Computational Validation of a 2-D Semi-empirical Model for Inductive Coupling in a Conical Pulsed Inductive Thruster

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Outline of Talk

- Pulsed Inductive Plasma Thrusters
  - MAD-IPA

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  - Analytical Model

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  - Finite element results indicate limit to applicability

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  - Finite element results indicate limit to applicability
- Non-dimensional Analysis

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  - Finite element results indicate limit to applicability
- Non-dimensional Analysis
- Conclusions

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Pulsed Inductive Plasma Thrusters
Lack of Cavity Decreases Propellant Utilization

Idealized thruster operation:

- Capacitors
- Inductive Coil
- Propellant Nozzle
- Propellant Injection
- Magnetic Field
- Coils Current
- Ionization & Acceleration
- Plasma
Lack of Cavity Decreases Propellant Utilization

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Analytical Model
Model Thruster-Plasma System as Circuits

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Governing Equations via Kirchhoff’s Law

\[
\frac{dI_1}{dt} = \frac{L_C V - L_C R_e I_1 - M R_p I_2 + (L_C I_2 + M I_1) \frac{dM}{dt}}{L_C (L_0 + L_C) - M^2}
\]

\[
\frac{dI_2}{dt} = \frac{M \frac{dI_1}{dt} + I_1 \frac{dM}{dt}}{L_C} - R_p I_2
\]

\[
\frac{dV}{dt} = -\frac{I_1}{C}
\]

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Equations Governing Current Sheet Motion

\[ L_{tot} = L_0 + L_C - \frac{M^2}{L_C} \]

\[ L_{tot}(\bar{r}, z) = L_0 + L_C \left( 1 - \exp \left( -\frac{z}{z_0} \right) \left( \frac{\bar{r}}{r_{coil}} \right)^N \right) \]

\[ M = L_C \exp \left( -\frac{z}{2z_0} \right) \left( \frac{\bar{r}}{r_{coil}} \right)^{N/2} \]

\[ F_i = \frac{I^2}{2} \frac{\partial L}{\partial x_i} \]

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\[ F_i = \frac{l^2}{2} \frac{\partial L}{\partial x_i} \]
Equations Governing Current Sheet Motion

\[
\frac{dv_z}{dt} = \frac{L_C I_1^2}{2z_0} \exp\left(-\frac{z}{z_0}\right) \left(\frac{\bar{r}}{r_{coil}}\right)^N \quad m_{\text{bit}}
\]

\[
\frac{dv_r}{dt} = \frac{P_2 2\pi \bar{r} l_{coi}}{2r_{coil}^N} - \frac{L_C I_1^2 N}{2r_{coil}^N} \exp\left(-\frac{z}{z_0}\right) (\bar{r})^{N-1} \quad m_{\text{bit}}
\]

\[
\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} \left[ M^2 - 1 \right]
\]

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Equations Governing Current Sheet Motion

\[ \frac{d v_z}{d t} = \frac{L_c I_1^2}{2z_0} \exp \left( -\frac{z}{z_0} \right) \left( \frac{\bar{r}}{r_{coil}} \right)^N \]

\[ \frac{d v_r}{d t} = \frac{P_2 2\pi \bar{r} l_{coil} - \frac{L_c I_1^2 N}{2 r_{coil}^N} \exp \left( -\frac{z}{z_0} \right) \left( \bar{r} \right)^{N-1}}{m_{bit}} \]

\[ \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} \left[ \mathcal{M}^2 - 1 \right] \]

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Equations Governing Current Sheet Motion

\[
\frac{dv_z}{dt} = \frac{\left[ \frac{L_c I_1^2}{2z_0} \exp \left( -\frac{z}{z_0} \right) \left( \frac{\bar{r}}{r_{coil}} \right)^N \right]}{m_{bit}}
\]

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\frac{dv_r}{dt} = \frac{\left[ P_2 2\pi \bar{r} l_{coil} - \frac{L_c I_1^2 N}{2r_{coil}^N} \exp \left( -\frac{z}{z_0} \right) \left( \bar{r} \right)^{N-1} \right]}{m_{bit}}
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\frac{dv_r}{dt} = \left[\frac{P_2 2\pi \bar{r} l_{coil}}{2r_{coil}^N} - \frac{L CI_1^2 N}{2r_{coil}^N} \exp\left(-\frac{z}{z_0}\right) (\bar{r})^{N-1}\right] m_{bit}
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\[ \frac{dv_z}{dt} = \frac{1}{m_{bit}} \left[ \frac{L_C l_1^2}{2z_0} \exp \left(-\frac{z}{z_0}\right) \left(\frac{\bar{r}}{r_{coil}}\right)^N \right] \]

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\[ \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} \left[ M^2 - 1 \right] \]
Model Relies on Semi-Empirical Expression

\[ L_{tot}(\bar{r}, z) = L_0 + L_C \left( 1 - \exp\left(-\frac{z}{z_0}\right) \left(\frac{r}{r_{coil}}\right)^N\right) \]
Model Relies on Semi-Empirical Expression

\[ L_{\text{tot}}(\bar{r}, z) = L_0 + L_C \left( 1 - \exp\left(-\frac{z}{z_0}\right) \left( \frac{\bar{r}}{r_{\text{coil}}} \right)^N \right) \]

Applicable to all inductive coil geometries?

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Computational Validation

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Simulation Configuration for Radial Compression
Simulation Configuration for Radial Compression

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Good Agreement from $20^\circ$-55$^\circ$
Error Function Better Fit at Angles less than 20°

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Error Function Better Fit at Angles less than 20°
Non-dimensional Analysis

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Substitutions

\[ l_1^* = \frac{1}{V_0} \sqrt{\frac{L_C}{C}} l_1 \]

\[ V^* = \frac{V}{V_0} \]

\[ v_z^* = \frac{\sqrt{L_0 C}}{z_0} v_z \]

\[ v_r^* = \frac{\sqrt{L_0 C}}{r_{\text{coil}}} v_r \]

\[ t^* = \frac{t}{\sqrt{L_0 C}} \]

\[ l_2^* = \frac{1}{V_0} \sqrt{\frac{L_C}{C}} l_2 \]

\[ M^* = \frac{M}{L_C} \]

\[ z^* = \frac{z}{z_0} \]

\[ r^* = \frac{r}{r_{\text{coil}}} \]

\[ P^* = \frac{P}{P_1} \]
Substitutions

\[ I_1^* = \frac{1}{V_0} \sqrt{\frac{L_C}{C}} I_1 \]

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\[ t^* = \frac{t}{\sqrt{L_0 C}} \]

\[ I_2^* = \frac{1}{V_0} \sqrt{\frac{L_C}{C}} I_2 \]

\[ M^* = \frac{M}{L_C} \]

\[ z^* = \frac{z}{z_0} \]

\[ r^* = \frac{r}{r_{coil}} \]

\[ P^* = \frac{P}{P_1} \]

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\[ l_1^* = \frac{1}{V_0} \sqrt{\frac{L}{C}} l_1 \]

\[ V^* = \frac{V}{V_0} \]

\[ v_{z^*} = \frac{\sqrt{L_0 C}}{z_0} v_z \]

\[ v_{r^*} = \frac{\sqrt{L_0 C}}{r_{coil}} v_r \]

\[ t^* = \frac{t}{\sqrt{L_0 C}} \]

\[ l_2^* = \frac{1}{V_0} \sqrt{\frac{L}{C}} l_2 \]

\[ M^* = \frac{M}{L_C} \]

\[ z^* = \frac{z}{z_0} \]

\[ r^* = \frac{r}{r_{coil}} \]

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\[ v_{r^*} = \frac{\sqrt{L_0 C}}{r_{coil}} v_r \]

\[ t^* = \frac{t}{\sqrt{L_0 C}} \]

\[ I_2^* = \frac{1}{V_0} \sqrt{\frac{L_C}{C}} I_2 \]

\[ M^* = \frac{M}{L_C} \]

\[ z^* = \frac{z}{z_0} \]

\[ r^* = \frac{r}{r_{coil}} \]

\[ P^* = \frac{P}{P_1} \]

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\[ l_1^* = \frac{1}{V_0} \sqrt{\frac{L C}{C}} l_1 \]

\[ v^* = \frac{V}{V_0} \]

\[ v_z^* = \frac{\sqrt{L_0 C}}{z_0} v_z \]

\[ v_r^* = \frac{\sqrt{L_0 C}}{r_{\text{coil}}} v_r \]

\[ t^* = \frac{t}{\sqrt{L_0 C}} \]

\[ l_2^* = \frac{1}{V_0} \sqrt{\frac{L C}{C}} l_2 \]

\[ M^* = \frac{M}{L_C} \]

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\[ I_2^* = \frac{1}{V_0} \sqrt{\frac{L_C}{C}} l_2 \]

\[ M^* = \frac{M}{L_C} \]

\[ z^* = \frac{z}{z_0} \]

\[ r^* = \frac{r}{r_{coil}} \]

\[ P^* = \frac{P}{P_1} \]

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Resulting Non-dimensional Equation Set

\[
\frac{d l_1^*}{d t^*} = \left[ L^* V^* + (M^* l_1^* + l_2^*) \frac{d M^*}{d t^*} \right] / \left( L^* + 1 - M^* \right) \\
- \left[ \psi_1 L^* l_1^* - \psi_2 L^* l_2^* M^* \right] / \left( L^* + 1 - M^* \right) \\
\frac{d l_2^*}{d t^*} = M^* \frac{d l_1^*}{d t^*} + l_1^* \frac{d M^*}{d t^*} - l_2^* L^* \psi_2 \\
\frac{d V^*}{d t^*} = -l_1^* \\
\frac{dr^*}{d t^*} = v_r^* \\
\frac{dz^*}{d t^*} = v_z^* 
\]

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\[
\frac{dl_1^*}{dt^*} = \left[ L^* V^* + (M^* l_1^* + l_2^*) \frac{dM^*}{dt^*} \right]/\left( L^* + 1 - M^*^2 \right) - \left[ \psi_1 L^* I_1^* - \psi_2 L^* I_2^* M^* \right]/\left( L^* + 1 - M^*^2 \right)
\]

\[
\frac{dl_2^*}{dt^*} = M^* \frac{dl_1^*}{dt^*} + I_1^* \frac{dM^*}{dt^*} - I_2^* L^* \psi_2
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\[
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\[
\frac{dz^*}{dt^*} = v_z^*
\]
Resulting Non-dimensional Equation Set

\[
\frac{dl_1^*}{dt^*} = \left[ L^* V^* + (M^* l_1^* + l_2^*) \frac{dM^*}{dt^*} \right] \div \left( L^* + 1 - M^{*2} \right) \\
- \left[ \psi_1 L^* l_1^* - \psi_2 L^* l_2^* M^* \right] \div \left( L^* + 1 - M^{*2} \right)
\]

\[
\frac{dl_2^*}{dt^*} = M^* \frac{dl_1^*}{dt^*} + l_1^* \frac{dM^*}{dt^*} - l_2^* L^* \psi_2
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- \left[ \psi_1 L^* l_1^* - \psi_2 L^* l_2^* M^* \right] / \left( L^* + 1 - M^* \right)^2
\]

\[
\frac{dl_2^*}{dt^*} = M^* \frac{dl_1^*}{dt^*} + l_1^* \frac{dM^*}{dt^*} - l_2^* L^* \psi_2
\]

\[
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Resulting Non-dimensional Equation Set

\[
\frac{dl_1^*}{dt^*} = \left[ L^* V^* + (M^* l_1^* + l_2^*) \frac{dM^*}{dt^*} \right] / \left( L^* + 1 - M^*^2 \right) \\
- \left[ \psi_1 L^* l_1^* - \psi_2 L^* l_2^* M^* \right] / \left( L^* + 1 - M^*^2 \right)
\]

\[
\frac{dl_2^*}{dt^*} = M^* \frac{dl_1^*}{dt^*} + l_1^* \frac{dM^*}{dt^*} - l_2^* L^* \psi_2
\]

\[
\frac{dV^*}{dt^*} = -l_1^*
\]

\[
\frac{dr^*}{dt^*} = v_r^*
\]

\[
\frac{dz^*}{dt^*} = v_z^*
\]
Resulting Non-dimensional Equation Set

\[
\frac{dl_1^*}{dt^*} = \left[ L^* V^* + (M^* l_1^* + l_2^*) \frac{dM^*}{dt^*} \right] \div \left( L^* + 1 - M^*^2 \right) \\
- \left[ \psi_1 L^* l_1^* - \psi_2 L^* l_2^* M^* \right] \div \left( L^* + 1 - M^*^2 \right)
\]

\[
\frac{dl_2^*}{dt^*} = M^* \frac{dl_1^*}{dt^*} + l_1^* \frac{dM^*}{dt^*} - l_2^* L^* \psi_2
\]

\[
\frac{dV^*}{dt^*} = - l_1^*
\]

\[
\frac{dr^*}{dt^*} = v_r^*
\]

\[
\frac{dz^*}{dt^*} = v_z^*
\]

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Resulting Non-dimensional Equation Set

\[
\frac{dM^*}{dt^*} = \frac{N}{2} r^* \frac{N}{2} - 1 v_r^* \exp\left(-\frac{z^*}{2}\right) - \frac{1}{2} r^* \frac{N}{2} v_z^* \exp\left(-\frac{z^*}{2}\right)
\]

\[
\frac{dv_r^*}{dt^*} = \lambda P^* r^* - \phi l_1^* r^* N - 1 \exp\left(-z^*\right)
\]

\[
\frac{dv_z^*}{dt^*} = \alpha l_1^* r^* N \exp\left(-z^*\right)
\]

\[
\frac{dP^*}{dt^*} = \Xi v_r^* \frac{dv_r^*}{dt^*}
\]
Resulting Non-dimensional Equation Set

\[
\frac{dM^*}{dt^*} = \frac{N}{2} r_*^{N/2 - 1} v_r^* \exp\left(-\frac{z^*}{2}\right) - \frac{1}{2} r_*^{N/2} v_z^* \exp\left(-\frac{z^*}{2}\right)
\]

\[
\frac{dv_r^*}{dt^*} = \lambda P^* r_* - \phi l_1^* v_r^* r_*^{N-1} \exp(-z^*)
\]

\[
\frac{dv_z^*}{dt^*} = \alpha l_1^* v_z^* r_*^N \exp(-z^*)
\]

\[
\frac{dP^*}{dt^*} = \Xi v_r^* \frac{dv_r^*}{dt^*}
\]
Resulting Non-dimensional Equation Set

\[
\frac{dM^*}{dt^*} = \frac{N}{2} r_*^{N - 1} \nu_r^* \exp\left(-\frac{z_*^*}{2}\right) - \frac{1}{2} r_*^{N - 1} \nu_z^* \exp\left(-\frac{z_*^*}{2}\right)
\]

\[
\frac{dv_r^*}{dt^*} = \lambda P^* r_*^2 - \phi l_1^* r_*^{N - 1} \exp(-z_*^*)
\]

\[
\frac{dv_z^*}{dt^*} = \alpha l_1^* r_*^N \exp(-z_*^*)
\]

\[
\frac{dP^*}{dt^*} = \Xi v_r^* \frac{dv_r^*}{dt^*}
\]
Resulting Non-dimensional Equation Set

\[
\frac{dM^*}{dt^*} = \frac{N}{2} r_2^{N-1} v_r^* \exp\left(-\frac{z^*}{2}\right) - \frac{1}{2} r_2^{N} v_z^* \exp\left(-\frac{z^*}{2}\right)
\]

\[
\frac{dv_r^*}{dt^*} = \lambda P^* r^* - \phi l_1^2 r_1^{N-1} \exp\left(-z^*\right)
\]

\[
\frac{dv_z^*}{dt^*} = \alpha l_1^2 r_1^{N} \exp\left(-z^*\right)
\]

\[
\frac{dP^*}{dt^*} = \Xi v_r^* \frac{dv_r^*}{dt^*}
\]

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Resulting Non-dimensional Equation Set

\[
\frac{dM^*}{dt^*} = \frac{N}{2} r^*_2 N - 1 v_r^* \exp\left(-\frac{z^*}{2}\right) - \frac{1}{2} r^*_2 v_z^* \exp\left(-\frac{z^*}{2}\right)
\]

\[
\frac{dv_r^*}{dt^*} = \lambda P^* r^* - \phi l_1^* r^* \exp(-z^*)
\]

\[
\frac{dv_z^*}{dt^*} = \alpha l_1^* r^N \exp(-z^*)
\]

\[
\frac{dP^*}{dt^*} = \Xi v_r^* \frac{dv_r^*}{dt^*}
\]
Non-dimensional Parameters

\[ \alpha = \frac{V_0^2 C^2 L_C}{2 m_{bit} z_0^2} \]

\[ \phi = \frac{V_0^2 C^2 L_C}{2 m_{bit} r_{coil}^2} \]

\[ \lambda = \frac{L_0 CP_1 2\pi l_{coil}}{2 m_{bit}} \]

\[ \psi_1 = R_e \sqrt{\frac{C}{L_0}} \]

\[ \psi_2 = R_p \sqrt{\frac{C}{L_0}} \]

\[ \Xi = \frac{4 \gamma}{\gamma + 1} \frac{m_i}{\gamma k T_1 r_{coil}^2 L_0 C} \]

\[ L^* = \frac{L_0}{L_C} \]

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New Non-dimensional Parameters

\[ \alpha = \frac{V_0^2 C^2 L_C}{2 m_{bit} z_0^2} \]
\[ \phi = \frac{V_0^2 C^2 L_C}{2 m_{bit} r_{coil}^2} \]
\[ \lambda = \frac{L_0 C P_1 2 \pi l_{coil}}{2 m_{bit}} \]
\[ L^* = \frac{L_0}{L_C} \]
\[ \psi_1 = R_e \sqrt{\frac{C}{L_0}} \]
\[ \psi_2 = R_p \sqrt{\frac{C}{L_0}} \]
\[ \Xi = \frac{4 \gamma m_i}{\gamma + 1} \frac{1}{\gamma k T_1 r_{coil}^2 L_0 C} \]
New Non-dimensional Parameters

\[ \alpha = \frac{V_0^2 C^2 L_C}{2m_{bit} z_0^2} \]

\[ \phi = \frac{V_0^2 C^2 L_C}{2m_{bit} r_{coil}^2} \]

\[ \lambda = \frac{L_0 CP_1 2\pi l_{coil}}{2m_{bit}} \]

\[ \frac{L^*}{L_C} = \frac{L_0}{L_C} \]

\[ \psi_1 = R_e \sqrt{\frac{C}{L_0}} \]

\[ \psi_2 = R_p \sqrt{\frac{C}{L_0}} \]

\[ \Xi = \frac{4\gamma}{\gamma + 1} \frac{m_i}{\gamma k T_1} \frac{1}{r_{coil}^2 L_0 C} \]

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New Non-dimensional Parameters

\[ \alpha = \frac{V_0^2 C^2 L_C}{2 m_{bit} z_0^2} \]
\[ \phi = \frac{V_0^2 C^2 L_C}{2 m_{bit} r_{coil}^2} \]
\[ \lambda = \frac{L_0 C P_1 2 \pi l_{coil}}{2 m_{bit}} \]
\[ \Xi = \frac{4 \gamma}{\gamma + 1} \frac{m_i}{\gamma k T_1} \frac{1}{r_{coil}^2 L_0 C} \]

\[ \psi_1 = R_e \sqrt{\frac{C}{L_0}} \]
\[ \psi_2 = R_p \sqrt{\frac{C}{L_0}} \]

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Physical Meaning of Scaling Parameters

\[ \alpha = \frac{C^2 V_0^2 L_C}{2 m_{bit} z_0^2} = \frac{1}{8 \pi^2} \frac{CV_0^2}{2 m_{bit} v_z^2 / 2} L^* \left( \frac{2 \pi \sqrt{L_0 C}}{L_0 / L_z} \right)^2 \]

\[ \phi = \frac{C^2 V_0^2 L_C}{2 m_{bit} r_{coil}^2} = \frac{1}{8 \pi^2} \frac{CV_0^2}{2 m_{bit} v_r^2 / 2} L^* \left( \frac{2 \pi \sqrt{L_0 C}}{L_0 / L_r} \right)^2 \]
Physical Meaning of Scaling Parameters

\[ \alpha = \frac{C^2 V_0^2 L_C}{2m_{bit} z_0^2} = \frac{1}{8\pi^2} \frac{CV_0^2/2}{m_{bit} v_z^2/2} L^* \left( \frac{2\pi \sqrt{L_0 C}}{L_0/L_z} \right)^2 \]

\[ \phi = \frac{C^2 V_0^2 L_C}{2m_{bit} r_{\text{coil}}^2} = \frac{1}{8\pi^2} \frac{CV_0^2/2}{m_{bit} v_r^2/2} L^* \left( \frac{2\pi \sqrt{L_0 C}}{L_0/L_r} \right)^2 \]

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Physical Meaning of Scaling Parameters

\[ \alpha = \frac{C^2 V_0^2 L_C}{2 m_{bit} z_0^2} = \frac{1}{8 \pi^2} \frac{CV_0^2/2}{m_{bit} v_z^2/2} L^* \left( \frac{2\pi \sqrt{L_0 C}}{L_0/L_z} \right)^2 \]

Axial Decoupling Timescale

\[ \phi = \frac{C^2 V_0^2 L_C}{2 m_{bit} r_{coil}^2} = \frac{1}{8 \pi^2} \frac{CV_0^2/2}{m_{bit} v_r^2/2} L^* \left( \frac{2\pi \sqrt{L_0 C}}{L_0/L_r} \right)^2 \]
Physical Meaning of Scaling Parameters

\[ \alpha = \frac{C^2 V_0^2 L_C}{2m_{bit} z_0^2} = \frac{1}{8\pi^2} \frac{CV_0^2}{2m_{bit} v_z^2} L^* \left( \frac{2\pi \sqrt{L_0 C}}{L_0 / \dot{L}_z} \right)^2 \]

\[ \phi = \frac{C^2 V_0^2 L_C}{2m_{bit} r_{coil}^2} = \frac{1}{8\pi^2} \frac{CV_0^2}{2m_{bit} v_r^2} L^* \left( \frac{2\pi \sqrt{L_0 C}}{L_0 / \dot{L}_r} \right)^2 \]

Radial Decoupling Timescale

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Radial Motion Shifts Peak in Thrust Efficiency

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Thrust Efficiency Maximum at Lower Values of Phi

\[ \eta_t \]

\[ \phi \]

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Combined Effects of $\alpha$ and $\phi$ on $\eta_t$
Conclusions
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Conclusions

- Radial current sheet motion causes slower axial current sheet acceleration
- This leads to dynamic impedance matching at longer characteristic circuit times
- Thrust efficiency is maximized when the axial decoupling timescale is shorter than the radial decoupling timescale

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Questions?