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Abstract

A detailed analysis and examples are presented that show how to enrich the kinematics of classical Kirchhoff plate theory by appending them with a set of continuous piecewise-cubic functions. This analysis is used to obtain functions that contain the effects of laminate heterogeneity and asymmetry on the variations of the inplane displacements and transverse shearing stresses, for use with a $\{3, 0\}$ plate theory in which these distributions are specified a priori. The functions used for the enrichment are based on the improved zig-zag plate theory presented recently by Tessler, Di Scuva, and Gherlone. With the approach presented herein, the inplane displacements are represented by a set of continuous piecewise-cubic functions, and the transverse shearing stresses and strains are represented by a set of piecewise-quadratic functions that are discontinuous at the ply interfaces.

Introduction

Many variants of refined theories for laminated-composite and sandwich plates have appeared in the technical literature for many years.¹⁻¹⁰ Each of these theories generally have a different degree of complexity and a corresponding range of validity. Plates that exhibit a relatively large amount of transverse shearing flexibility typically require a refined theory in order to obtain accurate predictions of their structural response, particularly when the characteristic dimension of the response is on the order of the plate thickness. In contrast, for plate-response phenomena that consist of deformations with relatively large characteristic dimensions, simpler theories that are less complex and computationally expensive can be utilized to obtain useful results.

The most basic plate theory that accounts for transverse shearing deformations is known as the first-order shear-deformation theory (FSDT). With regards to its origins, it is also commonly referred to as the Reissner-Mindlin plate theory.¹¹⁻¹³ In this theory, the deformation of a plate is expressed in terms of three translational displacements and two rotations associated with through-the-thickness shearing, in contrast to the classical plate theory (CLPT) conceived by Kirchhoff that uses only the three translational displacements. A drawback associated with the relative simplicity of FSDT is that the through-the-thickness distributions of the transverse shearing strains are approximated as uniform distributions. These approximate strain distributions correspond to transverse shearing stresses that violate the traction boundary conditions on the top and bottom bounding surfaces of a plate. As a result, a shear-correction factor must be used to obtain adequate results.

One step up from FSDT, is a group of relatively popular plate theories that satisfy the traction boundary conditions on the top and bottom bounding surfaces of a plate and avoid the need for a shear-correction factor.¹⁴⁻⁴¹ This boundary-condition problem is mitigated in these theories, at the expense of more complexity, by appending the kinematics of CLPT with terms that account for transverse shearing by using two additional kinematic variables and presumed distributions of through-the thickness shearing strains. Typically, these distributions are presumed to be symmetric over the plate thickness, even for laminated plates with asymmetric ply distributions.

Although, variations in actual shape of the transverse-shearing strain distributions for moderately thick laminated and sandwich plates may have a relatively small effect on the prediction of global response phenomena like buckling loads and fundamental frequencies, the mismatch between the distribution symmetry and ply lay-up symmetry is a fundamental issue that should be addressed. This issue is addressed in the present paper by enriching the CLPT kinematics with zig-zag variations to obtained transverse shearing strain distributions that are based on the actual through-the-thickness construction of a plate and without introducing any additional kinematic variables. To accomplish this task, the baseline plate kinematics are presented first. Then, the details of the procedure used to enrich the kinematics are given. Lastly, examples are presented for an unsymmetric laminate and a sandwich plate with an isotropic core and two identical laminated face plates.

The Baseline Plate Kinematics

Consider a uniform-thickness plate and the coordinate system shown in figure 1. Points of the plate are located by the orthogonal Cartesian coordinates (x, y, z) , where $(x, y, 0)$, $0 \leq x \leq a$, and $0 \leq y \leq b$ define points of the rectangular midplane. The plate lengths in the x - and y -directions are denoted by a and b , respectively. Points above and below the midplane are given by the nonzero z -coordinate values within $-h/2 \leq z \leq h/2$, where h is the plate thickness dimension. The principal material coordinate system at the point (x, y, z) is shown in figure 2. For laminated plates, the fiber orientation is denoted by the angle $\theta(z)$ shown in figure 2.

The baseline plate kinematics used in the present study are defined by

$$U(x, y, z) = u(x, y) - z \frac{\partial w(x, y)}{\partial x} + \Lambda_x(z) H_{xz}(x, y) \quad (1a)$$

$$V(x, y, z) = v(x, y) - z \frac{\partial w(x, y)}{\partial y} + \Lambda_y(z) H_{yz}(x, y) \quad (1b)$$

$$W(x, y, z) = w(x, y) \quad (1c)$$

where $U(x, y, z)$, $V(x, y, z)$, and $W(x, y, z)$ are the displacement-field components in the x -, y -, and z -coordinate directions, respectively, of the material point (x, y, z) . The functions $u(x, y)$ and $v(x, y)$ are the corresponding inplane displacements of the material point $(x, y, 0)$ of the midplane, and $w(x, y)$ is the out-of-plane displacements of the material point $(x, y, 0)$. The functions $\Lambda_x(z)$ and $\Lambda_y(z)$ are selected to satisfy the traction-free boundary conditions on the transverse shear stresses at the bounding surfaces of the plate given by the coordinates $(x, y, \pm \frac{h}{2})$. In addition, $\Lambda_x(z)$ and $\Lambda_y(z)$ are required to satisfy the conditions $U(x, y, 0) = u(x, y)$ and $V(x, y, 0) = v(x, y)$, but are otherwise arbitrary. The unknown functions $H_{xz}(x, y)$ and $H_{yz}(x, y)$ account for the effects of transverse shear deformation. Inspection of equations (1) reveals that CPT is contained as a proper, well-defined subset that is obtained by setting $\Lambda_x(z)$ and $\Lambda_y(z)$ equal to zero. Several choices for $\Lambda_x(z)$ and $\Lambda_y(z)$ have appeared in the technical literature; for example, see reference 32. When $\Lambda_x(z)$ and $\Lambda_y(z)$ are specified as cubic polynomials in z , the plate theory is sometimes

referred to as a $\{3, 0\}$ theory, which indicates that the inplane displacements $U(x, y, z)$ and $V(x, y, z)$ are cubic and the out-of-plane displacement $W(x, y, z)$ is constant across the plate thickness.

The linear strain-displacement relations used in the present study are obtained by substituting equations (1) into linear strain-displacement relations of the linear theory of elasticity, given by

$$\varepsilon_{xx}(x, y, z) = \frac{\partial U}{\partial x} \quad (2a)$$

$$\varepsilon_{yy}(x, y, z) = \frac{\partial V}{\partial y} \quad (2b)$$

$$\varepsilon_{zz}(x, y, z) = \frac{\partial W}{\partial z} \quad (2c)$$

$$\gamma_{xy}(x, y, z) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \quad (2d)$$

$$\gamma_{xz}(x, y, z) = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \quad (2e)$$

$$\gamma_{yz}(x, y, z) = \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \quad (2f)$$

First, consider the transverse shearing strains defined by equations (2e) and (2f). Substituting the presumed displacement field, given by equations (1), into these strain-displacement expressions gives

$$\begin{Bmatrix} \gamma_{xz}(x, y, z) \\ \gamma_{yz}(x, y, z) \end{Bmatrix} = \begin{Bmatrix} \Lambda_x'(z) H_{xz}(x, y) \\ \Lambda_y'(z) H_{yz}(x, y) \end{Bmatrix} \quad (3)$$

At the plate midplane $z = 0$, the transverse shearing strains are given by

$$\begin{Bmatrix} \gamma_{xz}^\circ(x, y) \\ \gamma_{yz}^\circ(x, y) \end{Bmatrix} \equiv \begin{Bmatrix} \gamma_{xz}(x, y, 0) \\ \gamma_{yz}(x, y, 0) \end{Bmatrix} = \begin{Bmatrix} \Lambda_x'(0) H_{xz}(x, y) \\ \Lambda_y'(0) H_{yz}(x, y) \end{Bmatrix} \quad (4)$$

Using equation (4), the inplane displacement field is expressed as

$$U(x, y, z) = u(x, y) - z \frac{\partial w(x, y)}{\partial x} + F_x(z) \gamma_{xz}^\circ(x, y) \quad (5a)$$

$$V(x, y, z) = v(x, y) - z \frac{\partial w(x, y)}{\partial y} + F_y(z) \gamma_{yz}^\circ(x, y) \quad (5b)$$

where

$$F_x(z) = \frac{\Lambda_x(z)}{\Lambda_x'(0)} \quad (5c)$$

$$F_y(z) = \frac{\Lambda_y(z)}{\Lambda_y'(0)} \quad (5d)$$

In these equations, $\Lambda_x'(0) \neq 0$ and $\Lambda_y'(0) \neq 0$ are required, and $F_x(0) = F_y(0) = 0$ are required to obtain $U(x, y, 0) = u(x, y)$ and $V(x, y, 0) = v(x, y)$. Substituting equations (1c), (5a), and (5b) into the remaining strain-displacement expressions yields the following expressions for the nonzero strains

$$\begin{pmatrix} \varepsilon_{xx}(x, y, z) \\ \varepsilon_{yy}(x, y, z) \\ \gamma_{xy}(x, y, z) \end{pmatrix} = \begin{pmatrix} \frac{\partial u(x, y)}{\partial x} \\ \frac{\partial v(x, y)}{\partial y} \\ \frac{\partial v(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} \end{pmatrix} - z \begin{pmatrix} \frac{\partial^2 w(x, y)}{\partial x^2} \\ \frac{\partial^2 w(x, y)}{\partial y^2} \\ 2 \frac{\partial^2 w(x, y)}{\partial x \partial y} \end{pmatrix} + \begin{pmatrix} F_x(z) \frac{\partial \gamma_{xz}^\circ(x, y)}{\partial x} \\ F_y(z) \frac{\partial \gamma_{yz}^\circ(x, y)}{\partial y} \\ F_y(z) \frac{\partial \gamma_{yz}^\circ(x, y)}{\partial x} + F_x(z) \frac{\partial \gamma_{xz}^\circ(x, y)}{\partial y} \end{pmatrix} \quad (6)$$

and

$$\begin{pmatrix} \gamma_{xz}(x, y, z) \\ \gamma_{yz}(x, y, z) \end{pmatrix} = \begin{pmatrix} F_x'(z) \gamma_{xz}^\circ(x, y) \\ F_y'(z) \gamma_{yz}^\circ(x, y) \end{pmatrix} \quad (7)$$

Next, to get the complete picture of the plate deformation, as shown in figures 3-7, consider the components of the linear rotation vector given by

$$\omega_x(x, y, z) = \frac{1}{2} \left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} \right) \quad (8a)$$

$$\omega_y(x, y, z) = \frac{1}{2} \left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} \right) \quad (8b)$$

$$\omega_z(x, y, z) = \frac{1}{2} \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) \quad (8c)$$

Substituting equations (1c), (5a), and (5b) into these expressions yields

$$\omega_x(x, y, z) = \frac{\partial w(x, y)}{\partial y} - \frac{1}{2} F_y'(z) \gamma_{yz}^\circ(x, y) \quad (9a)$$

$$\omega_y(x, y, z) = \frac{1}{2} F_x'(z) \gamma_{xz}^\circ(x, y) - \frac{\partial w(x, y)}{\partial x} \quad (9b)$$

$$\omega_z(x, y, z) = \frac{1}{2} \left(\frac{\partial v(x, y)}{\partial x} - \frac{\partial u(x, y)}{\partial y} + F_y(z) \frac{\partial \gamma_{yz}^\circ(x, y)}{\partial x} - F_x(z) \frac{\partial \gamma_{xz}^\circ(x, y)}{\partial y} \right) \quad (9c)$$

At the plate midplane,

$$\omega_x(x, y, 0) \equiv \omega_x^\circ(x, y) = \frac{\partial w(x, y)}{\partial y} - \frac{1}{2}\gamma_{yz}^\circ(x, y) \quad (10a)$$

$$\omega_y(x, y, 0) \equiv \omega_y^\circ(x, y) = \frac{1}{2}\gamma_{xz}^\circ(x, y) - \frac{\partial w(x, y)}{\partial x} \quad (10b)$$

$$\omega_z(x, y, 0) \equiv \omega_z^\circ(x, y) = \frac{1}{2}\left(\frac{\partial v(x, y)}{\partial x} - \frac{\partial u(x, y)}{\partial y}\right) \quad (10c)$$

where $F_x(0) = F_y(0) = 0$ have been used and, as shown by equations (5c) and (5d), it is noted that $F_x'(0) = F_y'(0) = 1$.

The constitutive equations used in the present study are those for a plate made of one or more layers of linear elastic, specially orthotropic materials that are in a state of plane stress. These equations, referred to the plate (x, y, z) coordinate system are given by

$$\begin{Bmatrix} \sigma_{xx}(x, y, z) \\ \sigma_{yy}(x, y, z) \\ \sigma_{xy}(x, y, z) \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11}^{(k)} & \bar{Q}_{12}^{(k)} & \bar{Q}_{16}^{(k)} \\ \bar{Q}_{12}^{(k)} & \bar{Q}_{22}^{(k)} & \bar{Q}_{26}^{(k)} \\ \bar{Q}_{16}^{(k)} & \bar{Q}_{26}^{(k)} & \bar{Q}_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}(x, y, z) \\ \varepsilon_{yy}(x, y, z) \\ \gamma_{xy}(x, y, z) \end{Bmatrix} \quad (11a)$$

$$\begin{Bmatrix} \sigma_{yz}(x, y, z) \\ \sigma_{xz}(x, y, z) \end{Bmatrix} = \begin{bmatrix} \bar{C}_{44}^{(k)} & \bar{C}_{45}^{(k)} \\ \bar{C}_{45}^{(k)} & \bar{C}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}(x, y, z) \\ \gamma_{xz}(x, y, z) \end{Bmatrix} \quad (11b)$$

where the superscript (k) denotes the k^{th} ply of a laminated plate with a total of N plies. The $\bar{Q}_{ij}^{(k)}$ terms are the transformed, reduced (plane stress) stiffnesses of classical laminated-plate theory, for the k^{th} ply, and the $\bar{C}_{ij}^{(k)}$ terms are the corresponding stiffnesses of a generally orthotropic ply.⁴² For a heterogeneous, laminated plate with N plies, both the $\bar{Q}_{ij}^{(k)}$ and the $\bar{C}_{ij}^{(k)}$ terms are functions of the through-the-thickness coordinate z , as indicated by the superscript (k) . In particular, the fiber orientation of the k^{th} ply within a laminated plate is given by the angle $\theta(z)$, shown in figure 2, where the values of z correspond to points within the thickness of the k^{th} ply. From equation (11b), it follows that the transverse shearing stresses on the top and bottom plate surfaces vanish provided that the corresponding transverse shearing strains vanish. Thus, from equation (7), it also follows that

$$F_x'\left(\pm \frac{h}{2}\right) = F_y'\left(\pm \frac{h}{2}\right) = 0 \quad (12)$$

is required for the transverse shearing stresses on the top and bottom plate surfaces to vanish.

Equations (1c), (5a), and (5b); the conditions defined by equation (12); and $F_x(0) = F_y(0) = 0$ define completely the kinematics of the baseline theory used herein. The kinematics of CLPT are recovered from these equations by specifying $F_x(z) = F_y(z) = 0$. Likewise, the kinematics of FSDT are recovered from these equations by specifying $F_x(z) = F_y(z) = z$ such that equations (5a) and (5b) become

$$U(x, y, z) = u(x, y) + z \left[\gamma_{xz}^o(x, y) - \frac{\partial w(x, y)}{\partial x} \right] \quad (13a)$$

$$V(x, y, z) = v(x, y) + z \left[\gamma_{yz}^o(x, y) - \frac{\partial w(x, y)}{\partial y} \right] \quad (13b)$$

The corresponding FSDT kinematics, in the x-z plane, are illustrated in figures 3 and 4. In contrast, the kinematics in the x-z plane of the baseline theory presented herein are illustrated in figures 5-7.

Zig-Zag Enrichment of the Kinematics

In the baseline kinematics, the functions $F_x(z)$ and $F_y(z)$ are typically specified as continuous functions, with continuous derivative, such as those given in reference 32. Moreover, for the most part, the functions are specified as antisymmetric functions; that is, $F_x(-z) = -F_x(z)$ and $F_y(-z) = -F_y(z)$. This antisymmetry with respect to the plate midplane appears to be inconsistent with laminates that exhibit a high degree of asymmetry in their ply constitutive properties. To obtain expressions for $F_x(z)$ and $F_y(z)$ that are based on actual laminate construction, particularly for asymmetric laminates, these functions are partitioned herein as follows

$$F_x(z) = H_x(z) + \Phi_x^{(k)}(z) \quad (14a)$$

$$F_y(z) = H_y(z) + \Phi_y^{(k)}(z) \quad (14b)$$

where $H_x(z)$ and $H_y(z)$ are continuous functions, with continuous derivatives, such as those given in reference 32 which include linear, cubic, trigonometric, hyperbolic, and exponential functions.

The terms $\Phi_x^{(k)}(z)$ and $\Phi_y^{(k)}(z)$ are piecewise linear functions with piecewise constant derivatives that are referred to herein as zig-zag functions. In these equations, the superscript (k) also denotes the ply number of a laminate composed of N plies. Moreover, for the k^{th} ply, it follows that $z_{(k-1)} \leq z \leq z_{(k)}$, where $z_{(k-1)}$ and $z_{(k)}$ are the ply coordinates shown in figure 8 for the x-z plane. Substituting equations (14) into equations (5a) and (5b) yields the inplane displacements

$$U(x, y, z) = u(x, y) - z \frac{\partial w(x, y)}{\partial x} + \left[H_x(z) + \Phi_x^{(k)}(z) \right] \gamma_{xz}^o(x, y) \quad (15a)$$

$$V(x, y, z) = v(x, y) - z \frac{\partial w(x, y)}{\partial y} + [H_x(z) + \Phi_x^{(k)}(z)] \gamma_{yz}^o(x, y) \quad (15b)$$

The user-specified functions $H_x(z)$ and $H_y(z)$ define a smooth distribution of displacements associated with transverse shearing deformations and $\Phi_x^{(k)}(z)$ and $\Phi_y^{(k)}(z)$ are corrections associated with laminate heterogeneity and asymmetry.

Following the refined zig-zag theory of Tessler et. al.,⁴³ ply interface displacements $\Delta_x^{(k)}$ (see figure 8) and $\Delta_y^{(k)}$ are defined in terms of the zig-zag functions by

$$\Delta_x^{(k)} = \Phi_x^{(k)}(z_{(k)}) \quad (16a)$$

$$\Delta_y^{(k)} = \Phi_y^{(k)}(z_{(k)}) \quad (16b)$$

for $k = 0, 1, \dots, N$; with $\Delta_x^{(0)} = \Delta_y^{(0)} = \Delta_x^{(N)} = \Delta_y^{(N)} = 0$. Within the k^{th} ply, $z_{(k-1)} \leq z \leq z_{(k)}$ and linear interpolation gives

$$\Phi_x^{(k)}(z) = \Delta_x^{(k-1)} + \beta_x^{(k)}(z - z_{(k-1)}) \quad (17a)$$

$$\Phi_y^{(k)}(z) = \Delta_y^{(k-1)} + \beta_y^{(k)}(z - z_{(k-1)}) \quad (17b)$$

where

$$\beta_x^{(k)} = \frac{\Delta_x^{(k)} - \Delta_x^{(k-1)}}{h^{(k)}} \quad (18a)$$

$$\beta_y^{(k)} = \frac{\Delta_y^{(k)} - \Delta_y^{(k-1)}}{h^{(k)}} \quad (18b)$$

and where $h_{(k)} = z_{(k)} - z_{(k-1)}$ is thickness of the k^{th} ply, as shown in figure 8. For $z_{(k-1)} \leq z \leq z_{(k)}$, it follows that the piecewise-constant derivatives of the zig-zag functions are given by

$$\Phi_x^{(k)'}(z) = \beta_x^{(k)} \quad (19a)$$

$$\Phi_y^{(k)'}(z) = \beta_y^{(k)} \quad (19b)$$

Thus, differentiating equations (14) gives

$$F_x'(z) = H_x'(z) + \beta_x^{(k)} \quad (20a)$$

$$F_y'(z) = H_y'(z) + \beta_y^{(k)} \quad (20b)$$

Substituting these results into equation (7) yields

$$\begin{Bmatrix} \gamma_{xz}(x, y, z) \\ \gamma_{yz}(x, y, z) \end{Bmatrix} = \begin{Bmatrix} [H_x'(z) + \beta_x^{(k)}] \gamma_{xz}^o(x, y) \\ [H_y'(z) + \beta_y^{(k)}] \gamma_{yz}^o(x, y) \end{Bmatrix} \quad (21)$$

for the transverse shearing strains.

At this point in the analysis, it is convenient to use equation (21) to express constitutive equations (11b) as

$$\begin{Bmatrix} \sigma_{yz}(x, y, z) \\ \sigma_{xz}(x, y, z) \end{Bmatrix} = \begin{bmatrix} \bar{C}_{44}^{(k)} & \bar{C}_{45}^{(k)} \\ \bar{C}_{45}^{(k)} & \bar{C}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} [H_y'(z) + \beta_y^{(k)}] \gamma_{yz}^o(x, y) \\ [H_x'(z) + \beta_x^{(k)}] \gamma_{xz}^o(x, y) \end{Bmatrix} \quad (22)$$

for laminated plates, where $\bar{C}_{44}^{(k)}$, $\bar{C}_{45}^{(k)}$, and $\bar{C}_{55}^{(k)}$ are constant-valued for material points of the k^{th} ply located by $z_{(k-1)} \leq z \leq z_{(k)}$. To proceed following the refined theory of Tessler et. al.,⁴³ the transverse shearing stresses are partitioned into

$$\begin{Bmatrix} \sigma_{yz}(x, y, z) \\ \sigma_{xz}(x, y, z) \end{Bmatrix} = \begin{bmatrix} \bar{C}_{44}^{(k)} & \bar{C}_{45}^{(k)} \\ \bar{C}_{45}^{(k)} & \bar{C}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} [H_y'(z) - 1] \gamma_{yz}^o(x, y) \\ [H_x'(z) - 1] \gamma_{xz}^o(x, y) \end{Bmatrix} + \begin{Bmatrix} \sigma_{yz}^{zz} \\ \sigma_{xz}^{zz} \end{Bmatrix} \quad (23a)$$

where

$$\begin{Bmatrix} \sigma_{yz}^{zz} \\ \sigma_{xz}^{zz} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{44}^{(k)} & \bar{C}_{45}^{(k)} \\ \bar{C}_{45}^{(k)} & \bar{C}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} [1 + \beta_y^{(k)}] \gamma_{yz}^o(x, y) \\ [1 + \beta_x^{(k)}] \gamma_{xz}^o(x, y) \end{Bmatrix} \quad (23b)$$

Next, the ply parameters

$$G_x^{(k)} = \bar{C}_{55}^{(k)} [1 + \beta_x^{(k)}] \quad (24a)$$

$$G_y^{(k)} = \bar{C}_{44}^{(k)} [1 + \beta_y^{(k)}] \quad (24b)$$

are introduced to eliminate $\beta_x^{(k)}$ and $\beta_y^{(k)}$ from equation (23b). Because $\beta_x^{(k)}$ and $\beta_y^{(k)}$ appear as independent unknowns in the analysis, they can be normalized by requiring $G_x^{(k)} = G_x$ and $G_y^{(k)} = G_y$ for $k = 1, 2, \dots, N$ such that

$$\beta_x^{(k)} = \frac{G_x}{\bar{C}_{55}^{(k)}} - 1 \quad (25a)$$

$$\beta_y^{(k)} = \frac{G_y}{\bar{C}_{44}^{(k)}} - 1 \quad (25b)$$

and

$$\sigma_{xz}^{zz} = G_x \gamma_{xz}^o(x, y) + G_y \frac{\bar{C}_{45}^{(k)}}{\bar{C}_{44}^{(k)}} \gamma_{yz}^o(x, y) \quad (26a)$$

$$\sigma_{yz}^{zz} = G_x \frac{\bar{C}_{45}^{(k)}}{\bar{C}_{55}^{(k)}} \gamma_{xz}^o(x, y) + G_y \gamma_{yz}^o(x, y) \quad (26b)$$

Following Tessler et. al.,⁴³ the constants G_x and G_y are found by first noting that

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \beta_x^{(k)} \\ \beta_y^{(k)} \end{Bmatrix} dz = \sum_{k=1}^N \begin{Bmatrix} \beta_x^{(k)} \\ \beta_y^{(k)} \end{Bmatrix} \int_{z_{(k-1)}}^{z_{(k)}} dz = \sum_{k=1}^N \begin{Bmatrix} \beta_x^{(k)} \\ \beta_y^{(k)} \end{Bmatrix} h_{(k)} \quad (27)$$

Then, using equations (18) gives

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \beta_x^{(k)} \\ \beta_y^{(k)} \end{Bmatrix} dz = \begin{Bmatrix} \Delta_x^{(N)} \\ \Delta_y^{(N)} \end{Bmatrix} - \begin{Bmatrix} \Delta_x^{(0)} \\ \Delta_y^{(0)} \end{Bmatrix} \quad (28)$$

and using $\Delta_x^{(0)} = \Delta_y^{(0)} = \Delta_x^{(N)} = \Delta_y^{(N)} = 0$ gives the desired result

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \beta_x^{(k)} \\ \beta_y^{(k)} \end{Bmatrix} dz = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (29)$$

Therefore, the constants G_x and G_y are found by integrating equations (25) to get

$$G_x = \left[\frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{dz}{\bar{C}_{55}^{(k)}} \right]^{-1} = \left[\frac{1}{h} \sum_{k=1}^N \frac{1}{\bar{C}_{55}^{(k)}} \int_{z_{(k-1)}}^{z_{(k)}} dz \right]^{-1} = \left[\frac{1}{h} \sum_{k=1}^N \frac{h_{(k)}}{\bar{C}_{55}^{(k)}} \right]^{-1} \quad (30a)$$

$$G_y = \left[\frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{dz}{\bar{C}_{44}^{(k)}} \right]^{-1} = \left[\frac{1}{h} \sum_{k=1}^N \frac{1}{\bar{C}_{44}^{(k)}} \int_{z_{(k-1)}}^{z_{(k)}} dz \right]^{-1} = \left[\frac{1}{h} \sum_{k=1}^N \frac{h_{(k)}}{\bar{C}_{44}^{(k)}} \right]^{-1} \quad (30b)$$

With G_x and G_y known, equations (20) are written as

$$F_x'(z) = H_x'(z) + \frac{G_x}{\bar{C}_{55}^{(k)}} - 1 \quad (31a)$$

$$F_y'(z) = H_y'(z) + \frac{G_y}{\bar{C}_{44}^{(k)}} - 1 \quad (31b)$$

and constitutive equations (22) become

$$\begin{Bmatrix} \sigma_{yz}(x, y, z) \\ \sigma_{xz}(x, y, z) \end{Bmatrix} = \begin{bmatrix} \bar{C}_{44}^{(k)} & \bar{C}_{45}^{(k)} \\ \bar{C}_{45}^{(k)} & \bar{C}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \left[H_y'(z) + \frac{G_y}{\bar{C}_{44}^{(k)}} - 1 \right] \gamma_{yz}^o(x, y) \\ \left[H_x'(z) + \frac{G_x}{\bar{C}_{55}^{(k)}} - 1 \right] \gamma_{xz}^o(x, y) \end{Bmatrix} \quad (32)$$

The next step in the analysis is to obtain expressions for the zig-zag functions $\Phi_x^{(k)}(z)$ and $\Phi_y^{(k)}(z)$ in terms of the constants G_x and G_y . Toward this goal, equations (18) are re-written as

$$\Delta_x^{(k)} = \Delta_x^{(k-1)} + h^{(k)} \beta_x^{(k)} \quad (33a)$$

$$\Delta_y^{(k)} = \Delta_y^{(k-1)} + h^{(k)} \beta_y^{(k)} \quad (33b)$$

Examining these equations for successive values of the index k , starting with $k = 1$ and using

$\Delta_x^{(0)} = \Delta_y^{(0)} = 0$, it is seen by mathematical induction that

$$\Delta_x^{(k)} = \sum_{p=1}^k \beta_x^{(p)} h_{(p)} \quad (33c)$$

$$\Delta_y^{(k)} = \sum_{p=1}^k \beta_y^{(p)} h_{(p)} \quad (33d)$$

Substituting these expressions into equations (17) gives

$$\Phi_x^{(k)}(z) = \sum_{p=1}^{k-1} \beta_x^{(p)} h_{(p)} + \beta_x^{(k)} (z - z_{(k-1)}) \quad (34a)$$

$$\Phi_y^{(k)}(z) = \sum_{p=1}^{k-1} \beta_y^{(p)} h_{(p)} + \beta_y^{(k)}(z - z_{(k-1)}) \quad (34b)$$

Now, consider $h_{(k)} = z_{(k)} - z_{(k-1)}$ for successive values of the index k , starting with $k = 1$. Noting that $z_{(0)} = -\frac{h}{2}$, mathematical induction yields

$$z_{(k)} = -\frac{h}{2} + \sum_{p=1}^k h_{(p)} \quad (35)$$

Thus, using equations (25) and (35), equations (34) are expressed as

$$\Phi_x^{(k)}(z) = \left(z + \frac{h}{2}\right) \left[\frac{G_x}{\bar{C}_{55}^{(k)}} - 1 \right] + G_x \sum_{p=1}^{k-1} \left[\frac{1}{\bar{C}_{55}^{(p)}} - \frac{1}{\bar{C}_{55}^{(k)}} \right] h_{(p)} \quad (36a)$$

$$\Phi_y^{(k)}(z) = \left(z + \frac{h}{2}\right) \left[\frac{G_y}{\bar{C}_{44}^{(k)}} - 1 \right] + G_y \sum_{p=1}^{k-1} \left[\frac{1}{\bar{C}_{44}^{(p)}} - \frac{1}{\bar{C}_{44}^{(k)}} \right] h_{(p)} \quad (36b)$$

Let M denote the ply that corresponds to the plate midplane, $z = 0$. At $z = 0$, equations (36) reduce to

$$\Phi_x^{(M)}(0) = \frac{h}{2} \left[\frac{G_x}{\bar{C}_{55}^{(M)}} - 1 \right] + G_x \sum_{p=1}^{M-1} \left[\frac{1}{\bar{C}_{55}^{(p)}} - \frac{1}{\bar{C}_{55}^{(M)}} \right] h_{(p)} \quad (37a)$$

$$\Phi_y^{(M)}(0) = \frac{h}{2} \left[\frac{G_y}{\bar{C}_{44}^{(M)}} - 1 \right] + G_y \sum_{p=1}^{M-1} \left[\frac{1}{\bar{C}_{44}^{(p)}} - \frac{1}{\bar{C}_{44}^{(M)}} \right] h_{(p)} \quad (37b)$$

Examination of equations (14) indicates that the constraints $F_x(0) = F_y(0) = 0$ require

$$H_x(0) + \Phi_x^{(M)}(0) = 0 \quad (38a)$$

$$H_y(0) + \Phi_y^{(M)}(0) = 0 \quad (38b)$$

Similarly, the constraints given by equations (12), which enforce the traction boundary conditions on the top and bottom plate surface, yield

$$H_x' \left(-\frac{h}{2}\right) + \frac{G_x}{\bar{C}_{55}^{(1)}} - 1 = 0 \quad (39a)$$

$$H_x' \left(+\frac{h}{2}\right) + \frac{G_x}{\bar{C}_{55}^{(N)}} - 1 = 0 \quad (39b)$$

$$H_y' \left(-\frac{h}{2} \right) + \frac{G_y}{\bar{C}_{44}^{(1)}} - 1 = 0 \quad (39c)$$

$$H_y' \left(+\frac{h}{2} \right) + \frac{G_y}{\bar{C}_{44}^{(N)}} - 1 = 0 \quad (39d)$$

where equations (31) have been used. For convenience, let the continuous functions have the form

$$H_x(z) = f_x(z) - \Phi_x^{(M)}(0) + a_x z + b_x \frac{z^2}{2} \quad (40a)$$

$$H_y(z) = f_y(z) - \Phi_x^{(M)}(0) + a_y z + b_y \frac{z^2}{2} \quad (40b)$$

which satisfy equations (38) provided $f_x(0) = f_y(0) = 0$. Substituting equations (40) into equations (39) gives

$$a_x - \frac{h}{2} b_x = 1 - \frac{G_x}{\bar{C}_{55}^{(1)}} - f_x' \left(-\frac{h}{2} \right) \quad (41a)$$

$$a_x + \frac{h}{2} b_x = 1 - \frac{G_x}{\bar{C}_{55}^{(N)}} - f_x' \left(+\frac{h}{2} \right) \quad (41b)$$

$$a_y - \frac{h}{2} b_y = 1 - \frac{G_y}{\bar{C}_{44}^{(1)}} - f_y' \left(-\frac{h}{2} \right) \quad (41c)$$

$$a_y + \frac{h}{2} b_y = 1 - \frac{G_y}{\bar{C}_{44}^{(N)}} - f_y' \left(+\frac{h}{2} \right) \quad (41d)$$

Solving these equations yields

$$a_x = 1 - \frac{1}{2} \left(\frac{G_x}{\bar{C}_{55}^{(1)}} + \frac{G_x}{\bar{C}_{55}^{(N)}} + f_x' \left(-\frac{h}{2} \right) + f_x' \left(+\frac{h}{2} \right) \right) \quad (42a)$$

$$b_x = \frac{1}{h} \left[\frac{G_x}{\bar{C}_{55}^{(1)}} - \frac{G_x}{\bar{C}_{55}^{(N)}} + f_x' \left(-\frac{h}{2} \right) - f_x' \left(+\frac{h}{2} \right) \right] \quad (42b)$$

$$a_y = 1 - \frac{1}{2} \left(\frac{G_y}{\bar{C}_{44}^{(1)}} + \frac{G_y}{\bar{C}_{44}^{(N)}} + f_y' \left(-\frac{h}{2} \right) + f_y' \left(+\frac{h}{2} \right) \right) \quad (42c)$$

$$b_y = \frac{1}{h} \left[\frac{G_y}{\bar{C}_{44}^{(1)}} - \frac{G_y}{\bar{C}_{44}^{(N)}} + f_y' \left(-\frac{h}{2} \right) - f_y' \left(+\frac{h}{2} \right) \right] \quad (42d)$$

Substituting these results into equations (40) gives

$$H_x(z) = f_x(z) - \Phi_x^{(M)}(0) + z - \frac{z}{2} \left[\frac{G_x}{\bar{C}_{55}^{(I)}} + f_x' \left(-\frac{h}{2} \right) \right] \left(1 - \frac{z}{h} \right) - \frac{z}{2} \left[\frac{G_x}{\bar{C}_{55}^{(N)}} + f_x' \left(+\frac{h}{2} \right) \right] \left(1 + \frac{z}{h} \right) \quad (43a)$$

$$H_y(z) = f_y(z) - \Phi_y^{(M)}(0) + z - \frac{z}{2} \left[\frac{G_y}{\bar{C}_{44}^{(I)}} + f_y' \left(-\frac{h}{2} \right) \right] \left(1 - \frac{z}{h} \right) - \frac{z}{2} \left[\frac{G_y}{\bar{C}_{44}^{(N)}} + f_y' \left(+\frac{h}{2} \right) \right] \left(1 + \frac{z}{h} \right) \quad (43b)$$

In addition,

$$H_x'(z) = f_x'(z) + 1 - \frac{1}{2} \left[\frac{G_x}{\bar{C}_{55}^{(I)}} + f_x' \left(-\frac{h}{2} \right) \right] \left(1 - \frac{2z}{h} \right) - \frac{1}{2} \left[\frac{G_x}{\bar{C}_{55}^{(N)}} + f_x' \left(+\frac{h}{2} \right) \right] \left(1 + \frac{2z}{h} \right) \quad (44a)$$

$$H_y'(z) = f_y'(z) + 1 - \frac{1}{2} \left[\frac{G_y}{\bar{C}_{44}^{(I)}} + f_y' \left(-\frac{h}{2} \right) \right] \left(1 - \frac{2z}{h} \right) - \frac{1}{2} \left[\frac{G_y}{\bar{C}_{44}^{(N)}} + f_y' \left(+\frac{h}{2} \right) \right] \left(1 + \frac{2z}{h} \right) \quad (44b)$$

In the present study, the functional form of $f_x(z)$ and $f_y(z)$ are obtained by requiring $H_x(z)$ and $H_y(z)$ to reduce to the distributions expected for a homogenous, orthotropic plate. For this very special class of plates, $\bar{C}_{44}^{(k)} = G_{23}$ and $\bar{C}_{55}^{(k)} = G_{13}$, where G_{13} and G_{23} are the shear moduli. Thus, equations (30) give $G_x = G_{13}$ and $G_y = G_{23}$. In addition, $\Phi_x^{(M)}(0) = \Phi_y^{(M)}(0) = 0$. Equations (43) reduce to

$$H_x(z) = f_x(z) - \frac{z}{2} \left[f_x' \left(-\frac{h}{2} \right) \left(1 - \frac{z}{h} \right) + f_x' \left(+\frac{h}{2} \right) \left(1 + \frac{z}{h} \right) \right] \quad (45a)$$

$$H_y(z) = f_y(z) - \frac{z}{2} \left[f_y' \left(-\frac{h}{2} \right) \left(1 - \frac{z}{h} \right) + f_y' \left(+\frac{h}{2} \right) \left(1 + \frac{z}{h} \right) \right] \quad (45b)$$

and equations (44) reduce to

$$H_x'(z) = f_x'(z) - \frac{1}{2} \left[f_x' \left(-\frac{h}{2} \right) \left(1 - \frac{2z}{h} \right) + f_x' \left(+\frac{h}{2} \right) \left(1 + \frac{2z}{h} \right) \right] \quad (46a)$$

$$H_y'(z) = f_y'(z) - \frac{1}{2} \left[f_y' \left(-\frac{h}{2} \right) \left(1 - \frac{2z}{h} \right) + f_y' \left(+\frac{h}{2} \right) \left(1 + \frac{2z}{h} \right) \right] \quad (46b)$$

Moreover, the transverse shearing stresses given by equation (32) become

$$\begin{Bmatrix} \sigma_{yz}(x, y, z) \\ \sigma_{xz}(x, y, z) \end{Bmatrix} = \begin{bmatrix} G_{23} & 0 \\ 0 & G_{13} \end{bmatrix} \begin{Bmatrix} H_y'(z) \gamma_{yz}^0(x, y) \\ H_x'(z) \gamma_{xz}^0(x, y) \end{Bmatrix} \quad (47)$$

For a homogenous plate with traction-free shear-stress boundary conditions on the top and bottom

surfaces, the distributions of transverse-shear stresses are symmetric about the plate midplane.

Thus, $H_x'(-z) = H_x'(z)$ and $H_y'(-z) = H_y'(z)$. Enforcing these symmetry requirements yields

$$f_x'(z) - f_x'(-z) - \frac{2z}{h} \left[f_x' \left(+\frac{h}{2} \right) - f_x' \left(-\frac{h}{2} \right) \right] = 0 \quad (48a)$$

$$f_y'(z) - f_y'(-z) - \frac{2z}{h} \left[f_y' \left(+\frac{h}{2} \right) - f_y' \left(-\frac{h}{2} \right) \right] = 0 \quad (48b)$$

for all values of $-h/2 \leq z \leq h/2$. Therefore, the functions $f_x(z)$ and $f_y(z)$ are required to satisfy:

$$f_x(0) = f_y(0) = 0 \quad (49a)$$

$$f_x'(-z) = f_x'(z) \quad (49b)$$

$$f_y'(-z) = f_y'(z) \quad (49c)$$

for all values of $-h/2 \leq z \leq h/2$. Additional simplifications are obtained by specifying

$$f_x' \left(\pm \frac{h}{2} \right) = f_y' \left(\pm \frac{h}{2} \right) = 0 \quad (50)$$

Enforcing these conditions in equations (43) and (44) yields, without a loss in generality,

$$H_x(z) = f_x(z) + \frac{z}{2} \left[2 - \frac{G_x}{\bar{C}_{55}^{(I)}} \left(1 - \frac{z}{h} \right) - \frac{G_x}{\bar{C}_{55}^{(N)}} \left(1 + \frac{z}{h} \right) \right] - \Phi_x^{(M)}(0) \quad (51a)$$

$$H_y(z) = f_y(z) + \frac{z}{2} \left[2 - \frac{G_y}{\bar{C}_{44}^{(I)}} \left(1 - \frac{z}{h} \right) - \frac{G_y}{\bar{C}_{44}^{(N)}} \left(1 + \frac{z}{h} \right) \right] - \Phi_y^{(M)}(0) \quad (51b)$$

and

$$H_x'(z) = f_x'(z) + \frac{1}{2} \left[2 - \frac{G_x}{\bar{C}_{55}^{(I)}} \left(1 - \frac{2z}{h} \right) - \frac{G_x}{\bar{C}_{55}^{(N)}} \left(1 + \frac{2z}{h} \right) \right] \quad (52a)$$

$$H_y'(z) = f_y'(z) + \frac{1}{2} \left[2 - \frac{G_y}{\bar{C}_{44}^{(I)}} \left(1 - \frac{2z}{h} \right) - \frac{G_y}{\bar{C}_{44}^{(N)}} \left(1 + \frac{2z}{h} \right) \right] \quad (52b)$$

Using these expressions with equations (36), equations (14) are expressed as

$$F_x(z) = f_x(z) + \Psi_x^{(k)}(z) \quad (53a)$$

$$F_y(z) = f_y(z) + \Psi_y^{(k)}(z) \quad (53b)$$

where $f_x(z)$ and $f_y(z)$ are continuous functions with continuous derivatives that satisfy equations (47) and (48). The functions $\Psi_x^{(k)}(z)$ and $\Psi_y^{(k)}(z)$ are the zig-zag enrichment functions given by

$$\Psi_x^{(k)}(z) = \Phi_x^{(k)}(z) - \Phi_x^{(M)}(0) + \frac{z}{2} \left[2 - \frac{G_x}{\bar{C}_{55}^{(1)}} \left(1 - \frac{z}{h} \right) - \frac{G_x}{\bar{C}_{55}^{(N)}} \left(1 + \frac{z}{h} \right) \right] \quad (54a)$$

$$\Psi_y^{(k)}(z) = \Phi_y^{(k)}(z) - \Phi_y^{(M)}(0) + \frac{z}{2} \left[2 - \frac{G_y}{\bar{C}_{44}^{(1)}} \left(1 - \frac{z}{h} \right) - \frac{G_y}{\bar{C}_{44}^{(N)}} \left(1 + \frac{z}{h} \right) \right] \quad (54b)$$

where $\Phi_x^{(k)}(z)$ and $\Phi_y^{(k)}(z)$ are given by equations (36), $\Phi_x^{(M)}(0)$ and $\Phi_y^{(M)}(0)$ are given by equations (37), and G_x and G_y are given by equations (30). The derivatives of $F_x(z)$ and $F_y(z)$ are obtained by substituting equations (52) into equations (31); which yields,

$$F_x'(z) = f_x'(z) + \Psi_x^{(k)'}(z) \quad (55a)$$

$$F_y'(z) = f_y'(z) + \Psi_y^{(k)'}(z) \quad (55b)$$

where

$$\Psi_x^{(k)'}(z) = \frac{G_x}{\bar{C}_{55}^{(k)}} - \frac{1}{2} \left[\frac{G_x}{\bar{C}_{55}^{(1)}} \left(1 - \frac{2z}{h} \right) + \frac{G_x}{\bar{C}_{55}^{(N)}} \left(1 + \frac{2z}{h} \right) \right] \quad (56a)$$

$$\Psi_y^{(k)'}(z) = \frac{G_y}{\bar{C}_{44}^{(k)}} - \frac{1}{2} \left[\frac{G_y}{\bar{C}_{44}^{(1)}} \left(1 - \frac{2z}{h} \right) + \frac{G_y}{\bar{C}_{44}^{(N)}} \left(1 + \frac{2z}{h} \right) \right] \quad (56b)$$

For the parabolic distribution of transverse shearing stresses commonly found in the technical literature for homogeneous plates,

$$f_x(z) = f_y(z) = z \left[1 - \frac{1}{3} \left(\frac{2z}{h} \right)^2 \right] \quad (57a)$$

$$f_x'(z) = f_y'(z) = 1 - \left(\frac{2z}{h} \right)^2 \quad (57b)$$

These functions are illustrated in figure 9.

Examples

Results obtained from equations (53) and (55), based on equations (57), are presented for two examples in this section. The first example is a $[-15/30/0/90]_T$ laminated composite plate that exhibits the full extent of plate anisotropy. The second example is a sandwich plate with a 0.42-inch-thick isotropic core and identical $[\pm 45_4]$ laminated composite face plates, denoted by $[\pm 45_4/\text{core}]_S$. For both examples, the principal transverse-shear moduli of a ply are given by $G_{13} = 0.864 \times 10^6$ psi and $G_{23} = 0.368 \times 10^6$ psi, and the thickness of each ply is 0.005 in. For the core of the sandwich plate, $G_{13} = G_{23} = G_{\text{core}} = 0.100 \times 10^6$ psi.

For the $[-15/30/0/90]_T$ laminated composite plate, equations (30) yield $G_x/G_{13} = 0.72$ and $G_y/G_{23} = 1.29$. In addition, $\Phi_x^{(M)}(0)$ and $\Phi_y^{(M)}(0)$ defined by equations (37) are given by -0.0021 and 0.0008 in., respectively, where $M = 2$ and 3 contain the plate midplane and give identical results. Graphs of $F_x(z)$ and $F_y(z)$ given by equation (53a) and (53b) are shown in figures 10 and 11, respectively. These graphs clearly show the effects of laminate asymmetry and inplane-coordinate direction on the inplane-displacement distributions. Moreover, both graphs depict piecewise-cubic curves that are continuous at the ply interfaces. Similarly, graphs of the derivatives of $F_x(z)$ and $F_y(z)$ given by equation (55a) and (55b) are also shown in figures 10 and 11, respectively. These two derivative graphs depict piecewise-quadratic curves that are discontinuous at the ply interfaces, consistent with the refined zig-zag theory of Tessler et. al.⁴³

For the seventeen-ply $[\pm 45_4/\text{core}]_S$ sandwich plate, the core comprises 84% of the plate thickness. Equations (30) yield $G_x/G_{\text{core}} = G_y/G_{\text{core}} = 1.16$. In addition, $\Phi_x^{(M)}(0)$ and $\Phi_y^{(M)}(0)$ defined by equations (37) are equal to zero, where $M = 9$ corresponds to the plate midplane. For this plate, $F_x(z)$ and $F_y(z)$ given by equation (53a) and (53b) are identical, and a graph of $F_x(z)$ is shown in figure 12. This graph which clearly shows the presence of laminate symmetry is composed piecewise-cubic curves that are also continuous at the ply interfaces, with the most variation occurring in the more flexible core, as expected. Note that no variation in $F_x(z)$ is shown across the faceplates, consistent with the fact that the $\bar{C}_{44}^{(k)}$, $\bar{C}_{45}^{(k)}$, and $\bar{C}_{55}^{(k)}$ are identical for two adjacent $+45$ and -45 degree plies. The corresponding graph of the derivatives of $F_x(z)$ is also shown in figure 12. This derivative graph also depicts piecewise-quadratic curves that are discontinuous at the ply interfaces. Moreover, the distribution of the curves is symmetric across the plate thickness.

CONCLUDING REMARKS

A detailed analysis has been presented that shows how to enrich the kinematics of classical Kirchhoff plate theory by using a set of continuous piecewise-cubic functions referred to in the technical literature as zig-zag functions. This work was motivated by the desire to obtain realistic estimates of the effects of laminate heterogeneity and asymmetry on the variations of the inplane displacements and transverse shearing stresses, for use with a $\{3, 0\}$ plate theory in which these

distributions are specified apriori. The enrichment is based on the improved zig-zag functions, and the corresponding analysis methodology, presented recently by Tessler, Di Scuva, and Gherlone. With the approach that has been presented herein, the inplane displacements are represented by a set of continuous piecewise-cubic functions. In contrast, the transverse shearing stresses and strains are represented by a set of piecewise-quadratic curves that are discontinuous at the ply interfaces. Corresponding results have also been presented for a general asymmetric laminate and a sandwich plate with identical laminated composite face plates and an isotropic core. These results demonstrate clearly that the zig-zag enrichment captures the expected effects of laminate heterogeneity and asymmetry.

References

1. Kreja, I.: A Literature Review on Computational Models for Laminated Composite and Sandwich Panels. *Central European Journal of Engineering*, vol. 1, no. 1, 2011, pp. 59-80.
2. Ghugal, Y. M. and Shimpi, R. P.: A Review of Refined Shear Deformation Theories of Isotropic and Anisotropic Laminated Plates. *Journal of Reinforced Plastics and Composites*, vol. 21, no. 9, 2002, pp. 775-813.
3. Carrera, E.: Theories and Finite Elements for Multilayered, Anisotropic, Composite Plates and Shells. *Archives of Computational Methods in Engineering*, vol. 9, no. 2, 2002, pp. 87-140.
4. Kant, T. and Swaminathan, K.: Estimation of Transverse/Interlaminar Stresses in Laminated Composites - a Selective Review and Survey of Current Developments. *Composite Structures*, vol. 49, 2000, pp. 65-75.
5. Altenbach, H.: Theories for Laminated and Sandwich Plates. *Mechanics of Composite Materials*, vol. 34, no. 3, 1998, pp. 243-252.
6. Reddy, J. N.: *Mechanics of Laminated Composite Plates - Theory and Analysis*. CRC Press, 1997.
7. Liu, D. and Li, X.: An Overall View of Laminate Theories Based on Displacement Hypothesis. *Journal of Composite Materials*, vol. 30, no. 14, 1996, pp. 1539-1561.
8. Burton, W. S. and Noor, A. K.: Assessment of Computational Models for Sandwich Panels and Shells. *Computer Methods in Applied Mechanics and Engineering*, vol. 124, 1995, pp. 125-151.
9. Reddy, J. N. and Robbins, D. H.; Jr.: Theories and Computational Models for Composite Laminates. *ASME Applied Mechanics Reviews*, vol. 47, no. 6, part 1, 1994, pp. 147-169.
10. Reddy, J. N.: A General Non-Linear Third-Order Theory of Plates with Moderate Thickness. *International Journal of Non-Linear Mechanics*, vol. 25, no. 6, 1990, pp. 677-686.

11. Reissner, E.: On the Theory of Bending of Elastic Plates. *Journal of Mathematics and Physics*, vol. 23, 1944, pp. 184-191.
12. Reissner, E.: The Effect of Transverse Shear Deformation on the Bending of Elastic Plates. *ASME Journal of Applied Mechanics*, vol. 12, 1945, pp. 69-77.
13. Mindlin, R. D.: Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates. *ASME Journal of Applied Mechanics*, vol. 18, 1951, pp. 31-38.
14. Stein, M.: Nonlinear Theory of Plates and Shells Including the Effects of Transverse Shearing. *AIAA Journal*, vol. 24, no. 9, 1986, pp. 1537-1544.
15. Soldatos, K. P.: On Certain Refined Theories for Plate Bending. *ASME Journal of Applied Mechanics*, vol. 55, December, 1988, pp. 994-995.
16. Sun, L. X. and Hsu, T. R.: Thermal Buckling of Laminated Composite Plates with Transverse Shear Deformation. *Computers and Structures*, vol. 36, no. 5, 1990, pp. 883-889.
17. Touratier, M.: An Efficient Standard Plate Theory. *International Journal of Engineering Science*, vol. 29, no. 8, 1991, pp. 901-916.
18. Soldatos, K. P.: Nonlinear Analysis of Transverse Shear Deformable Laminated Composite Cylindrical Shells - Part I: Derivation of Governing Equations. *ASME Journal of Pressure Vessel Technology*, vol. 114, no. 1, 1992, pp. 105-109.
19. Soldatos, K. P.: Nonlinear Analysis of Transverse Shear Deformable Laminated Composite Cylindrical Shells - Part II: Buckling of Axially Compressed Cross-Ply Circular and Oval Cylinders. *ASME Journal of Pressure Vessel Technology*, vol. 114, no. 1, 1992, pp. 110-114.
20. Hadian, J. and Nayfeh, A. H.: Free Vibration and Buckling of Shear-Deformable Cross-Ply Laminated Plates Using the State-Space Concept. *Computers and Structures*, vol. 48, no. 4, 1993, pp. 677-693.
21. Shu, X. and Sun, L.: Thermomechanical Buckling of Laminated Composite Plates with Higher-Order Transverse Shear Deformation. *Computers and Structures*, vol. 53, no. 1, 1994, pp. 1-7.
22. Touratier, M. and Faye, J.-P.: On a Refined Model in Structural Mechanics: Finite Element Approximation and Edge Effect Analysis for Axisymmetric Shells. *Computers and Structures*, vol. 54, no. 5, 1995, pp. 897-920.
23. Shu, X.: An Improved Simple Higher-Order Theory for Laminated Composite Shells. *Computers and Structures*, vol. 60, no. 3, 1996, pp. 343-350.
24. Ziyaeifar, M. and Elwi, A. E.: Degenerate Plate-Shell Elements with Refined Transverse Shear Strains. *Computers and Structures*, vol. 60, no. 6, 1996, pp. 1079-1091.

25. Cheng, Z.-Q. and Kitipornchai, S.: Exact Connection Between Deflections of the Classical and Shear Deformation Laminated Plate Theories. *ASME Journal of Applied Mechanics*, vol. 66, March, 1999, pp. 260-262.
26. Aydogdu, M. and Timarci, T.: Vibration Analysis of Cross-Ply Laminated Square Plates with General Boundary Conditions. *Composites Science and Technology*, vol. 63, 2003, pp. 1061-1070.
27. Karama, M. and Mistou, S.: Experimental Validation of Refined Models of Composite Plates. *Science and Engineering of Composite Materials*, vol. 11, no. 1, 2004, pp. 1-7.
28. Zenkour, A. M.: Buckling of Fiber-Reinforced Viscoelastic Composite Plates Using Various Plate Theories. *Journal of Engineering Mathematics*, vol. 50, 2004, pp. 75-93.
29. Zenkour, A. M.: Analytical Solution for Bending of Cross-Ply Laminated Plates Under Thermo-Mechanical Loading. *Composite Structures*, vol. 65, 2004, pp. 367-379.
30. Zenkour, A. M.: Thermal Effects on the Bending Response of Fiber-Reinforced Viscoelastic Composite Plates Using a Sinusoidal Shear Deformation Theory. *Acta Mechanica*, vol. 171, 2004, pp. 171-187.
31. Kim, J.-S.: Reconstruction of First-Order Shear Deformation Theory for Laminated and Sandwich Shells. *AIAA Journal*, vol. 42, no. 8, 2004, pp. 1685-1697.
32. Timarci, T. and Aydogdu, M.: Buckling of Symmetric Cross-Ply Square Plates with Various Boundary Conditions. *Composite Structures*, vol. 68, 2005, pp. 381-389.
33. Aydogdu, M.: Comparison of Various Shear Deformation Theories for Bending, Buckling, and Vibration of Rectangular Symmetric Cross-Ply Plate with Simply Supported Edges. *Journal of Composite Materials*, vol. 40, no. 23, 2006, pp. 2143-2155.
34. Girish, J. and Ramachandra, L. S.: Thermomechanical Postbuckling Analysis of Cross-Ply Laminated Cylindrical Shell Panels. *ASCE Journal of Engineering Mechanics*, vol. 132, no. 2, 2006, pp. 133-140.
35. Shi, G.: A New Simple Third-Order Shear Deformation Theory of Plates. *International Journal of Solids and Structures*, vol. 44, 2007, pp. 4399-4417.
36. Akavci, S. S. and Tanrikulu, A. H.: Buckling and Free Vibration Analyses of Laminated Composite Plates by Using Two New Hyperbolic Shear-Deformation Theories. *Mechanics of Composite Materials*, vol. 44, no. 2, 2008, pp. 145-154.
37. Aydogdu, M.: A New Shear Deformation Theory for Laminated Composite Plates. *Composite Structures*, vol. 89, 2009, pp. 94-101.

38. Fu, Y.; Li, S.; and Jiang, Y.: Nonlinear Free Vibration Analysis of Piezoelectric Laminated Plates with Interface Damage. *Applied Mathematics and Mechanics (English Edition)*, vol. 30, no. 2, 2009, pp. 129-144.
39. Xiang S. and Wang, K.: Free Vibration Analysis of Symmetric Laminated Composite Plates by Trigonometric Shear Deformation Theory and Inverse Multiquadric RBF. *Thin-Walled Structures*, vol. 47, 2009, pp. 304-310.
40. Panda, S. K. and Ramachandra, L. S.: Buckling and Postbuckling Behavior of Cross-Ply Composite Plate Subjected to Nonuniform In-Plane Loads. *ASCE Journal of Engineering Mechanics*, vol. 137, no. 9, 2011, pp. 589-597.
41. Barut, A.; Madenci, E.; and Nemeth, M. P.: Stress and Buckling Analyses of Laminates with a Cutout Using a $\{3, 0\}$ -Plate Theory. *Journal of Mechanics of Materials and Structures*, vol. 6, no. 6, 2011, pp. 827-868.
42. Jones, R. M.: *Mechanics of Composite Materials*. Second ed., Taylor & Francis, 1999.
43. Tessler, A.; Di Sciuva, M.; and Gherlone, M.: A Consistent Refinement of First-Order Shear Deformation Theory for Laminated Composite and Sandwich Plates Using Improved Zigzag Kinematics. *Journal of Mechanics of Materials and Structures*, vol. 5, no. 2, 2010, pp. 341-367.

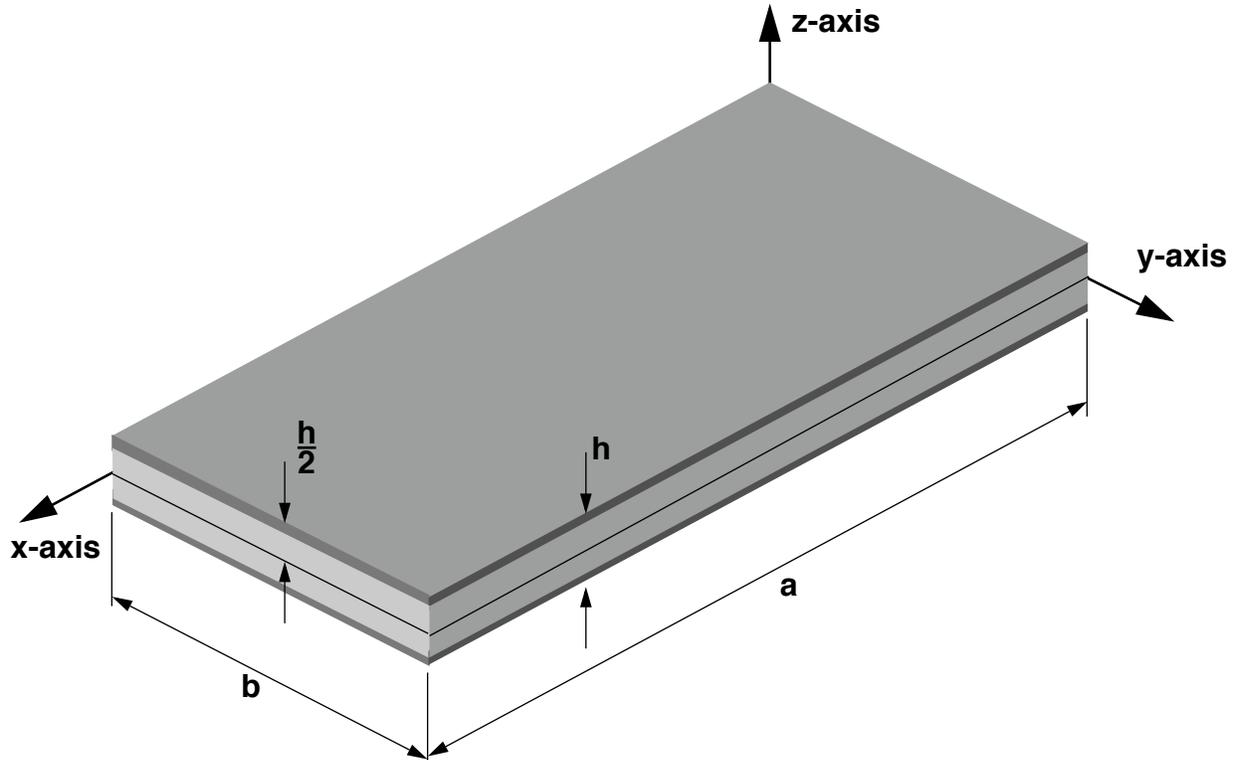


Fig. 1 Plate geometry and coordinate system.

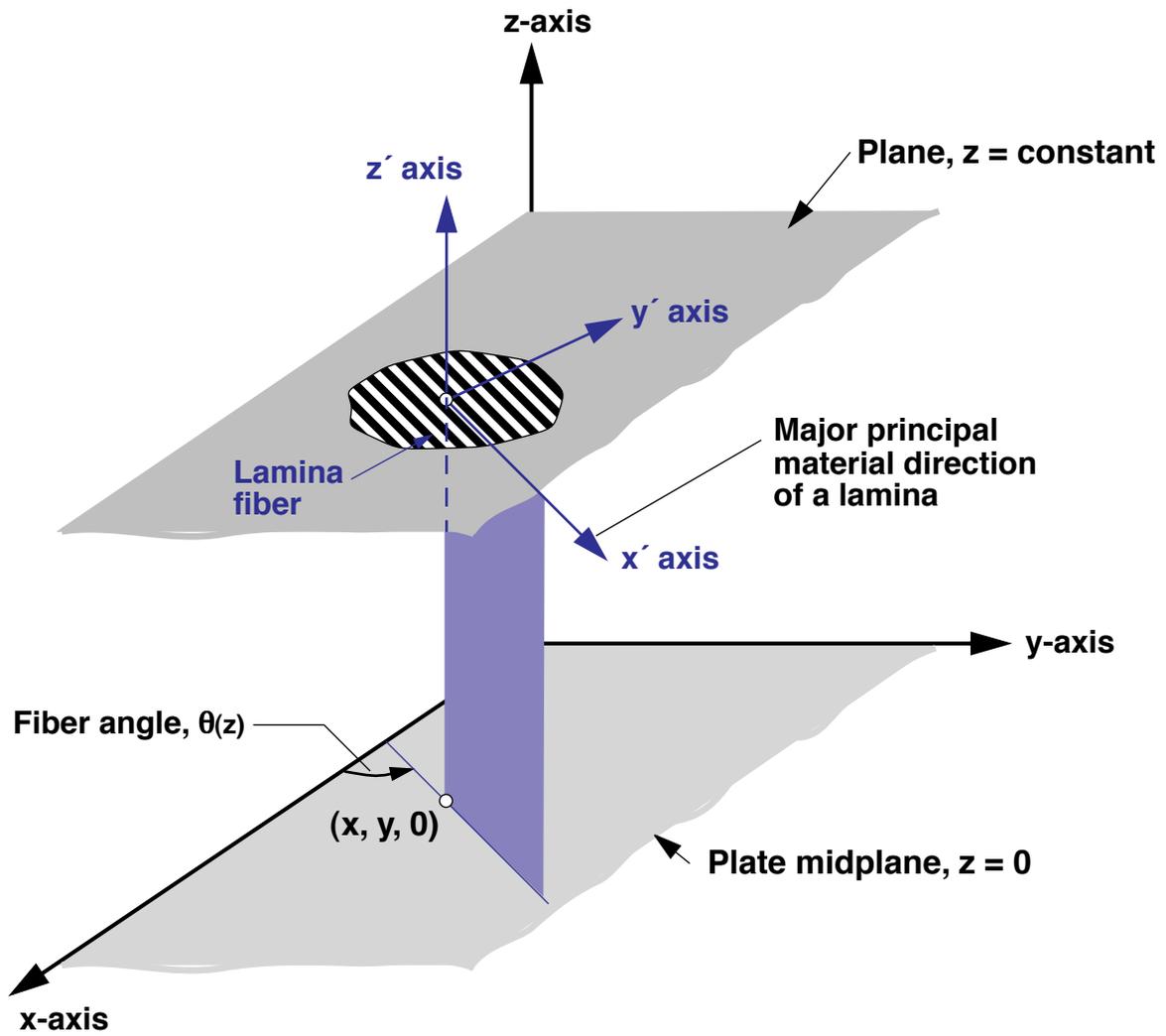


Fig. 2 Principal material coordinate system at $z = \text{constant}$.

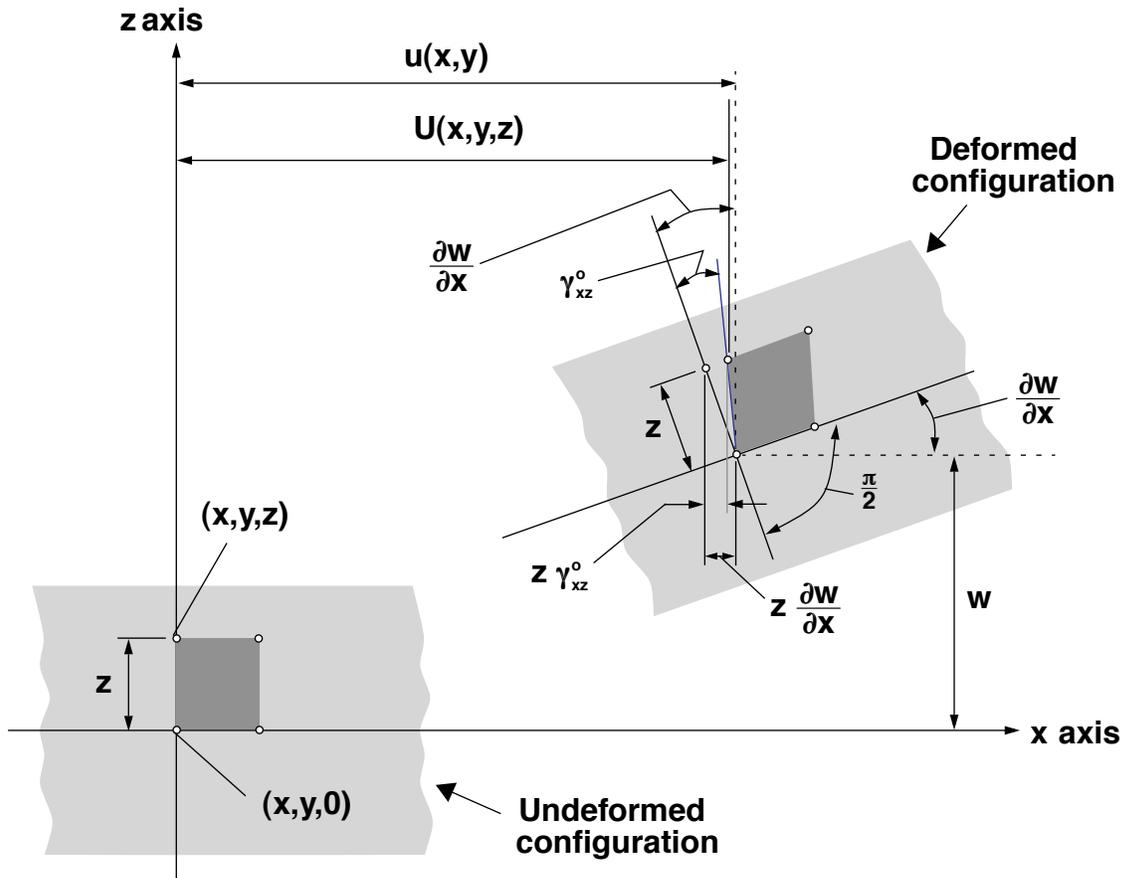


Fig. 3 Kinematics of first-order shear-deformation plate theory in the x-z plane.

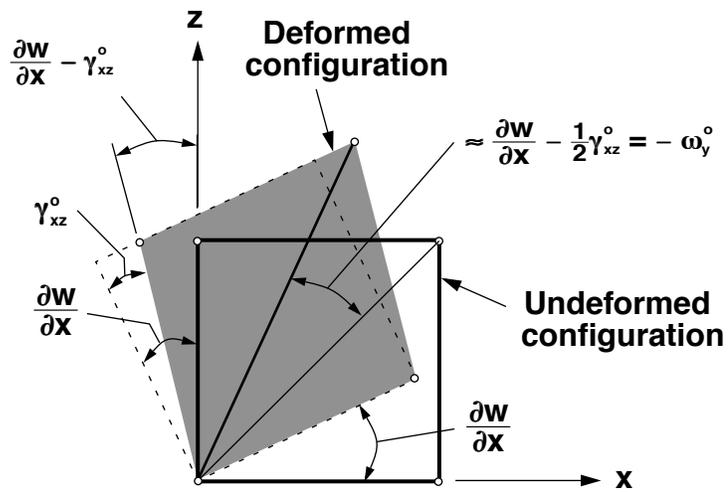


Fig. 4 Deformation of a differential element in the x-z plane, based on first-order shear-deformation plate theory.

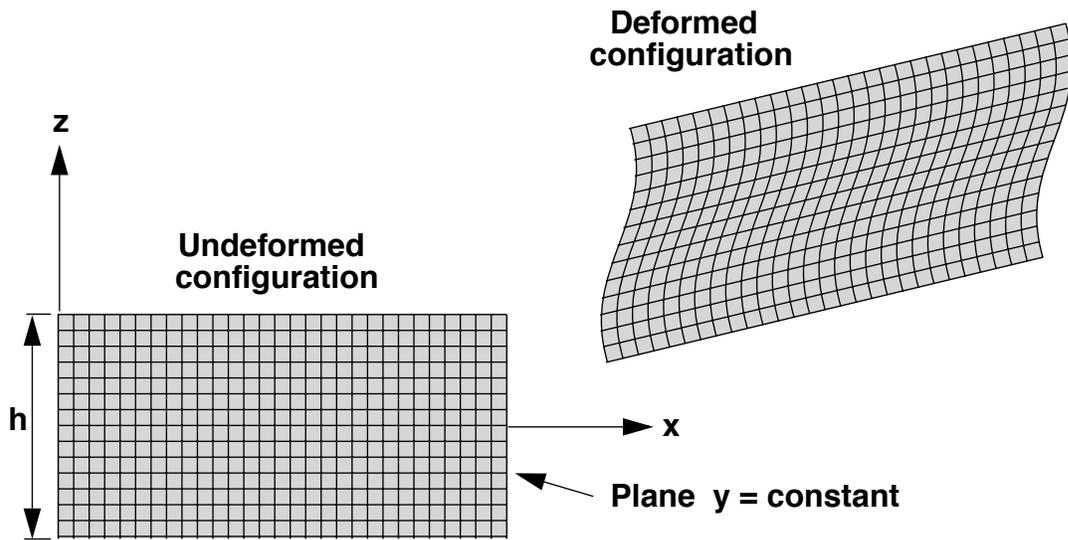


Fig. 5 Deformation of plate cross-section, based on the kinematics used in the present study.

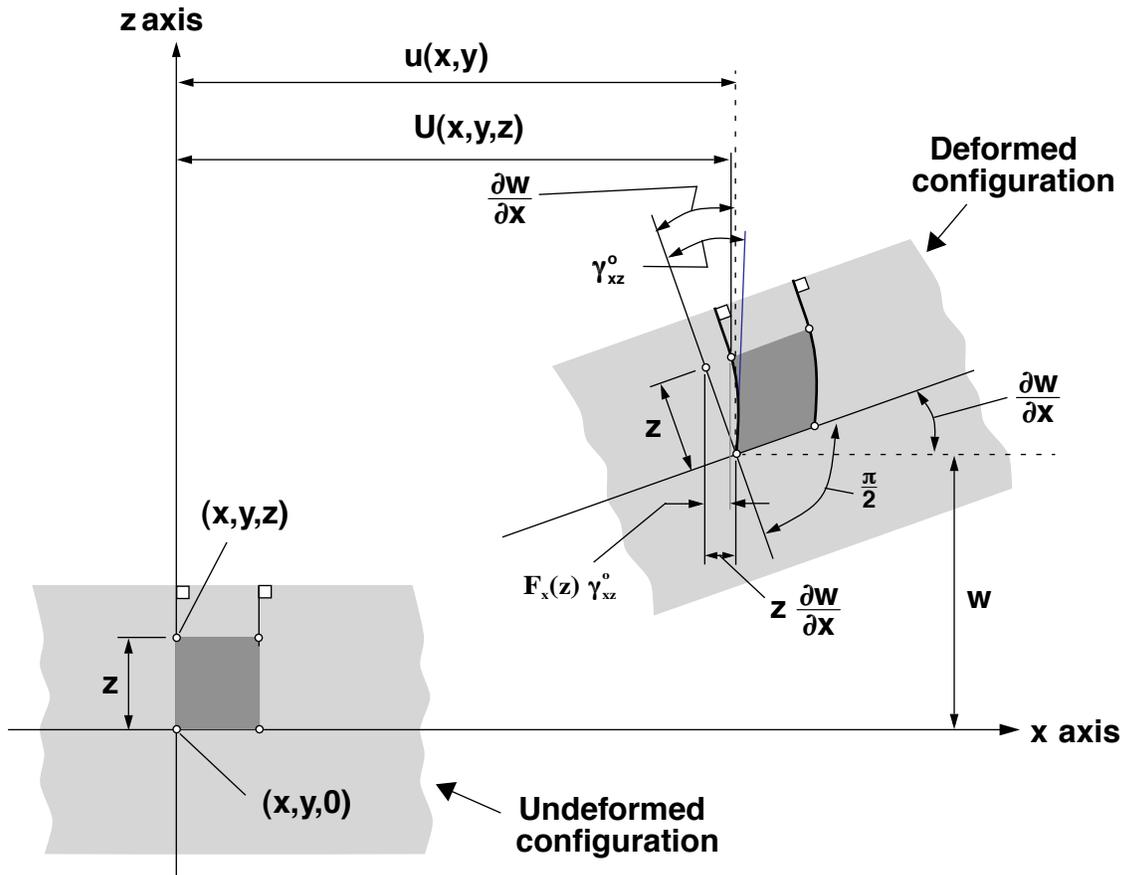


Fig. 6 Kinematics used in the present study.

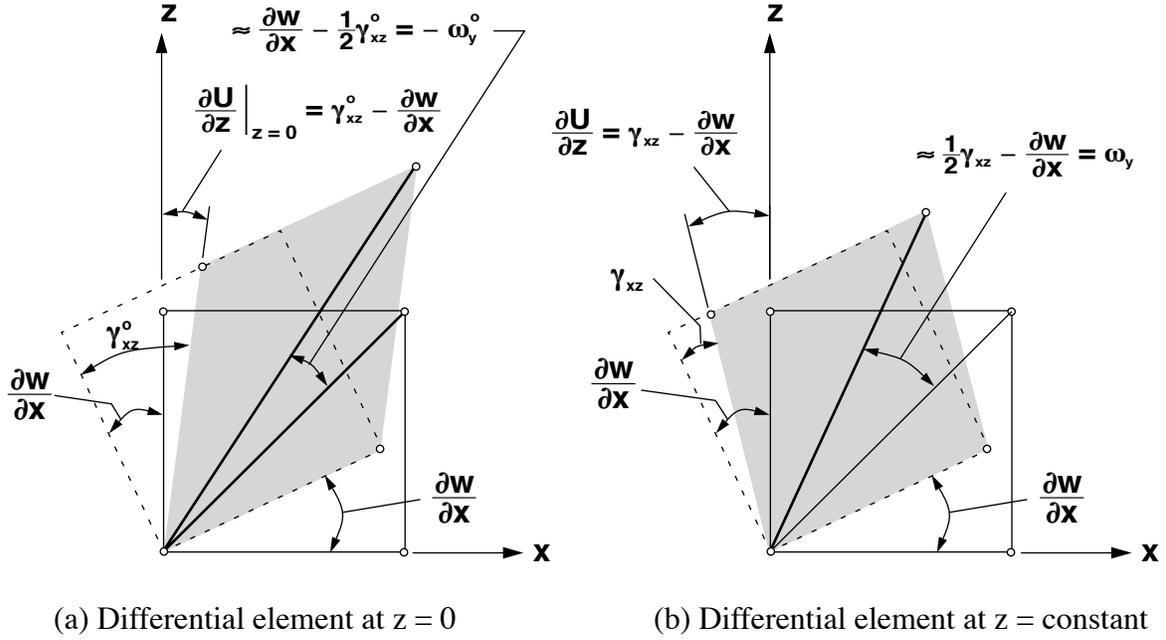


Fig. 7 Deformation of a differential element in the $x - z$ plane, based on the kinematics of the present study.

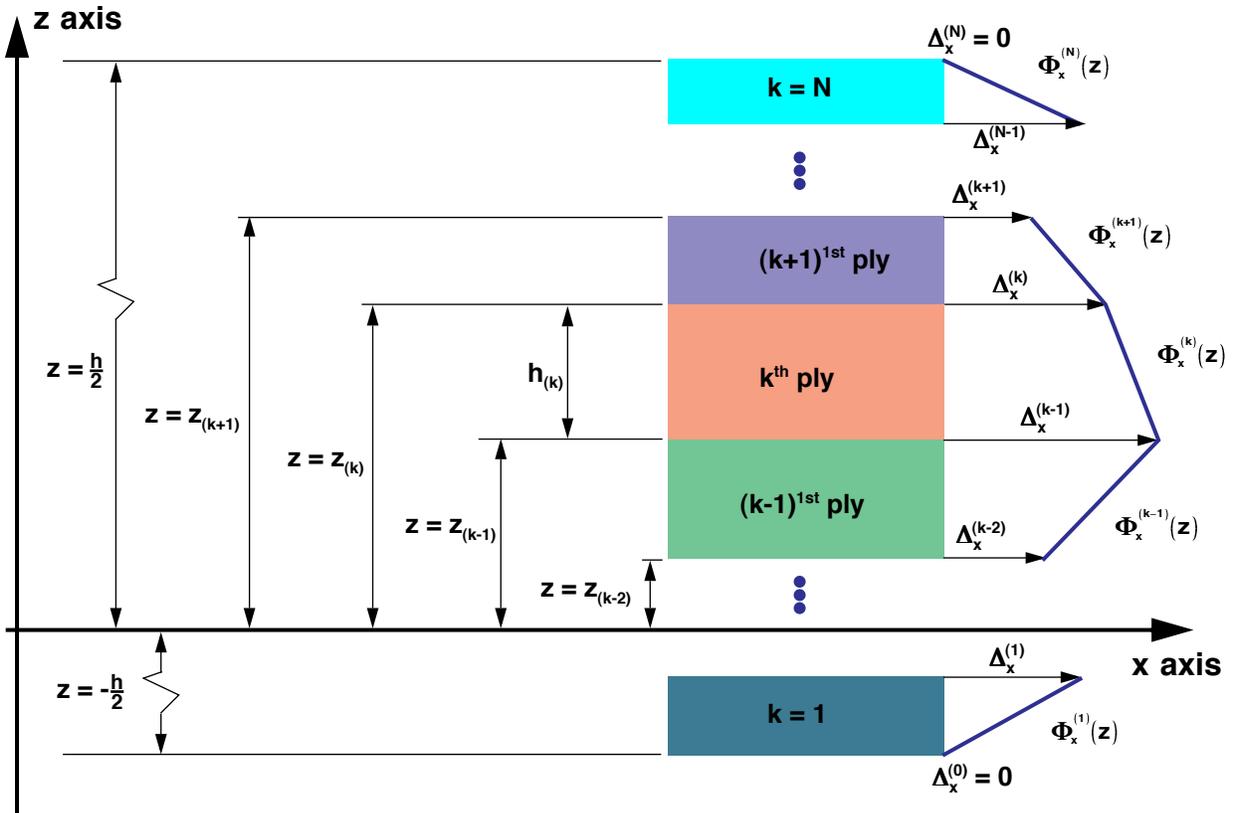


Fig. 8 Ply coordinates and zig-zag functions in the $x - z$ plane.

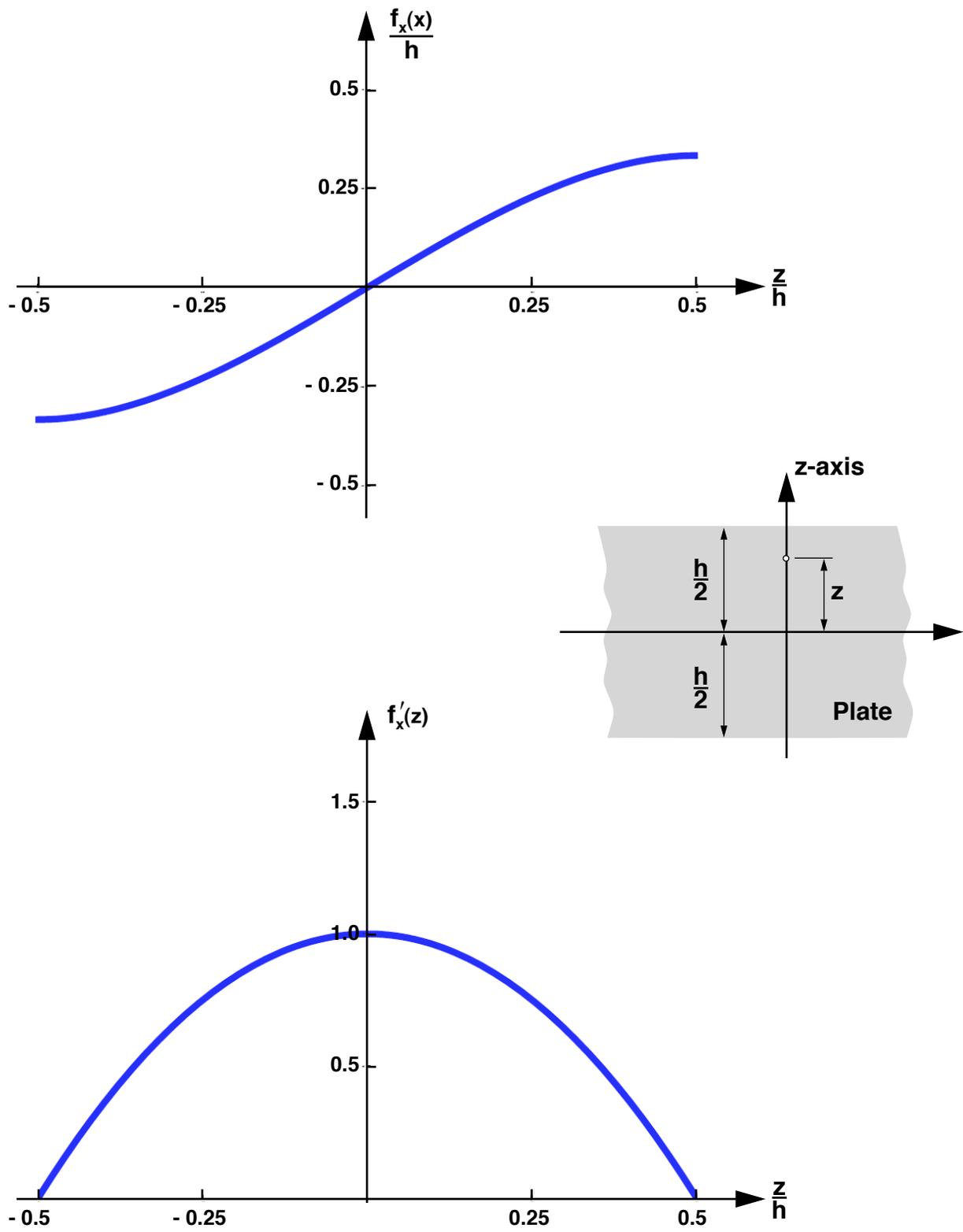


Fig. 9 Graphs of the function $f_x(z)$ and its derivative.

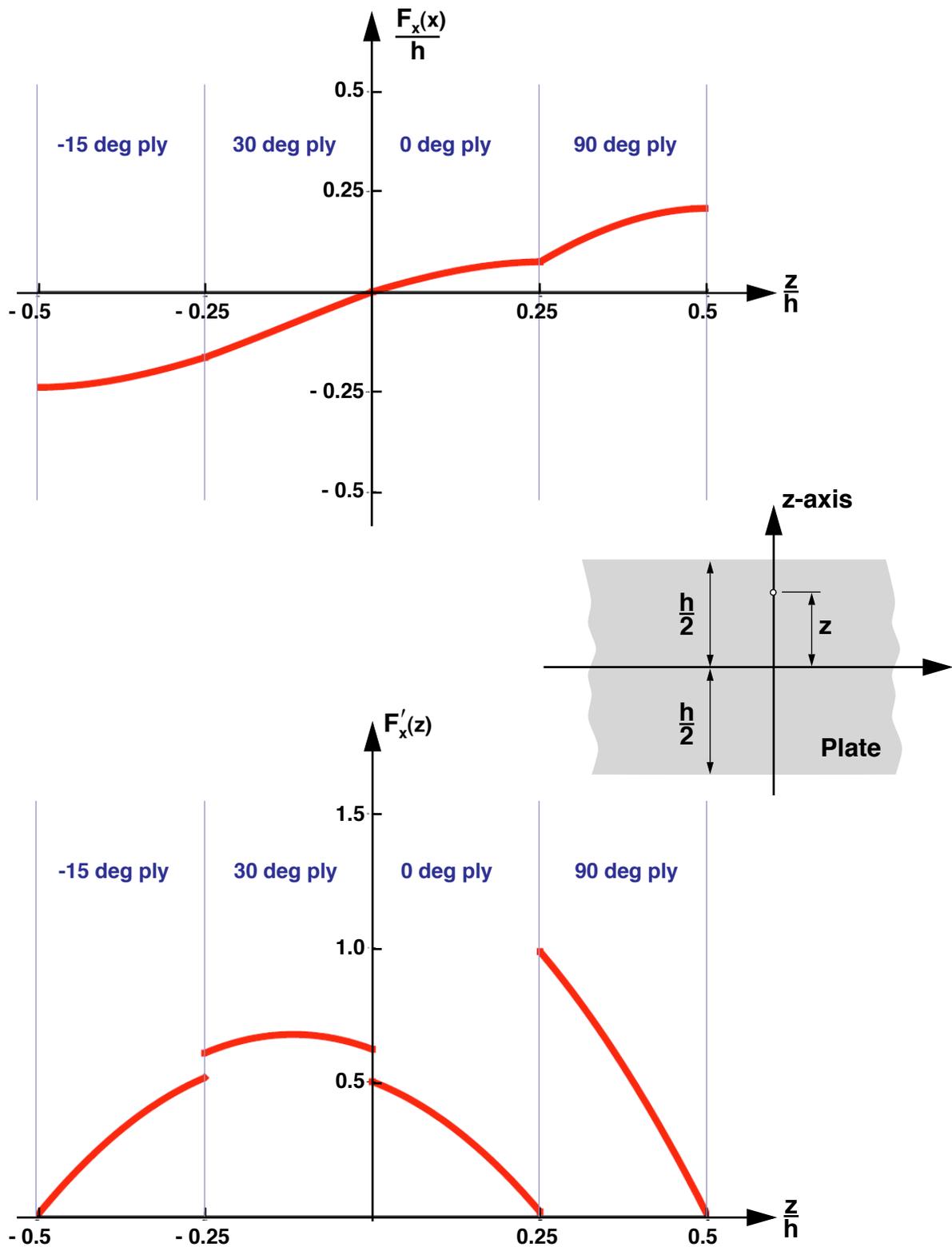


Fig. 10 Graphs of the function $F_x(z)$ and its derivative for $[-15/30/0/90]_T$ laminate.

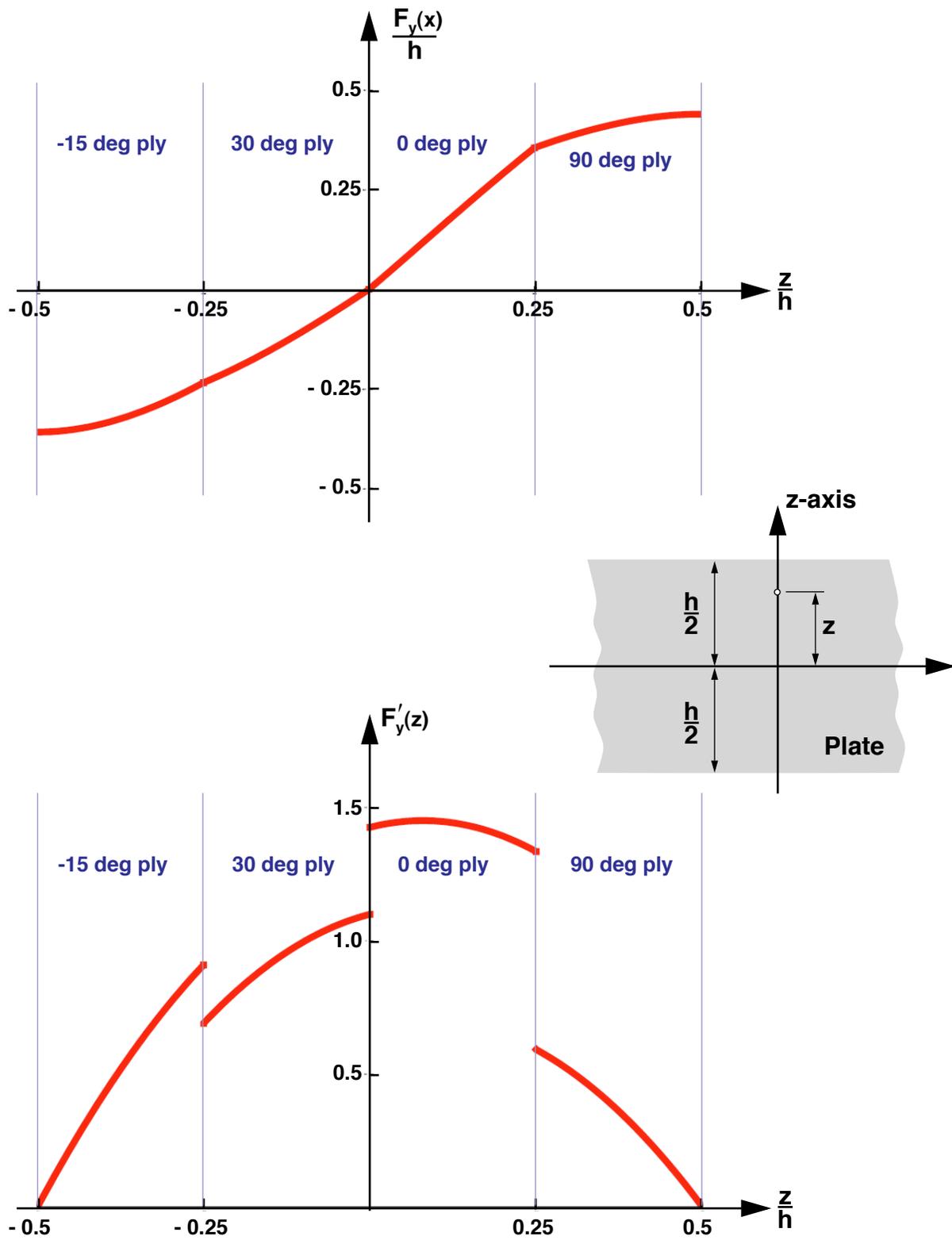


Fig. 11 Graphs of the function $F_y(z)$ and its derivative for $[-15/30/0/90]_T$ laminate.

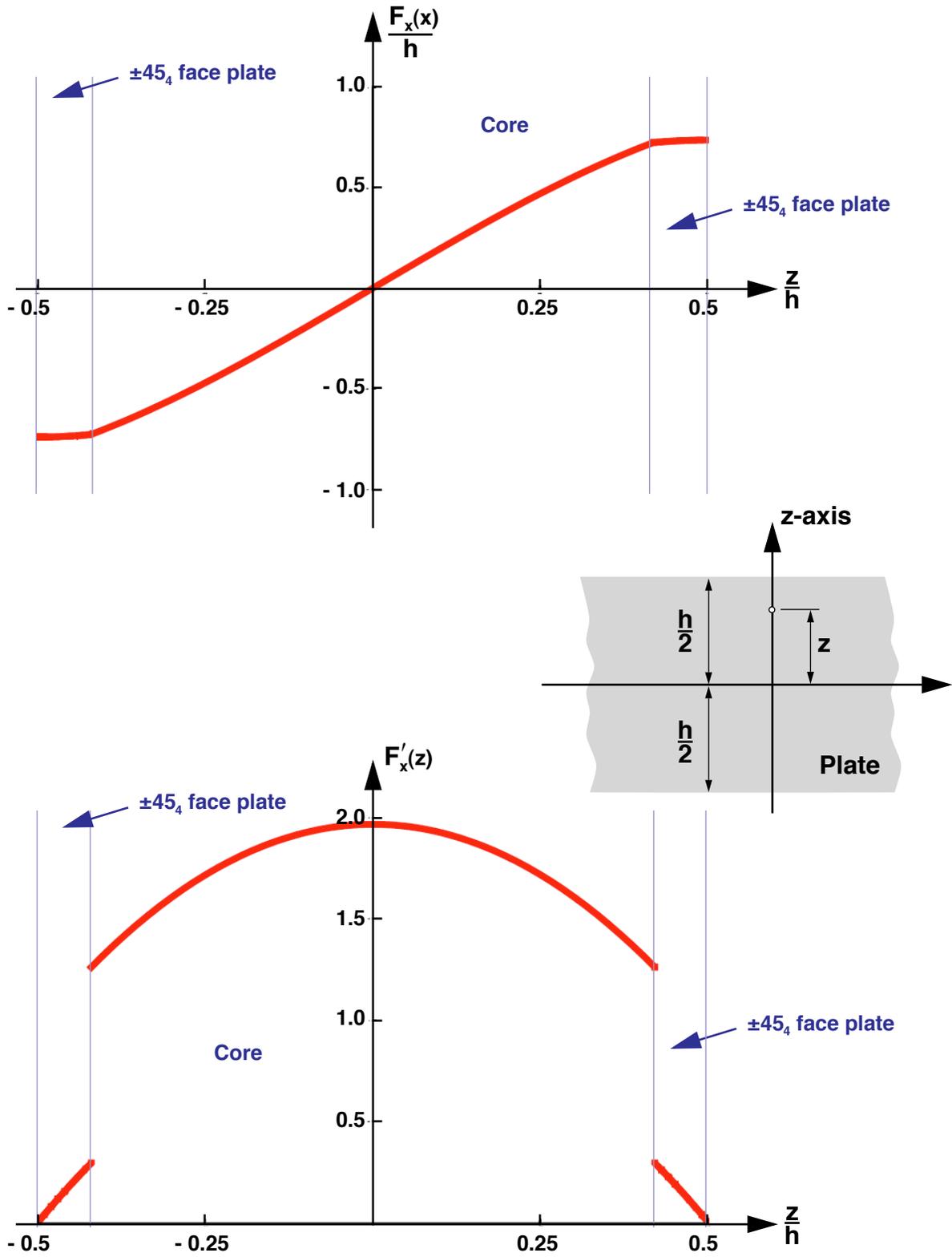


Fig. 12 Graphs of the function $F_x(z)$ and its derivative for $[\pm 45_4/\text{core}]_s$ sandwich plate.

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14. ABSTRACT A detailed analysis and examples are presented that show how to enrich the kinematics of classical Kirchhoff plate theory by appending them with a set of continuous piecewise-cubic functions. This analysis is used to obtain functions that contain the effects of laminate heterogeneity and asymmetry on the variations of the inplane displacements and transverse shearing stresses, for use with a {3, 0} plate theory in which these distributions are specified a priori. The functions used for the enrichment are based on the improved zig-zag plate theory presented recently by Tessler, Di Scuva, and Gherlone. With the approach presented herein, the inplane displacements are represented by a set of continuous piecewise-cubic functions, and the transverse shearing stresses and strains are represented by a set of piecewise-quadratic functions that are discontinuous at the ply interfaces.					
15. SUBJECT TERMS Refined plate theory; layerwise theories; plate kinematics; sandwich plates; zig-zag theories					
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