Transformer

3.1. Noise definitions

Johnson and Nyquist established the theoretical basis for Brownian-type electrical noise in 1928 [5,6]. Referred to as Johnson, Nyquist, white, thermal, etc., noise, all imply the same thing: a thermally excited vibration of the charge carriers in a conductor [3]. Other forms of transformer noise can be considered, e.g., \(1/f\) or Barkhausen noise [7], but for simplicity, only Johnson noise is examined here.

Although random vibrations give zero average currents in conductors, the instantaneous current fluctuations cause voltage fluctuations across any set of terminals. The available noise power in the conductor is,

\[P_t = k_B \cdot T \cdot \Delta f \]  

where \(k_B\) is the Boltzmann’s constant (1.38 X 10^{-23} \text{ J K}^{-1}), \(T\) is the conductor’s absolute temperature in Kelvin, and \(\Delta f\) is the noise bandwidth of the measurement system [3,6]. “Available” implies maximum power measured under conjugate-matched conditions.

Assume a noise source such as the resistance member of element \(Z_t\) as depicted in figure 4(a) in which \(E_t\) represents the generated noise voltage. Given that \(Z_t\) is a series equivalent resistance and reactance, \(Z_t(\omega) = R_t(\omega) + jX_t(\omega)\), to extract the available power, a conjugate load, \(Z_t^*(\omega) = R_t(\omega) - jX_t(\omega)\), is placed at the terminals as in figure 4(b). The reactances cancel and the real parts form a voltage divider such that \(E_o = E_t/2\), where \(E_o\) is the voltage measured at the load. The power dissipated at the load, the available power, is related to (23) by [6],
and this leads to an expression for the thermally generated r.m.s. voltage across any impedance element,

\[
E_t = \sqrt{4 \cdot k_B \cdot T \cdot \text{Re}(Z)} \cdot \Delta f
\]  

(25)

To discern the independence of the noise bandwidth, both sides of (25) are divided by \( \sqrt{\Delta f} \) obtaining the root normalized noise spectral density in volts per unit root hertz [8],

\[
S_t(f) = \sqrt{4 \cdot k_B \cdot T \cdot \text{Re}[Z(f)]}
\]

(26)

The approximation \( \sqrt{4 \cdot k_B \cdot T \cdot R} \) is common; that is, \( \text{Re}(Z) \) is replaced with resistor \( R \) leaving the spectral distribution flat at all frequencies. However, according to (26)—and in practice—the spectral density is only as flat as the real part of the source impedance producing the noise. The spectral density in (26) is usually referred to as narrow-band noise (the noise content within a 1Hz interval) and \( E_t \) in (25) is referred to as wide-band noise (the noise content in rectangular-shaped bandwidth \( \Delta f \) ) [3,8].

Now consider an amplifier that measures external noise with transfer function \( H(s) \). The noise power content in the interval \( \Delta f \) is not the same as that in the signal transfer bandwidth. The signal power content lies in the frequency span, \( B = f_H - f_L \), between the \( \frac{1}{2} \) power (or \(-3dB\)) points. However, noise bandwidth is described by a rectangular power content that is equivalent to the total area under the power gain curve throughout its entire frequency span \( (0, \infty) \) divided by its maximum—it is known as the equivalent noise bandwidth (ENB) because of the equivalent base x height interval. With the power gain equal to the square-magnitude of the transfer function, recalling (11) and (13), the ENB (or \( \Delta f \)) is defined here as,

\[
\Delta f(R_g) = \frac{1}{A_{\text{max}}^2(R_g)} \cdot \int_0^\infty A^2(R_g, \omega) d\omega
\]

(27)

The ENB is relative to the \(-3dB\) bandwidth, \( B \), except for a few filters such as a Chebyshev or Legendre of orders > 3 where it is larger than \( B \) [9]. Applying (27) to a 1\(^{\text{st}}\)-order lowpass filter reduces it to, \( \Delta f = (\pi/2) \cdot f_c \), where \( f_c \) is the \(-3dB\) cutoff frequency. For a \( m^{\text{th}} \)-order, a lowpass filter with \( m > 1 \) gives, \( \Delta f = (\pi/2)(1/2)^m \cdot f_c \).

To understand the role of (27) in noise amplification, assume resistor \( R_g \) is connected to the network input of figure 1(b) using inset parameters in figure 2. Although hypothetical, also assume that a true-r.m.s., infinite bandwidth, impedance meter measures the noise voltage at the output shown by the setup in figure 5. With \( E_t \) as the thermal noise voltage generated by \( R_g \) and its spectral density defined by (26), the r.m.s. output voltage with any given \( R_g \) is,

\[
e_{\text{no}}(R_g) = \sqrt{\int_0^\infty S_t^2(\omega) \cdot A^2(R_g, \omega) d\omega}
\]

(28)

Neglecting reactive elements, if the spectral density is constant with frequency such that \( S_t(f) = 4 \cdot k_B \cdot T \cdot R_g \), then (28) is modified to,

\[
e_{\text{no}}^2(R_g) = 4 \cdot k_B \cdot T \cdot R_g \cdot A_{\text{max}}^2(R_g) \cdot \left[ \frac{1}{A_{\text{max}}^2(R_g)} \cdot \int_0^\infty A^2(R_g, \omega) d\omega \right]
\]

(29)

where (28) is seen in the brackets on the right hand side. Through (29), equation (28) reduces to,

\[
e_{\text{no}}(R_g) = S_t \cdot A_{\text{max}}(R_g) \cdot \frac{\sqrt{\Delta f(R_g)}}{\Delta f(R_g)}
\]

(30)
Dividing both sides of (30) by √Δf produces the noise voltage spectral density, \( S_{no} \), at the output,

\[
S_{no}(R_g) = \frac{e_{no}(R_g)}{\sqrt{\Delta f(R_g)}} = S_1 \cdot A_{\text{max}}(R_g)
\]  

(31)

3.2. Noise definitions applied to low-noise transformer

Inserting the transformer elements and substituting \( A_{\text{max}} \) from (13) and the spectral density of (26) into (31) provides the maximum output noise spectral density,

\[
(a) \quad S_{no(\text{max})}(R_g) = \frac{R_L \cdot M \cdot \sqrt{4 \cdot k_B \cdot T \cdot R_g}}{(R_g + R_p) \cdot I_s + R_L \cdot I_p}
\]

Or, \( (b) \quad S_{no(\text{max})}(R_g) \big|_{R_L \to \infty} = n \cdot \sqrt{4 \cdot k_B \cdot T \cdot R_g}
\]

(32)

in which \( n \) is the turns-ratio and \( R_L \) is removed in (34b).

Although equations (29)-(32) are considered ideal and the output noise is calculated by knowledge of the input noise, in actual practice the reverse is calculated: the measured output noise is referred to the input by dividing by the gain. As a function of frequency and \( R_g \), the referred to input (RTI) noise spectral density is expressed as,

\[
S_{ni}(R_g, \omega) = \frac{S_{no}(R_g)}{A(R_g, \omega)}
\]

(33)

With the bandpass response of the transformer, the input spectral density \( S_{ni} \) will have a U-shaped response—1/f noise and other factors have effect on actual measurements, but they are discounted here.

To obtain \( S_{no}(R_g, \omega) \), Nyquist’s theorem is invoked: the output noise spectral density is dependent on the resistance member of the output impedance. Utilizing (26), the output impedance is the parallel combination of the \( Z_s \) given in (5b) and \( Z_L \). With \( Z_L = R_L \) such that \( R_L >> R_s \), the real part \( R_o = \text{Re}(Z_o) \) is closely approximated by,

\[
R_o(R_g, \omega) = \left( R_L \cdot \frac{M^2 - L_p \cdot L_s}{M^2 - L_p \cdot L_s} \right) \cdot \omega^4 + \frac{R_g}{(R_g + R_L)} \cdot \left[ \left( R_p + R_g \right) \cdot \left( 2 \cdot R_s + R_L \right) \cdot \frac{M^2 + R_s \cdot R_L \cdot L_p^2 + \left( R_p + R_g \right) \cdot \left( L_p \right) \cdot L_s^2}{\left( M^2 - L_p \cdot L_s \right) \cdot \omega^4 + 2 \cdot R_L \cdot \left( R_p + R_g \right) \cdot M^2 + \left( R_p + R_g \right) \cdot \left( 2 \cdot R_s + R_L \right) \cdot L_s^2 \cdot \omega^2 + R_s \cdot R_L \cdot \left( R_p + R_g \right) \cdot \left( R_p + R_g \right) \cdot \left( R_p + R_g \right) \cdot \left( R_p + R_g \right)} \right]
\]

(34)

Substitution of (34) into (26) then gives the output noise voltage spectral density in terms of \( R_g \),

\[
S_{no}(R_g, \omega) = \sqrt{4 \cdot k_B \cdot T \cdot R_o(\omega, R_g)}
\]

(35)

To obtain the input noise voltage spectral density, the placement of noise source \( E_i \) in series with the input signal source \( V_g \) in figure 1(b)—assuming that the remainder of the network is noiseless—gives the solution. Since the individual transformer noise sources are uncorrelated, by superposition, the root-sum-squares of the noise sources can be referred to the input as a single noise source, \( S_{ni} \), stated by (33). Based on model parameters (see inset of figure 2), for decade steps of \( R_g \) used in (35), figure 6(a) plots the output noise voltage spectral density curve family. In figure 6(b), the input noise voltage spectral density curves are derived by dividing (35) by the gain of (11). The results agree with PSpice simulations.

Note that \( R_g = 0 \Omega \) does not yield zero output noise; there is always intrinsic noise. For example, at \( f = 100\text{Hz} \) in figure 6(b) with \( R_g = 0 \Omega \), \( S_{ni} = 0.035\text{nV}/\sqrt{\text{Hz}} \). Thus, the thermal noise generated by \( R_g \) is mixed with the transformer’s noise floor. \( R_g \) noise can be somewhat obtained by subtracting in quadrature the spectral density baseline \( S_{ni}(R_g = 0, \omega) \) from the \( S_{ni}(R_g \neq 0, \omega) \) plots,
\[ S_t(R_g,\omega) = \sqrt{S_{n1}(R_g,\omega)^2 - S_{n0}(0,\omega)^2} \]  

(36)

From the results shown in figure 7, although this method corrects for low values, as \( R_g \) increases, another noise contribution becomes evident—the asymptotically flat “corrected” spectral density regions show this. The spectral density values are compared to the values generated by \( R_g \) with the relative errors and placed into table 1. It is clear that an extra noise source dominates as \( R_g \) increases—for example, a 5% error is surpassed before \( R_g = 100\Omega \); at 1kΩ the error is ~42%! Hence, maintaining as low as possible \( R_g \), yet remaining greater than the equivalent resistance (0.074Ω, as calculated from the baseline spectral density) is the proper course.

3.3. Defining intrinsic noise sources

To reiterate, if \( R_g \) is too low, an intrinsic, baseline noise voltage dominates the measurement; on the other hand, for large \( R_g \)—although the baseline noise is removed—another noise source dominates. This section describes these intrinsic noise sources as well-placed, independent and equivalent noise sources.

We first look at the network box in figure 1(b) without elements \( V_g, Z_g, \) and \( Z_l \) and treat each port as a connection to a one-port network with an opposite open-circuit port [8]. Through Thévenin’s theorem, a noiseless network port responds similar to signal input—except there are two noise voltage generators inserted at both ports as shown in figure 8(a). Noise is extracted out of the network box to the external noise sources; the spectral densities of these sources are measured at each port with the opposite port open. To define these, the primary and secondary noise voltage generators \( E_p \) and \( E_s \) are added to the voltage vector of (1) yielding the linear equations,

\[ (a) \ V_1 + E_p = z_{11} \cdot I_1 + z_{12} \cdot I_2 \quad \text{and} \quad (b) \ V_2 + E_s = z_{21} \cdot I_1 + z_{22} \cdot I_2 \]  

(37)

\( E_p \) and \( E_s \) are partially correlated since they represent different fractions of the same internal noise mechanisms [8].

Now consider the arrangement in figure 8(b) where voltage and current noise generators, \( E_n \) and \( I_n \), are placed at only the primary port. To show the relationship of this arrangement to that in figure 8(a), the statements derived from (1) for figure 8(b) are [8,10],

\[ (a) \ V = z_{11} \cdot I + z_{12} \cdot I_2 \quad \text{and} \quad (b) \ V_2 = z_{21} \cdot I + z_{22} \cdot I_2 \]  

(38)

Examination of figure 8(b) shows we have expressions \( V = V_1 + E_n \) and \( I = I_n + I_1 \) that when combined into (38a) and (38b) lead to,

\[ (a) \ V_1 + (E_n - z_{11} \cdot I_n) = z_{11} \cdot I_1 + z_{12} \cdot I_2 \quad \text{and} \quad (b) \ V_2 = z_{21} \cdot I_1 + z_{22} \cdot I_2 \]  

(39)

Comparing (39) and (37) reveal that \( E_p \) and \( E_s \) are related to \( E_n \) and \( I_n \) by,

\[ (a) \ E_p = E_n - z_{11} \cdot I_n \quad \text{and} \quad (b) \ E_s = -z_{21} \cdot I_n \]  

(40)

Substitution of (3a) and (3b) into (40a) and (40b) relates \( E_p \) and \( E_s \) to the transformer elements:

\[ (a) \ E_p(s) = E_n(s) - (R_p + sL_p) \cdot I_n(s) \quad \text{and} \quad (b) \ E_s(s) = -sM \cdot I(s) \]  

(41)

Since \( E_p \) and \( E_s \) of (41) are partially correlated, by (40) \( E_n \) and \( I_n \) are also partially correlated; however, to ease calculations, consider \( E_n \) and \( I_n \) as totally uncorrelated and adequate to represent the total noise [3,8].

The noise source configurations in figures 8(a) and 8(b) are both independent of \( R_g \), and, if considered, the arrangement of figure 8(b) proves more advantageous in analyzing noise. Figure 9 shows the complete equivalent transformer circuit that highlights the topology of the input port signal, \( V_g \), and noise sources \( E_t, E_n, \) and \( I_n; \) the single RTI noise source \( E_{ni} \) can replace \( V_g \) in order to represent to total
input noise seen at the source terminals. A salient point here is that this circuit is independent of the gain and the input impedance of the transformer.

Recalling (7) and (8), the relationship between the system and network gains in figure 9 is given by,

\[ |H| = \left| \frac{V_2}{V_1} \right| = \left| T \cdot \frac{Z_p}{Z_p + R_g} \right| \]  

(42)

The noise voltage \( E_{\text{no}} \) due to \( E_i \) at the input port or due to the RTI source \( E_{\text{ni}} \) assuming \( V_g = 0 \), is calculated by,

\[ E_{\text{no}}^2 = |T|^2 \cdot E_i^2 = |H|^2 \cdot E_{\text{ni}}^2 \]  

(43)

The last two terms are of interest and from this the input port noise is,

\[ E_{\text{no}}^2 = \left( E_i^2 + E_{\text{ni}}^2 \right) \left| T \right|^2 \cdot \left| \frac{Z_p}{Z_p + R_g} \right|^2 + I_n^2 \left| T \right|^2 \cdot \left| \frac{Z_p \cdot R_g}{Z_p + R_g} \right|^2 \]  

Inserting this into (43) yields the r.m.s.-squared noise at the output,

\[ E_{\text{no}}^2 = \left( E_i^2 + E_{\text{ni}}^2 \right) |T|^2 \cdot \left| \frac{Z_p}{Z_p + R_g} \right|^2 + I_n^2 |T|^2 \cdot \left| \frac{Z_p \cdot R_g}{Z_p + R_g} \right|^2 \]  

(44)

The RTI noise source then can be derived by (43), i.e., \( E_{\text{ni}} = E_{\text{no}}/|H| \), and by noting the right side of (42) appears inside (44). Then after reduction,

\[ E_{\text{ni}}^2 = E_i^2 + E_{\text{ni}}^2 + I_n^2 \cdot R_g^2 \]  

(45)

shows that the RTI noise source is independent of system gain, \( H \), and input impedance, \( Z_p \).

\( E_{\text{ni}} \) is defined by three noise generators: \( R_g \) noise voltage, \( E_n \) and two intrinsic noise generators, \( E_n \) and \( I_n \). Since \( E_{\text{ni}} \) and \( E_i \) can be established, solving for \( E_n \) and \( I_n \) is straightforward. Via equations (45) and (25), as \( R_g \to 0 \), \( E_{\text{ni}} \to E_n \); or as \( R_g \to \infty \), \( E_{\text{ni}} \to I_n \cdot R_g \). There may be an ideal range of \( R_g \) that \( E_i \) dominates; however, if \( E_n \) and/or \( I_n \) are considerably high for all \( R_g \), \( E_i \) may not be discerned [8]. Usually one cannot accurately measure the input noise to obtain (45) because of probe disturbances—the noise voltages can only be referred to by the output noise. After converting \( E_n \) and \( I_n \) to spectral densities, i.e.,

\[ S_{\text{En}} = E_n^2 \cdot \Delta f^{-\frac{1}{2}} \]  

and \( S_{\text{In}} = I_n^2 \cdot \Delta f^{-\frac{1}{2}} \) and applying these to (45), the following using (33) determines \( S_{\text{En}} \) and \( S_{\text{In}} \),

\[(a) \ S_{\text{En}}(\omega) = S_{\text{mi}}(\omega, R_g) \bigg|_{R_g = 0} \quad [\text{V} \sqrt{\text{Hz}}] \quad \text{and (b) } S_{\text{In}}(\omega) = \frac{S_{\text{mi}}(\omega, R_g)}{R_g} \bigg|_{R_g = \infty} \quad [\text{A} \sqrt{\text{Hz}}] \]  

(46)

Figure 10 displays plots of (46a) and (46b) using the model parameters from figure 2. The \( S_{\text{En}} \) curve has a minimum at \( 0.035\text{nV\sqrt{Hz}} \) from ~0.5Hz to ~1.3 kHz, whereas above 10Hz, \( S_{\text{In}} \) is flat at \( 4.075\text{pA\sqrt{Hz}} \). Now with the intrinsic noise sources identified, (36) is modified by procedures (46a) and (46b),

\[ S_i(R_g, \omega) = \sqrt{S_{\text{mi}}(R_g, \omega)^2 + \left( S_{\text{En}}(\omega)^2 + S_{\text{In}}(\omega)^2 \cdot R_g^2 \right)} \]  

(47)

With decade \( R_g \) values, equation (47) is plotted in figure 11 using the model parameters. One can readily notice the difference between this family of curves and those of figure 7: the spectral density values on the right side of the graph are equal to the \( S_i \) values listed in table 1—all now have 0% error! However, each curve grows unbounded for \( f < 0.1\text{Hz} \) in figure 11—this is because \( S_{\text{In}} \) is approximated, losing accuracy as \( f \to 0 \).

3.4. Closer look at equivalent intrinsic noise sources, \( E_n \) and \( I_n \)
Starting with (46a), \( S_{in} \) is the ratio of (35) and (11), i.e., \( S_{in}/A \). Since (35) contains (34), after setting \( R_g = 0 \), this reduces to \( S_{En} \), the equivalent input noise voltage spectral density,

\[
S_{En}(R_L, \omega) = \frac{\left( 4 \cdot k_B \cdot T \cdot \left[ \frac{L_p \cdot L_n - M^2}{R_L} \right] \cdot \omega^2 + \left( R_p \cdot M^2 + R_s \cdot L_p^2 + \frac{R_p^2}{R_L} \cdot L_s^2 \right) \cdot \omega^2 + R_s \cdot R_p^2 \right)^{1/2}}{M \cdot \omega}
\]

(48)

From (46b) we have the same ratio, \( S_{in}/A \), but it is divided by \( R_g \). In the limit as \( R_g \to \infty \), we obtain the equivalent input noise current spectral density,

\[
S_{In}(R_L, \omega) = \frac{\sqrt{4 \cdot k_B \cdot T \cdot \left( \frac{L_n^2}{R_L} \cdot \omega^2 + R_s \right)}}{M \cdot \omega}
\]

(49)

As functions of \( R_L \), (48) and (49) reveal that the load can considerably affect the spectral distribution of \( E_n \) and \( I_n \), as seen in figures 12(a) and 12(b). The dotted flat line at the bottom of figure 12(a) is the “ideal” \( S_{En} \) in the limit of (48) as \( R_L \to \infty \) and the diagonal line in figure 12(b) is the “ideal” \( S_{In} \) in the limit of (49) as \( R_L \to \infty \). It is evident that the loading effect on the input noises must considered when selecting the passband. As a rule derived from the graph—and practice, higher values of \( R_L \) result in lower input-related noise content in the interval \( \Delta f \).

Given \( R_g \) and \( R_L \), note that \( S_{En} \) and \( S_{In} \) are minimum within a mid-range of frequencies in figures 10 and 12. The \( S_{En}(min) \) is found by taking the square of (48) to derive an equivalent resistance. Setting the derivative to zero and solving leads to,

\[
R_{En}(R_L) = R_p + R_s \cdot \left( \frac{L_n^2}{M^2} \right) + \frac{R_p^2}{R_L} \cdot \left( \frac{L_n^2}{M^2} \right) - 2 \cdot R_p \cdot \sqrt{R_s} \cdot \left( \frac{L_p}{M^2} + \frac{L_n}{R_L} \right)
\]

(50)

Only the first two terms of (50) are usually significant—inserting them into equation (26) gives \( S_{En} \). Typically, \( R_L \gg R_p^2 \), thus one can use \( R_{En} = R_p + \frac{R_s}{n^2} \) within the mid-range passband range to approximate \( S_{En}(min) \)

\[
S_{En}(min) = \sqrt{4 \cdot k_B \cdot T \cdot \left( R_p + \frac{R_s}{n^2} \right)}
\]

(51)

The minimum spectral density from noise source \( I_n \) is found in the limit of (49) as \( f \to \infty \) as seen in figure 12(b). This leads to an equivalent resistance of,

\[
R_{In} = \left( \frac{M^2}{L_n^2} \right) \cdot R_L = \frac{R_L}{n^2}
\]

(52)

The approximated \( S_{In}(min) \) becomes,

\[
S_{In}(min) = \sqrt{4 \cdot k_B \cdot T \cdot \left( \frac{n^2}{R_L} \right)}
\]

(53)

At the intersection of (49) and the ideal \( S_{In} \) in figure 12(b), (53) is reasonable close to (49) with less than \(-5\%\) error for \( f > \frac{\sqrt{R_s \cdot R_g}}{2 \cdot \pi \cdot L_s} \). At frequencies below this, one should use the ideal case of (49),
\[ S_{in} = \frac{4 \cdot k_B \cdot T \cdot R_s}{2 \pi f \cdot M}, \]
to determine the spot frequency spectral density (the noise content within a unit bandwidth centered at \( f \)). For our model parameters, \( R_{En} \approx 0.074 \Omega \) and \( S_{En(min)} \approx 0.035 \text{nV} / \sqrt{\text{Hz}} \) within \( \sim 0.1 \text{Hz} - 1 \text{kHz} \). Also, \( R_{in} \approx 1 \text{k}\Omega \) and \( S_{in(min)} \approx 4.074 \text{pA} / \sqrt{\text{Hz}} \) for \( f > 5.15 \text{ Hz} \).

3.5. Relationship between RTI and intrinsic noise sources in terms of frequency and source resistance.

We now look graphically at the relationship between the RTI noise, \( E_{in} \), and noise sources \( E_n, E_s \), and \( I_n \) as a function of frequency and \( R_g \). After converting all to spectral density, (45) leads to,

\[ S_{in}(R_g, \omega) = \sqrt{S_i(R_g)^2 + S_{En}(\omega)^2 + S_{in}(\omega)^2 \cdot R_g^2} \quad (54) \]

Using the model parameters, (54) is mapped as a function of frequency along with noise levels caused by sources \( E_i, E_n, \) and \( I_n \) for \( R_g \) values of 8.6m\( \Omega \), 8.6\( \Omega \), and 8.6k\( \Omega \) in figures 13(a), 13(b), and 13(c).

In figure 13(a), as \( R_g \to 0 \), the spectral density is \( S_{in} \approx S_{En} \) for all \( f \). Observe that in this situation \( S_i \) as well as \( S_{in}R_g \) fall well below the \( S_{En} \) curve and also note that here as well as elsewhere \( S_{En} \) is independent of \( R_g \).

At \( R_g = R_n = 8.6 \Omega \) in figure 13(b), there are three regions in which to consider: i) \( f < f_a \) : \( S_{in} \approx S_{in}R_g \), ii) \( f_a \geq f \geq f_b \) : \( S_{in} \approx S_i \), and iii) \( f > f_b \) : \( S_{in} \approx S_{En} \). Frequencies \( f_a \) and \( f_b \) locate the points where \( S_i \) intersect \( S_{in}R_g \) and \( S_{En} \), respectively; thus by setting the product of (49) and \( R_n \) then (48) equal to \( \sqrt{4 \cdot k_B \cdot T \cdot R_n} \) and solving both in, \( f_a \) and \( f_b \) results in,

\[ (a) \quad f_a = \frac{1}{2\pi} \sqrt{\frac{R_L \cdot R_s \cdot R_n}{R_L \cdot M^2 - R_n \cdot L_s^2}} \quad \text{and} \quad (b) \quad f_b = \frac{1}{2\pi} \sqrt{\frac{R_L \cdot (R_n - R_p) \cdot M^2 - R_s \cdot L_p^2}{L_g \cdot L_s - M^2}} \quad (55) \]

Using the model parameters, \( f_a = 0.48 \text{ Hz} \) and \( f_b = 42 \text{ kHz} \). Both \( S_{En} \) and \( S_{in} \) contributions are well below \( S_i \) such that one can state \( S_{in} = S_i \) with little error. As will be shown later, this \( R_g \) value is \( R_n \), the optimum source resistance in the frequency span \( f_a \) to \( f_b \). Also note in figure 13(b) that the optimum frequency occurs when \( S_{En} = S_{in}R_g \). This is denoted by \( f_n \) and is equal to the geometric center frequency, \( f_n = \sqrt{f_a \cdot f_b} \) in our example, \( f_n = 142 \text{ Hz} \).

Finally, in figure 13(c), at \( R_g = 8.6 \text{k}\Omega \), because of \( R_g^2 \) in (54), the \( S_{in}R_g \) term increases faster than \( S_i \) for increasing \( R_g \), making \( I_n \) the dominant noise source such that \( S_{in} \approx S_{in} \cdot R_g \) for all \( f \).

Changing the abscissa in figure 13(b) to source resistance \( R_g \) and setting the operating frequency to \( f_n = 142 \text{ Hz} \), the optimal \( R_g \) values are depicted in figure 14; there are again three regions of dominance to consider: \( S_{En} \) at low values of \( R_g \), \( S_{in} \cdot R_g \) at high values of \( R_g \), and \( S_i \) in the middle [8]. Choosing the same intersections from above, these points coincide with \( R_{En} \) in (50) for the lower limit of \( R_g \) and with \( R_n \) in (53) for the upper limit. \( R_n \) also coincides with the geometric center of the lower and upper limit values, \( R_n = \sqrt{R_{En} \cdot R_{in}} \), a point that coincides with the intersection of \( S_{En} \) and \( S_{in} \cdot R_g \). The useful source resistance in figure 14 is \( 0.074 \Omega \leq R_g \leq 1 \text{k}\Omega \) at \( f_n = 142 \text{ Hz} \).

3.6. Optimum noise resistance of low-noise transformer

Setting \( R_g = R_n \) (or as close as possible) is the concept of noise matching and this results in the least possible overall noise power making the optimum noise resistance of much interest. Since \( R_n \) occurs at the intersection of \( S_{En} \) and \( S_{in} \cdot R_g \), then \( R_n = S_{En}/S_{in} = E_{in}/I_n \). Substituting in (48) and (49) here leads to \( R_n \) as a function of load resistance \( R_i \) and frequency—note that it does not depend on \( T \),

\[ 11 \]
$R_n(R_L, \omega) = \left[ \frac{\left[ L_p \cdot L_s - M^2 \right]^2}{R_L} \cdot \omega^4 + \left( R_p \cdot M^2 + R_s \cdot L_p^2 + \frac{R_p^2 \cdot L_s^2}{R_L} \right) \cdot \omega^2 + R_s \cdot R_p^2 \right]^{\frac{1}{2}}$ (56)

Looking back at figures 12(a) and 12(b), though not influenced by $R_g$, $R_n$ is dependent on $R_L$. Using the model parameters, figure 15 maps (56) for various load resistances. The curves reflect the input source resistance that ensures the least noise at a given frequency. As expected, $R_n$ increases with $R_L$, but also the plateau shifts right to a higher $f_n$ as well as having wider and higher plateau frequency regimes. If $R_L$ is removed, $R_n$ is a diagonal line in log-log scale throughout the spectrum—then there is not a unique optimum frequency point. There is the appearance then that lower $R_L$—hence lower $R_n$—is indicative of lower noise; however, as shown later, this is not the case.

Through the ratio of (51) and (53), with knowledge of the turns-ratio, winding resistances and load, the plateau value of $R_n$ of a 1:$n$ turns-ratio transformer can be approximated by,

$$R_{n(mid)} = \frac{1}{n} \cdot \sqrt{R_L \cdot \left( \frac{R_p + R_s}{n^2} \right)}$$ (57)

Inserting (57) into (56a) and (56b) under the radical gives the optimum frequency, $f_n = \sqrt{f_a \cdot f_b}$.

4. Noise Figure Analysis

4.1. Noise factor definitions applied to low-noise transformer

The study of noise generation in the above low-noise transformer model indicated that noise matching would reduce the intrinsic noise present in signal and source noise measurements; in addition, using the proper optimum frequency regime reduces the amount of noise observed. To utilize the previous analysis would always be tedious thus it is desirable to have a technique to obviate this. From literature, there is a figure of merit known as the noise factor that in graph form resolves the optimum noise performance by inspection [11,13]. According to IEEE: the noise factor of a two-port device is the ratio of the available output noise power per unit bandwidth to the portion of that noise caused by the actual source connected to the input terminals of the device, measured at the standard temperature of 290K [11]. In equation form, this would read [8],

$$F = \frac{\text{Total available noise output power}}{\text{Available noise output power arising from source thermal noise}}$$

Put into familiar terms, since this refers to available power, figure 1(b) is examined in a conjugate matched condition: $Z_L = Z_s^*$. Letting $P_{noa}$ be the total available noise power at the output and $P_{toa}$ be the portion at the output due to $E_{\text{ni}}$, (19b) is used to show that the ratio of the available powers at the input is identical to $F$,

$$F = \frac{P_{noa}}{P_{toa}} = \frac{G_n \cdot P_{nia}}{G_s \cdot P_{ta}} = \frac{P_{nia}}{P_{ta}}$$ (58)

where $P_{nia}$ and $P_{ta}$ are the available powers of the RTI noise and that due to Re($Z_g$), respectively. If we replace $V_g$ in (15c) with $E_{\text{ni}}$ and use (24) to define $E_{\text{r}}$, (58) develops into,
\[
F = \frac{\frac{E_n^2}{4\text{Re}(Z_g)}}{\frac{E_i^2}{4\text{Re}(Z_i)}} = \frac{E_n^2}{E_i^2}
\]

Incorporating (45) into (59) for \( E_n \) yields the usual expression seen for the noise factor [3,12],

\[
F = 1 + \frac{E_n^2 + I_n^2 \cdot R_g^2}{E_i^2}
\]

Equation (60) covers both frequency and source resistance and it demarcates between intrinsic \((E_n \text{ and } I_n)\) and thermal \((E_i)\) noise sources; moreover, it reveals that \( F - 1 \) is the ratio of intrinsic and source noise powers.

Primarily \( F \) compares the noise of different systems and does not necessarily indicate optimum noise performance; however, the noise factor is useful in that it not only indicates how close one is to the ideal noiseless network but also the degree that the actual network adds to the noise already present [8]. To aid us, the noise factor can be converted into decibels, \( NF = 10 \cdot \log_{10}(F) \), and called the noise figure. In this form, an ideal network yields \( NF = 0 \) dB. At \( NF = 10 \) dB, the noise power due to \( E_n \) and/or \( I_n \) are 10 times that of \( E_i \); at \( NF = 20 \) dB it is 100 times, and so on. At \( NF = 3 \) dB, source and intrinsic noise levels are equal and, from an engineering point of view, it is futile to make measurements for \( NF > 3 \) dB [3]. In practice, the nominal optimal performance is found at \( 0.5 \) dB \( \leq NF \leq 3 \) dB.

A signal-to-noise (SNR) aspect of \( F \) is found by multiplying both sides of (60) by the input signal power \( V_i^2 \) and power gain \( A^2 \) [3,13],

\[
F = \frac{\text{SNR}_{in}}{\text{SNR}_{out}}
\]

Hence, \( F \) measures the decrease in SNR through the network. At \( NF = 3 \) dB, \( \text{SNR}_{out} \) is one-half of \( \text{SNR}_{in} \), and at \( NF = 0.5 \) dB, \( \text{SNR}_{out} \) is 89.3% of the \( \text{SNR}_{in} \).

The advantage of \( F \) is the ability to display it as a contour map. With constant \( R_i \), multiplying both sides of (60) by \( \Delta f \) allows the spectral densities of (46a) and (46b) or (48) and (49) to define \( F \) as a function of frequency and source resistance,

\[
F(R_g, \omega) = 1 + \frac{S_{En}^2(\omega) + S_{ln}^2(\omega) \cdot R_g^2}{S_i^2(\omega)}
\]

where, \( S_i(R_g) = \sqrt{4 \cdot k_B \cdot T \cdot R_g} \). Based on the transformer model parameters, figure 16 depicts a contour map of the noise figure of a 1:1000 turns-ratio transformer for \( R_i = 1 \) GΩ. The contours are essentially the loci of points of constant \( NF \) as a function of source resistance and operating frequency [7]. Ideally, one keeps the source resistance and frequency selection inside the 3dB contour. The center point of the contour map, a local minimum, agrees with the above calculations for the optimum noise resistance and frequency, \( (R_n = 8.6 \Omega \text{ and } f_n = 142 \text{Hz}) \). The NF map presented here is model-based in that higher-order elements are neglected and it does not account for domain-fluctuations of the core [7], 1/f noise, etc. NF contour maps are often derived experimentally, (cf. Ref. [4]); however, figure 16 is adequate for tutorial purposes.

To demonstrate the utility of the NF map, consider a 10 Hz, 1 kΩ source resistance connected to the transformer primary input. At these values, several facts can be deduced in figure 16 at, say, \( NF \approx 10 \) dB: the intrinsic noise power swamps the source noise power by a factor of 10 (in voltage, by \( \sqrt{10} \)), the RTI noise is \( E_{ni} = E_i \cdot 10^{NF/10} = 12.7 \text{nV}/\sqrt{\text{Hz}}, \) and the SNR throughput is reduced by 10. The map indicates that, for example, to accurately measure \( E_i \) noise at 10 Hz, the source resistance must be reduced by a factor of 10 to meet noise-matched conditions.

This merits some words though: Noise matching is not power or impedance matching—yet they can
coincide. There is not a direct relationship between \( R_n \) and the input impedance \( Z_p \) in noise matching as in power matching [3]. Adding resistance in series (or in parallel) to increase (or decrease) the source resistance only introduces another thermal noise source making the situation worse. In all cases, the source resistance should be close as possible to \( R_n \).

To optimize noise matching, taking the derivative of (60) with respect to \( R_g \) and setting it to zero obtaining \( \frac{d}{dR_g} F_n = 0 \) leads to, again, the optimum noise resistance, \( R_g = R_n = \frac{E_n}{I_n} \). Substitution into (60) results in the minimum noise factor,

\[
F_{\text{min}} = 1 + \frac{E_n \cdot I_n}{2 \cdot k_B \cdot T} \tag{63}
\]

After converting to \( S_{En} \) and \( S_{in} \) via (46a) and (46b)—or (48) and (49), \( F_{\text{min}} \) can be expressed as,

\[
F_{\text{min}}(R_L, \omega) = 1 + \frac{S_{En}(R_L, \omega) \cdot S_{in}(R_L, \omega)}{2 \cdot k_B \cdot T} \tag{64}
\]

Note that \( F_{\text{min}} \) is indirectly affected by \( R_g \) since it is subsumed under noise-matching conditions. With our model parameters, (64) is plotted in figure 17 along with \( R_n \) for \( R_L = 1G\Omega \). \( NF_{\text{min}} \) is minimum in the same frequency regime as the \( R_n \) plateau. For instance, at \( f_n = 142Hz \) and \( R_n = 8.56\Omega \), \( NF_{\text{min}} = 0.074\text{dB} \), the thermal noise generated by \( R_g = R_n \) is \( E_n = 0.377\text{nV}/\sqrt{\text{Hz}} \), and the RT1 noise is \( E_{ni} = 0.381\text{nV}/\sqrt{\text{Hz}} \). Both intrinsic noise sources are minimum and together make r.m.s. noise voltage, \( E_n = \sqrt{E_n^2 + I_n^2 \cdot R_n^2} = \sqrt{\frac{NF}{10^{10}}} = 0.0496\text{nV}/\sqrt{\text{Hz}} \). The factor \( \sqrt{2} \cdot E_n \) appears because \( E_n = I_n R_g \) when noise-matched.

Since \( F_{\text{min}} \) varies with \( R_L \) according to (64), \( S_{En} \), \( S_{in} \) and \( R_g \) are written as functions of both frequency and load resistance. Figure 18 plots (63) using the model parameters. The family of curves reveals that with noise-matched conditions at all frequencies, \( F_{\text{min}} \) decreases markedly with increasing \( R_L \). Obviously, larger \( R_L \) has an advantage in bandwidth below the 3dB line; but on the other hand, driving down \( R_n \) by way of a lower \( R_L \) reduces the optimum frequency. At the minimum of each curve, the optimum frequency is determined by the geometric center, \( f_n = \sqrt{f_g \cdot f_b} \).

Another way to look at \( F \) is to bring out an order of \( E_n, I_n \), and \( R_g \) each in (60) and incorporate \( F_{\text{min}} \) of (63) and \( R_n = \frac{E_n}{I_n} \) to have,

\[
F = 1 + \left( \frac{F_{\text{min}} - 1}{2} \right) \left( \frac{R_n + R_g}{R_g} \right) \tag{65}
\]

where \( R_g \neq 0 \) [8]. With the presence of \( F_{\text{min}} \), (65) gives \( F \) at \( f_n \). Substituting (64) into (65) and treating \( S_{En} \) and \( S_{in} \) strictly as functions of \( R_L \) leads to,

\[
F\left( \frac{R_g}{R_n}, R_L \right) = 1 + \frac{S_{En}(R_L) \cdot S_{in}(R_L)}{4 \cdot k_B \cdot T} \left( \frac{R_g}{R_n} + \frac{R_n}{R_g} \right) \tag{66}
\]

\( F \) is mapped using the ratio \( R_g/R_n \) in (66) at different \( R_L \) values in figure 19, where the values of \( R_n \) and \( f_n \) are also dependent on \( R_L \). Because of \( f_n \), the minimum points \((R_g/R_n = 1)\) are equal to the \( F_{\text{min}} \).

The values of \( R_g \) in which \( F \) intersects the 3 dB line in figure 19 are two important points to determine. Setting \( F = 2 \) in (65) and rearranging gives a quadratic relationship between parameters \( R_g, R_n \), and \( F_{\text{min}}: \left( \frac{R_g}{R_n} \right)^2 - \frac{2}{(F_{\text{min}} - 1)} \left( \frac{R_g}{R_n} \right) + 1 = 0 \). The solutions of this are the intersection points, which after multiplying by \( R_n \) give the inequality statement,
Thus by (67) \( R_g \) is kept between the two boundaries, the source noise is equal to or greater than the intrinsic noise.

In conclusion, looking back at figures 12(a) and 12(b), although \( S_{En} \) is affected by \( R_L \), \( S_{En(min)} \) is not; but \( S_{In(min)} \) does change significantly with \( R_L \). However, when \( R_g = R_n \) within the plateau interval about \( f_n \), the product of (53) and (57) gives (51)—not a function of \( R_L \); therefore, \( S_{In}^2 + R_n^2 = S_{En}^2 \) is constant. In short, there seems to be no benefit of a large \( R_L \) other than the broader curves under the 3dB line. The apparent advantage of increasing \( R_L \) to decrease \( F_{\min} \) is offset by an increase of the thermal noise of \( R_g \), since it must increase to match \( R_n \) anyway.

4.2. Approximations of noise factor in low-noise transformer

We now approximate the above work for the mid-range passband about \( f_n \). We start by substituting (51) and (53) into (45) to give the RTI noise power expression in r.m.s.,

\[
E_r^2 = 4 \cdot k_B \cdot T \cdot \Delta f \cdot \left( \frac{n^2 \cdot R_g^2 + R_g + R_p + R_n}{n^2} \right)
\]  

(68)

a quadratic equation in \( R_g \) scaled by \( 4 \cdot k_B \cdot T \cdot \Delta f \). On the other hand, at \( f_n \), we can substitute (51) and (53) into (64) to obtain the approximation,

\[
F_{\min} \approx 1 + 2 \cdot \sqrt{\frac{n^2 \cdot R_g + R_p}{R_L}}
\]

(69)

Inserting this and (57) into (65) leads to the same result as dividing (68) by \( E_r^2 \) from (25),

\[
F(R_g) \approx 1 + \frac{n^2 \cdot R_p + R_g + R_n}{n^2 \cdot R_g}
\]

(70)

Or for large turns-ratio and low source resistance, since usually \( R_L >> n^2 \cdot R_g \) one can approximate with simply: \( F(R_g) \approx \frac{R_g + R_p + R_n}{n^2 \cdot R_g} \).

4.3. Noise factor of low-noise transformer system with source at different temperature

Everything so far has involved the source resistance at the same temperature as the transformer, but consider now the source at a different temperature. \( F \), then, has to be reworked in order to accommodate two temperatures: \( T \) (now reserved for \( R_g \)) and \( T_a \) (reserved for the ambient temperature). Modifying \( S_{En} \) and \( S_{In} \) in (48) and (49), as well as \( S_t \) of (26), allows (62) to be treated as an expression that accounts for all variables,

\[
F_x(R_g, R_L, T, T_a, \omega) = 1 + \frac{S_{En}^2(R_L, T_a, \omega) + S_{In}^2(R_L, T_a, \omega) \cdot R_g^2}{S_t^2(R_g, T)}
\]

(71)

Equation (71) has been relabeled to \( F_x \) to distinguish it from \( F \) of equation (62), which is at ambient temperature \( T_a \).

To construct a 2D NF map of (71), three parameters are set constant: \( T_a = 290K \), ambient (room) temperature; \( R_L = 1G\Omega \), the load; and, assuming that the passband frequency shifts with \( R_g \) such that it straddles \( f_n \), we use approximations (51) and (53) to reduce (71) to an expression that relates it to (70),
\[ F_x(R_g, T) = 1 + \frac{T_a}{T} \left( F(R_g) - 1 \right) \]  

(72)

Converted into a noise figure, (72) is plotted in figure 20. With the same constraints, applying (51) and (53) to (64) leads to,

\[ F_{\text{min}}(T) = 1 + \frac{T_a}{T} \left( F_{\text{min}} - 1 \right) \]  

(73)

where \( F_{\text{min}} - 1 \) is taken from (69). Note that figure 20 indicates that the best performance is achieved when the choices of \( R_g \) and \( T \) satisfy the contours above 3dB.

A thermistor-transformer example is now presented in which \( R_g \) and its temperature \( T \) are the only variables, i.e., the transformer temperature, \( T_a = 290 \text{K} \), remains constant. Using the model parameters, the results are placed into table 2. Included with the variables, \( F_x \) from (72) and \( F_{\text{min}} \) from (73) are also shown. The source spectral density \( S_t \) is given and, after rearranging (71), the RTI spectral density, \( S_{ni} \), and the \( S_{\text{En}} \) and \( S_{\text{In}} \) contributions, \( S_{\text{A} x} \), are also given. At \( T = T_a \) in table 2 (1st row), the temperatures cancel in (72), which, after making \( R_g = 30\Omega \), (62) gives the noise factor. The 2nd row sets \( R_g = 3\Omega \) at 77K, where the noise factors are well below 2 with a large difference between \( S_t \) and \( S_{\text{A} x} \). The last row sets \( R_g = 0.3\Omega \) at 40K and shows that the noise factor go up as well as the relative difference between \( S_t \) and \( S_{ni} \); the performance becomes worse because \( R_g \ll R_n \).

\[ 1/F_x \] gives the fractional presence of \( S_t \) in the total noise \( S_{ni} \). At \( T = 290 \text{K} \) and \( R_g = 30\Omega \), 98% of the total noise is source or \( R_g \) noise; at \( T = 77 \text{K} \) and \( R_g = 3\Omega \), it is 90.6% of the total noise; and, for \( T = 40 \text{K} \) and \( R_g = 0.3\Omega \), 35.8% of the total is \( R_g \) noise.

5. DC-blocking Capacitor and Noise Measurement

Some cases require a d.c. blocking capacitor between the source and transformer primary to prevent core magnetization due to bias current; this, however, may have an undesirable effect on the transfer function and noise level. The insertion of a d.c. blocking capacitor into figure 9 is redrawn in figure 21. The capacitor in series with \( R_g \) forms input impedance \( Z_g(\omega) = R_g - jX_g(\omega) \), where \( X_g(\omega) = 1/(\omega C_g) \) is the reactance. The substitution of \( R_g \) for impedance \( Z_g \) in (45) yields,

\[ E_n^2 = E_1^2 + E_n^2 + I_n^2 \left| Z_g \right|^2 = E_1^2 + E_n^2 + I_n^2 \cdot R_g^2 + I_n^2 \cdot X_g^2 \] 

(74)

The additional term signifies extra noise voltage across the capacitor due to \( I_n \); moreover, it is frequency dependent due to both \( I_n \) and \( X_g \). This noise has a distinct \( 1/\omega \) distribution (not excess noise) and its contribution is inversely proportional to \( C_g \).

To lower the cutoff frequency as well as reduce the noise contribution, use as large of \( C_g \) as possible. With the above transformer model’s unusual low cutoff frequency, finding a large value capacitor that does not introduce other unwanted factors is difficult. Electrolytic capacitors, popular for large values, are known for both low accuracy and temperature stability [14]; however, they contribute noise due to leakage currents and should be avoided [12]. Double layer capacitors are currently a better choice: They have slightly better accuracy and stability and, with higher series resistance, the leakage is much less [14].

Consider, say, \( C_g = 15 \text{mF} \) and its effect on the transfer function. Inserting \( Z_g(s) = R_g + 1/(s \cdot C_g) \) into (8) produces figure 22, where only frequencies below 1kHz are shown—higher frequency results are similar to Figure 2(a). The plots for \( R_g < 10\Omega \) reveal peaking centered at \( \frac{1}{2 \cdot \pi \cdot \sqrt{I_n \cdot C_g}} \approx 3.15 \text{Hz} \) and cutoff frequencies insensitive to \( R_g \), e.g., \( f_c \approx 2 \text{Hz} \). By comparison, the cutoff frequencies in figure 22 are relatively the same as figure 2(a) for \( R_g \geq 10\Omega \). Also, the curve’s ENB is quite different from those of figure 2(a), e.g., at \( f < 1 \text{kHz} \), the ENB is larger for \( R_g \leq 1\Omega \) plots.

Substituting \( R_g + 1/s \cdot C_g \) into \( Z_g(s) \) in (5b) with \( R_t \) in parallel gives \( R_n \), which when placed into (35) yields the output spectral densities of figure 23(a). In contrast to figure 6(a), there is obvious peaking of
the output noise. Divided by the corresponding amplitudes (figure 22), the family input spectral density curves are given in figure 23(b). Comparing figure 23(b) to figure 6(b), the converging noise values are larger as \( f \to 0 \) for all \( R_g \). This is due to the noise across \( X_g \) and can be compensated by applying (36) to figure 23(b) for results similar to figure 7.

Applying the techniques given in (46a) and (46b) is tantamount to either shorting the left side of \( C_g \) in figure 21 to ground to derive \( S_{in} \) or setting \( R_g \) to a large value (e.g. 1G\( \Omega \)) to determine \( S_{in} \). Afterwards, applying (47) yields results similar to figure 11. Using \( S_{in} \) and \( S_{ln} \) with (74) and modifying (62) to include \( C_g \) leads to,

\[
F(R_g, \omega) = 1 + \frac{S_{in}^2(\omega) + S_{ln}^2(\omega) \cdot \left( \frac{R_g^2 + \frac{1}{\omega^2 \cdot C_g^2}}{S_{in}^2(R_g)} \right)}{S_{in}^2(R_g)}
\]  

From (75), a NF map for our transformer model with \( C_g = 2.3F \) along with the inset parameters from figure 2 is displayed in figure 24. Compared to figure 16, the contours on the lower left half of the map shift right in response to the extra capacitance. This is expected since the bandwidth is narrower and there is extra noise power due to \( X_g \). Despite this, \( R_a \) and \( f_n \) undergo very little change in this example.

6. The Amplifier Chain and Noise

6.1. Secondary resistance without load

The cascade arrangement of the low-noise transformer and amplifiers and/or meter devices is examined in this section. These extra devices generate additional noise that propagates through the system; hence, their noise resistances should be matched—or closely matched. In most cases though, the applied or input noise is already greater than the device’s intrinsic noise by a factor of \( \sqrt{2} \). Ideally, all connected stages—whether a transformer or an amplifier/meter—should operate at noise resistance or \( R_n \) input conditions and its output resistance should be matched to the following stage’s \( R_n \).

With that said, before examining other devices, we isolate first the transformer’s output resistance member. Removing \( R_L \) and evaluating (5b) gives,

\[
R_{os}(R_g, \omega) = \frac{\left( R_g + R_p \right) \cdot M^2 + R_g \cdot L_p^2 \cdot \omega^2 + R_g \cdot \left( R_g + R_p \right)}{L_p^2 \cdot \omega^2 + \left( R_g + R_p \right)^2}
\]  

Figure 25 plots (76) for decade values of \( R_g \) using the model parameters. Notice that \( R_{os} \) increases whereas \( f_n \) shifts to the right with larger \( R_g \); otherwise, the family of curves is flat in the optimum frequency regime and approximated by,

\[
R_{os}(R_g) = R_g + n^2 \cdot \left( R_g + R_p \right)
\]  

Equations (76) or (77) are useful for noise matching whereas equation (34), by using \( R_L \), derives the total resistance \( R_{os} \)—eventually, both sets of equations use (35) to obtain \( S_{no} \).

Although meant for transformers, figure 9 serves as a schematic as well for a low-noise amplifier (LNA) or a meter, such as a spectrum analyzer (S/A). The transformer, LNA, and S/A thus use similar equations: (45) for \( E_{ni} \); (46a) and (46b) for \( S_{in} \) and \( S_{ln} \); and (62), (64), and (67) for \( F, F_{(min)} \) and the \( R_g \) boundaries, respectively. For a LNA or S/A, \( R_g \) in figure 9 is substituted by the preceding stage drive resistance, e.g., the transformer’s output resistance. In the same way, the load resistance, \( R_L \), of a transformer can be the input resistance member of a BJT base or FET gate impedance.

6.2. Cascade power gain, noise power, and noise factors
When comparing systems it is more convenient to evaluate the system gain and noise factor of the entire chain as a single network rather than in piecemeal. Because the individual noise powers are uncorrelated, the derivation of the system noise factor is not as straightforward as the system signal power gain. H. T. Friis, in 1944, was the first to describe a method that evaluates a power transfer system in terms of individual power gains and noise factors [13]. The method also accounts for terminal impedance mismatch; yet the load impedance and input/output resistance thermal noise are of no consequence [8,13].

Figure 26 presents a cascaded power transfer system consisting of three stages: A low-noise transformer of gain $A_1$ and noise factor $F_1$; a LNA of $A_2$ and $F_2$; and a S/A of $A_3$ and $F_3$. Also shown are the intrinsic noise voltage sources, $E_{nA_k}$, of each $k$th stage that through (45) are represented by $E_{nA_k}^2 = E_{nA_k}^2 + I_{nA_k}^2 \cdot R_{nA_k}^2$, in which $R_{nA_k}$ is the previous stage’s output resistance. The available power gain, $G_a$, is a function of the network $z$-parameters and source impedance; yet, looking at (19b), $G_a$ is independent of the load. Combining the magnitude-square of (6b) into (19b) simplifies $G_a$ for each $k$th stage to,

$$G_a = A_k^2 \left( \frac{R_{o(k-1)}}{R_{o_k}} \right)$$

where $A_k$ is the voltage gain and $R_{o_k}$ is the output resistance of the $k$th stage, respectively. For $k = 1$, $R_{o_1} = R_g$ and $R_{o_1} = R_{os}$ according to (77). Evaluating (78) for the transformer and the LNA in figure 26 gives the following results:

$$(a) \quad G_{a_1} = \frac{n^2 \cdot R_g}{R_g + n^2 \cdot (R_g + R_p)} = n^2 \left( \frac{R_g}{R_{os}} \right) \quad \text{and} \quad (b) \quad G_{a_2} = \frac{R_{os} \cdot R_{l_2} \cdot A_{amp}^2}{R_{o_2} \cdot (R_{os} + R_{l_2})^2} = A_{LNA}^2 \left( \frac{R_{o_2}}{R_{o_2}} \right)$$

where (79a) is the mid-band approximation of (21b) and though not shown it is a function of $R_p$. In (79b) $A_2 = A_{LNA} \cdot \frac{R_{l_2}}{R_{os} + R_{l_2}}$, where $A_{LNA}$ is the LNA voltage gain setting; the optimal case, $R_{l_2} >> R_{os}$, is assumed for the right-most approximation in (79b). For the S/A, the gain is simply set to unity or $G_{a_3} = 1$.

The total power gain of any measurement system with $N$ gain/attenuator elements is the product of the individual power gains,

$$G_{sys} = \prod_{k=1}^{N} G_{a_k} = G_{a_1} \cdot G_{a_2} \cdot \ldots \cdot G_{a_N}$$

The application of (80) to equations (79a), (79b), and the S/A gain results in the mid-band approximation of the system in figure 26,

$$G_{sys}(R_g) = n^2 \cdot A_{LNA}^2 \left( \frac{R_g}{R_{o_2}} \right)$$

Setting $R_g$ constant in figure 26 as well as removing $E_i$ and $E_{nA_k}$ noise sources, the input and output available powers are simply $P_g = V_g^2 / 4 \cdot R_g$ and $P_o = P_g \cdot G_{sys}$, respectively. Now if $E_i$ is reinstated and $P_g = 0$, (80) gives the power transfer of external noise $P_i$ from (23) such that, $P_{no} = P_i \cdot G_{sys}$. Figure 27 details the total power transfer through figure 26 after reestablishing noise sources $E_{nA_k}$. With $F_k$ as the individual noise factors, the intrinsic noise power at each $k$th stage is,

$$P_{nA_k} = k_B T \Delta f \cdot (F_k - 1)$$

With all parameters constant, (82) depends on resistance only through $F_k$. Setting signal $P_g = 0$ in figure 27, the individual output powers are $P_{no} = G_{a_1} \cdot k_B T \Delta f \cdot F_1$, $P_{no_2} = G_{a_2} \cdot [P_{no_1} + k_B T \Delta f \cdot (F_2 - 1)]$, and $P_{no} = P_{no_3} = G_{a_3} \cdot [P_{no_2} + k_B T \Delta f \cdot (F_3 - 1)]$, where $k_B T \Delta f$ is the noise power of each source resistance.
Expanding $P_{no}$ results in the total noise power output:

$$P_{no} = k_B \cdot T \cdot \Delta f \cdot G_{sys} \left( F_1 + \frac{F_2 - 1}{G_{a_1}} + \frac{F_3 - 1}{G_{a_1} \cdot G_{a_2}} \right)$$  \hspace{1cm} (83)$$

From (58) with $P_{toa} = k_B \cdot T \cdot \Delta f \cdot G_{sys}$, the system noise factor is the enclosed terms in (83) and can be further generalized as,

$$F_{sys} = F_1 + \frac{F_2 - 1}{G_{a_1}} + \frac{F_3 - 1}{G_{a_1} \cdot G_{a_2}} + \ldots + \frac{F_N - 1}{G_{a_1} \cdot G_{a_2} \ldots G_{a_{N-1}}} = F_1 + \sum_{n=2}^{N} \frac{F_{n+1} - 1}{\prod_{m=1}^{n} G_m}$$ \hspace{1cm} (84)

Known as Friis’ formula, (84) is a canonical statement for $N$ gain/attenuator elements in a power transfer system [8,13]. For an optimal system, typically only the first two terms are significant; however, the best performance is obtained when $F_1$ has significantly more influence over the summation term in (84) [3,8].

6.3. SNR and conversion of cascade transformer/amplifier/meter system

The system of figure 27 can be reduced to the simple power transfer network shown in figure 28. Three input ports represent the system intrinsic and source resistance noise powers with the bottom port for the signal power. $P_{out}$, the power transfer output, contains both the signal output power, $P_o = P_g \cdot G_{sys}$, and the output noise power now defined as, $P_{no} = P_t \cdot G_{sys} \cdot F_{sys}$. Equations (82) and (84) furnish the system equivalent intrinsic noise power,

$$P_{nA} = k_B \cdot T \cdot \Delta f \cdot F_{sys}$$ \hspace{1cm} (85)$$

Once $G_{sys}$, $F_{sys}$, and the system ENB are determined, (58) can be used to determine the SNR using the available power. Applying definitions $SNR_{out} = P_o/P_{no}$, $SNR_{in} = P_g/P_{ni}$, and the above work to (61) leads to,

$$SNR_{out} = \frac{P_g}{k_B \cdot T \cdot \Delta f \cdot F_{sys}}$$ \hspace{1cm} (86)$$

To maintain $SNR_{out} \geq 1$, $P_g \geq k_B \cdot T \cdot \Delta f \cdot F_{sys}$, signal $V_g$ then must be greater than or equal to the system RTI noise voltage,

$$V_g \geq E_{ni} = \sqrt{4 \cdot k_B \cdot T \cdot \Delta f \cdot R_g \cdot F_{sys}}$$ \hspace{1cm} (87)$$

We convert to spectral density by examining figure 29, it is similar to figure 9, except $E_{nA}$ replaces $E_{ni}$ and $I_n \cdot R_g$ and there is not a load impedance. Dividing (87) by $\Delta f$ gives the system noise voltage spectral density of the equivalent input noise source for a low noise transformer-driven system affected by $R_g$,

$$S_{ni}(R_g, \omega) = \sqrt{4 \cdot k_B \cdot T \cdot R_g \cdot F_{sys}(R_g, \omega)}$$ \hspace{1cm} (88)$$

Under the same conditions, since $P_{nA}$ of (85) is dependent on $R_g$ via $F_{sys}$, the system intrinsic noise voltage source is,

$$S_{nA}(R_g, \omega) = \sqrt{4 \cdot k_B \cdot T \cdot R_g \cdot (F_{sys}(R_g, \omega) - 1)}$$ \hspace{1cm} (89)$$

To obtain the thermal noise $S_t$ of (26), if $A_{sys}$ and $F_{sys}$ are known at $V_g = 0$, $A_{sys}$ is divided into measurement $S_{no}$ yielding (88), i.e., $S_{ni} = S_{no}/A_{sys}$. Rearranging for $S_t$ then gives,
\[
S_t(R_g, \omega) = \frac{S_{no}(R_g, \omega)}{A_{sys}(R_g, \omega) \cdot \sqrt{F_{sys}(R_g, \omega)}}
\]  
(90)

Strictly as a real-number process between zero and one, multiplying \(S_{no}\) over \(A_{sys}\) by the conversion and correction factor in (90), \(F_{sys}^{-\frac{1}{2}}\), gives the noise voltage spectral density of the sensor.

7. Conclusion

One of the underlying themes of this work has been the effect on a low-noise transformer due to variable input source or sensor resistance, \(R_g\). Two concerns were: i) a transfer function highly dependent upon input/output impedances; the output impedance, however, is usually constant. And ii) noise characteristics are determined by source and load impedances; that is, the spectral density distribution is shaped by the transfer function and the magnitude levels depend on termination impedances. We discerned, also, between the two transfer functions: \(T(s)\), the network gain that is independent of \(R_g\), and \(H(s)\), the system transfer function that is dependent on \(R_g\). Then we demonstrated with \(H(s)\) that the lower frequency cutoff shifts with respect to the sensor or source resistance, \(R_g\).

Nyquist's theorem was applied to acquire the low noise transformer's output noise voltage spectral density; this divided by the gain gave the RTI noise voltage spectral density. It was demonstrated that the magnitude of the noise varies with \(R_g\). Further calculations, where the \(R_g = 0\Omega\) baseline was subtracted in quadrature from the \(R_g \neq 0\Omega\) curves, revealed that the noise magnitudes changed disproportional to \(R_g\), indicative of intrinsic noises other than the input sensor noise. Two-port analysis defined intrinsic equivalent voltage and current noise sources, \(E_n\) and \(I_n\), at the input, which when added in quadrature to the sensor thermal noise gave the correct expression for the RTI noise. The spectral densities for \(E_n\) and \(I_n\) are derived by zero/high resistance measurement techniques or by calculation. For post-processing, (47) and (27) can be utilized to derive the correct sensor r.m.s. noise, \(E_i\). Equation (54) was mapped at various \(R_g\) values to show the advantage of matching the source to \(R_n\), the optimum noise resistance. The optimum frequency, \(f_n\), is the geometric center of endpoints where \(S_i = S_{En} = S_{In}R_g\) — at the center point, \(R_g = R_n\). Operating beyond the endpoints allows \(S_{En}\) or \(S_{In}R_g\) to dominate \(S_i\). We also looked at the nexus between \(R_n\) and \(R_t\); as a note, although not mentioned in above, for large \(R_t\), \(R_n\) is on the order of \(\sqrt{R_t}/n\).

The noise factor (and noise figure) absorbs a number of calculations into a single expression that is dependent upon source resistance and frequency. \(NF\) contour maps visually identify the optimal operating region, the area inside the 3dB contour. Although a figure of merit to compare different systems, there are other uses: one can derive RTI, source (sensor), and intrinsic noises, quantify SNR loss, determine noise-matched conditions, state the range of \(R_g\) in which \(NF \leq 3\text{dB}\), and relate it to temperatures other than ambient. The latter was plotted as a sensor temperature vs. sensor resistance NF contour map where the optimal performance was the region above the 3dB line. Throughout this work, there was composite dependence of \(F_{min}\) on \(R_t\), which implied that lower \(F\) values (and higher \(R_g\)) follow higher \(R_t\) values.

Dc blocking issues were also examined. An improper d.c. blocking capacitor value selection could place a resonant peak at or near a region of interest altering the measurement. It was shown that the effect on the \(NF\) reshaped the contours such that there was more intrinsic noise at lower frequencies. Conclusion: if a dc-blocking capacitor is needed, choose a reasonably large value.

We concluded by reducing the multi-element gain system to a single element. Generalizing the power gain with the unloaded transfer function and the ratio of input/output resistances, an indexed power gain formula for each element was generated. The product of these separate power gains gave the system power gain. Each element’s intrinsic noise power was also used to derive the system’s noise power and noise factor, \(F_{sys}\). The optimal \(F_{sys}\) is one in which the power gain of the first device far exceeds the following devices. Knowledge of \(F_{sys}\) provides a system noise and \(SNR_{out}\) expression, a system r.m.s. noise expression that dictates the minimum signal voltage, and the system RTI and intrinsic noise spectral densities. From the latter two, an equation was introduced that converts and corrects data from the low-noise transformer-driven system to give sensor noise without having to contend with complex values.
Acknowledgements

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References

Figure 1. Schematic diagrams of (a) transformer and (b) the two-port z-parameter network of the transformer.
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Figure 9. Equivalent noise sources $E_n$ and $I_n$ with input signal and noise source $V_g$ and $E_t$. 

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Figure 10. Equivalent noise voltage, $E_n$, and current, $I_n$, spectral densities based on model parameters.
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Figure 19: Noise figure (NF) versus $R_g/R_n$ of transformer model using the model parameters at various load resistance ($R_L$) values.
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Figure 25. Resistance member ($R_{os}$) of the transformer output impedance using the model parameters.
Figure 26. Measurement system of cascaded transformer, low noise amplifier, and spectrum analyzer networks with associated noise sources. LNA drive resistance is typically $R_{o2} = 50\,\Omega$ or $600\,\Omega$
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Figure 29: Diagram of a measurement system network in terms of signal- and noise-related voltages.
Table 1. Corrected RTI spectral densities vs. actual thermal noise spectral density, $S_t$, from figure 7.

<table>
<thead>
<tr>
<th>$R_g$</th>
<th>$S_t$(100 Hz)</th>
<th>$S_t$</th>
<th>error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 Ω</td>
<td>0.0407 nV/√Hz</td>
<td>0.0407 nV/√Hz</td>
<td>0.012 %</td>
</tr>
<tr>
<td>1 Ω</td>
<td>0.1288 nV/√Hz</td>
<td>0.1287 nV/√Hz</td>
<td>0.058 %</td>
</tr>
<tr>
<td>10 Ω</td>
<td>0.4091 nV/√Hz</td>
<td>0.407 nV/√Hz</td>
<td>0.508 %</td>
</tr>
<tr>
<td>100 Ω</td>
<td>1.35 nV/√Hz</td>
<td>1.287 nV/√Hz</td>
<td>4.91 %</td>
</tr>
<tr>
<td>1 kΩ</td>
<td>5.763 nV/√Hz</td>
<td>4.070 nV/√Hz</td>
<td>41.6 %</td>
</tr>
</tbody>
</table>
Table 2: Noise factors and noise spectral densities for various $R$ and $T$ values at source generator temperature $T_a = 290K$ and load resistance $R_L = 1G\Omega$—the operating frequency is $f = 100$ Hz.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$R_g$</th>
<th>$F_{\min \Delta}$ [Eq. (73)]</th>
<th>$F_{\Delta}$ [Eq. (72)]</th>
<th>$S_r$</th>
<th>$s_{\Delta r} = s \sqrt{F_r}$</th>
<th>$s_{\Delta f} = s \sqrt{F_r} - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30K</td>
<td>30 $\Omega$</td>
<td>1.017 (0.07dB)</td>
<td>1.032 (0.14dB)</td>
<td>0.693 nV/$\sqrt{\text{Hz}}$</td>
<td>0.704 nV/$\sqrt{\text{Hz}}$</td>
<td>0.125 nV/$\sqrt{\text{Hz}}$</td>
</tr>
<tr>
<td>77K</td>
<td>3 $\Omega$</td>
<td>1.065 (0.27dB)</td>
<td>1.104 (0.43dB)</td>
<td>0.113 nV/$\sqrt{\text{Hz}}$</td>
<td>0.119 nV/$\sqrt{\text{Hz}}$</td>
<td>0.0365 nV/$\sqrt{\text{Hz}}$</td>
</tr>
<tr>
<td>40K</td>
<td>0.3 $\Omega$</td>
<td>1.125 (0.51dB)</td>
<td>2.791 (4.46dB)</td>
<td>0.026 nV/$\sqrt{\text{Hz}}$</td>
<td>0.043 nV/$\sqrt{\text{Hz}}$</td>
<td>0.034 nV/$\sqrt{\text{Hz}}$</td>
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</tbody>
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