Analysis and Sizing for Transient Thermal Heating of Insulated Aerospace Vehicle Structures

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Acknowledgments

The author would like to thank NASA Langley Research Center for the opportunity to perform this work under the Floyd Thompson Fellowship. Fellowship advisors, Professors Raphael Haftka and Bhavani Sankar from the University of Florida and Dr. William Garver from Lockheed Martin provided much helpful advice and encouragement. Professor Haftka contributed essential guidance for nondimensionalizing the governing equations and boundary conditions. Dr. Garver initially indicated the strong need for the type of simple design tool developed in this paper. Colleagues, Dr. Kamran Daryabeigi and Carl Poteet provided valuable review and discussion of the work as it progressed.

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Abstract

An analytical solution was derived for the transient response of an insulated structure subjected to a simplified heat pulse. The solution is solely a function of two nondimensional parameters. Simpler functions of these two parameters were developed to approximate the maximum structural temperature over a wide range of parameter values. Techniques were developed to choose constant, effective thermal properties to represent the relevant temperature and pressure-dependent properties for the insulator and structure. A technique was also developed to map a time-varying surface temperature history to an equivalent square heat pulse. Equations were also developed for the minimum mass required to maintain the inner, unheated surface below a specified temperature. In the course of the derivation, two figures of merit were identified. Required insulation masses calculated using the approximate equation were shown to typically agree with finite element results within 10%-20% over the relevant range of parameters studied.

Nomenclature

\(a_1, a_2\) Coefficients of approximate equation for maximum structural temperature

\(b_n\) Coefficients of series solution for \(\tau > \tau_h\)

\(c_m\) Coefficients of series solution for \(0 < \tau \leq \tau_h\)

\(c_{pe}\) Effective insulator specific heat capacity

\(c_{ps}\) Structural specific heat capacity

\(d_e\) Insulator thickness

\(d_s\) Structural thickness

\(f_{thr}\) Fraction of the surface temperature range

\(k_e\) Effective insulator thermal conductivity

\(m\) Mass per unit area

\(m_e\) Insulation mass per unit area

\(m_s\) Structural mass per unit area

\(m_{opt}\) Minimum total mass per unit area

\(m_{so}\) Structural mass/area at which total mass/area is minimum

\(P_{avg}\) Average ambient pressure during equivalent square surface temperature pulse

\(T\) Temperature

\(t\) Time
$T_h$  Applied surface temperature rise of insulator
$t_h$  Duration of heating pulse
$T_i$  Initial temperature
$T_m$  Maximum structural temperature rise
$T_{ce}$  Temperature to use for calculating effective insulation specific heat capacity
$T_{cs}$  Temperature to use for calculating effective structural specific heat capacity
$T_{ha}$  Applied surface temperature of insulator
$T_{ke}$  Temperature to use for calculating effective insulation conductivity
$T_{ma}$  Maximum structural temperature
$T_{mx}$  Maximum surface temperature for a surface temperature history
$T_{thr}$  Threshold temperature for truncating insulator temperature rise integral
$x$  Nondimensionalized spatial variable
$x'$  Spatial variable, position through insulator thickness
$\beta$  Ratio of insulator conductance/area to insulator heat capacity/area
$\beta_s$  Figure of merit for thermal performance of structure
$\delta$  Dirac delta function
$\epsilon$  Error of approximate equation
$\gamma$  Ratio of insulator to structural heat capacity/area
$\kappa_e$  Figure of merit for thermal performance of insulator
$\lambda$  Eigenvalue
$\rho_e$  Effective insulator density
$\rho_s$  Structural density
$\tau$  Nondimensionalized time
$\tau_h$  Nondimensionalized duration of heat pulse
$\tau_m$  Nondimensionalized time of maximum structural temperature
$\theta$  Nondimensionalized temperature
1 Introduction

Thermal protection systems are a critical component of hypersonic and atmospheric entry vehicles. The reusable ceramic tiles and blankets of the Space Shuttle Orbiter work well as thermal insulators, but result in a fragile, high maintenance exterior surface. An intriguing approach to this problem is to build the thermal insulation into the exterior vehicle wall. This deceptively simple idea will be difficult to achieve because it requires a flight weight aerospace vehicle skin to not only carry the required mechanical loads, but also to accommodate severe transient heating with the corresponding hot outer surface and large temperature gradients through its thickness.

A thermally insulating structural panel will likely be of a sandwich construction as a result of both thermal and structural considerations. The outer, heated face sheet of the sandwich will typically be a thin layer of non-insulating material that will contribute little to the thermal response of the inner, unheated face sheet. Thus, ignoring edge closeouts and joints, the transient thermal response of an insulating sandwich panel will be similar in character to the thermal response of an insulated structure like the external tiles and blankets covering the aluminum structure of the Space Shuttle Orbiter. For a non-homogeneous sandwich core, it may be possible to calculate effective thermal properties using a rule of mixtures to approximate its thermal performance.

In this paper, a simplified transient thermal problem was investigated in an attempt to gain basic insight that will be required to develop optimum sandwich panels that can simultaneously insulate and carry structural loads. An analytical solution was derived for the transient response of an insulated structure to a simplified heat pulse. The solution is a function of two nondimensional quantities. Simpler functions of these two parameters were developed to approximate the maximum structural temperature over a wide range of parameter values. Techniques were developed to choose constant, effective properties to represent the relevant temperature and time-dependent thermal properties for the insulator and structure. A technique was also developed to map a time-varying surface temperature history to an equivalent square heat pulse. Using these techniques, the maximum structural temperature rise was calculated using the analytical solutions and compared with finite element simulations over a wide range of parameters. Equations were also developed for the minimum mass for an insulated structure required to maintain the inner, unheated surface below a specified temperature. A figure of merit which correlates to the effectiveness of the heat capacity of the underlying structural material in reducing the amount of required insulation was developed. A second figure of merit was identified for the combination of insulator thermal properties that minimizes the mass of an insulator required to limit the maximum temperature of an underlying structure subjected to a transient heating pulse. Insulation was sized for a number of surface heating histories, insulators, and structural materials by iterating a one-dimensional, finite element analysis and by using the simple equations derived in this paper. Finite element results were correlated to the two figures of merit developed in this paper.
2 Problem Definition

The simplified problem investigated in this paper is illustrated in Fig. 1. A thermal insulator with thickness, $d_e$, density, $\rho_e$, specific heat capacity, $c_{pe}$, and thermal conductivity, $k_e$, covers a structure (inner face sheet) of thickness, $d_s$, density, $\rho_s$, and specific heat capacity, $c_{ps}$. The inner surface of the structure is assumed to be perfectly insulated to simplify the mathematics of the problem and because it is a commonly used conservative assumption for sizing thermal protection systems. To further simplify the problem, the structure is treated as a lumped heat capacitance and the outer face sheet of the insulating sandwich panel is neglected. For this solution, the material properties are assumed to be constant, so effective, averaged properties would have to be used to approximate more complex material behavior.

A simple transient heat pulse is assumed. Initially the insulator and structure are assumed to be at a uniform temperature of 0. The outer surface of the insulator is assumed to instantaneously rise to a temperature $T_h$ at $t = 0$ and maintain that temperature until $t = t_h$ at which time it instantaneously returns to 0.

2.1 Mathematical Description

Using a nondimensional spatial variable, $x = \frac{x'}{d_e}$, the governing differential equation for heat conduction through the insulator can be written as

$$\frac{\partial T}{\partial t} = \beta \frac{\partial^2 T}{\partial x^2}$$  \hspace{1cm} (1)

where

$$\beta = \frac{k_e}{\rho_e c_{pe} d_e^2}$$  \hspace{1cm} (2)
The boundary condition at \( x = 1 \) is defined as
\[
\frac{\partial T(1,t)}{\partial t} = -\beta \gamma \frac{\partial T(1,t)}{\partial x}
\] (3)
where
\[
\gamma = \frac{\rho_e c_p e d_e}{\rho_s c_p s d_s}
\] (4)
The boundary condition at \( x = 0 \) is defined as
\[
T(0,t) = \begin{cases} T_h & \text{for } 0 < t \leq t_h \\ 0 & \text{for } t > t_h \end{cases}
\] (5)
Finally, the initial condition is a uniform temperature of 0.
\[
T(x,0) = 0
\] (6)
The mathematical problem defined by Eqs. 1 through 6 can be completely nondimensionalized [1] by defining a nondimensional time, \( \tau = \frac{\beta t}{\gamma} \) and a nondimensional temperature, \( \theta = \frac{T}{T_h} \). The nondimensionalized differential equation becomes
\[
\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial^2 x}
\] (7)
The boundary condition at \( x = 1 \) becomes
\[
\frac{\partial \theta(1,\tau)}{\partial \tau} = -\gamma \frac{\partial \theta(1,\tau)}{\partial x}
\] (8)
The boundary condition at \( x = 0 \) becomes
\[
\theta(0,\tau) = \begin{cases} 1 & \text{for } 0 < \tau \leq \tau_h \\ 0 & \text{for } \tau > \tau_h \end{cases}
\] (9)
where
\[
\tau_h = \frac{k_e}{\rho_e c_p e d_e} t_h
\] (10)
The initial condition becomes
\[
\theta(x,0) = 0
\] (11)

2.2 Nondimensional Governing Parameters

Inspection of the mathematical problem defined by Eqs. 7 through 11 reveals that any solution will be completely governed by two nondimensional parameters, \( \gamma \) and \( \tau_h \). Each of these parameters has clear physical significance. The first parameter, \( \gamma \), is defined by Eq. 4 which shows it to be the ratio of the heat capacity per unit area of the insulator to the heat capacity per unit area of the underlying structure.

Equation 10 defining the nondimensional heat pulse duration, \( \tau_h \), can be slightly rewritten to show that it is the ratio of insulator conductance per unit area to heat capacity per unit area times the duration of the heat pulse. The same ratio of
insulator conductance per unit area to heat capacity per unit area, \( \beta \), is used to nondimensionalize time.

\[
\tau_h = \frac{k_e}{\rho_e c_p e d_e} t_h
\]  

3 Analytical Solution

An analytical solution for the problem illustrated in Fig. 1 can be obtained by combining two existing solutions with modifications. The solution of the first part of the problem, \( 0 < t \leq t_h \), can be found in the classic heat transfer textbook by Carslaw and Yeager [2]. Converting the solution to the nomenclature used in this paper and nondimensionalizing produces

\[
\theta(x, \tau) = \frac{T(x, \tau)}{T_h} = 1 - \sum_{m=1}^{\infty} c_m \sin(\lambda_m x) e^{-\lambda_m^2 \tau}
\]  

where

\[
c_m = \frac{2(\lambda_m^2 + \gamma^2)}{\lambda_m(\lambda_m^2 + \gamma^2 + \gamma)}
\]  

The values for \( \lambda_m \) can be found by solving the equation

\[
\lambda_m \tan \lambda_m = \gamma
\]  

Solving the second part of the problem, \( t > t_h \), requires more effort. There is an existing solution for a similar problem in which the insulator and lumped mass are initially at a uniform temperature, but at \( t = 0 \) the temperature of the insulator outer surface is instantaneously reduced to 0. A solution to this similar problem, including its derivation, is presented in Reference [3]. A derivation of the solution to the second part of the current problem is presented in Appendix A. The derivation closely follows the approach used by De Chant, however, instead of a uniform temperature, the solution to Eq. 13 at time \( t_h \) is used as the initial temperature distribution.

So, for \( t > t_h \) or \( \tau > \tau_h \) the solution is

\[
\theta(x, \tau) = \frac{T(x, \tau)}{T_h} = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) e^{-\lambda_n^2 (\tau - \tau_h)}
\]  

where

\[
b_n = c_n(1 - e^{-\lambda_n^2 \tau_h}) - \sum_{\substack{m=1 \\text{to} \infty \\text{if} \ m \neq n}} \left( \frac{\sin(\lambda_m - \lambda_n)}{(\lambda_m - \lambda_n)} - \frac{\sin(\lambda_m + \lambda_n)}{(\lambda_m + \lambda_n)} + \frac{2 \sin \lambda_m \sin \lambda_n}{\gamma} \right) \frac{c_m e^{-\lambda_m^2 \tau_h}}{1 - \frac{\sin(2\lambda_n)}{2\lambda_n} + \frac{2 \sin^2 \lambda_n}{\gamma}}
\]  

6
The quantities \( c_m \) and \( c_n \) can be calculated using Eq. 14 and values for \( \lambda_m \) and \( \lambda_n \) are obtained by solving Eq. 15. In Eq. 17 the summation excludes the term \( m = n \) because the contribution of that term is captured by the exponential term preceding the summation.

4 Numerical Example for Series Solution

The series solution described by Eqs. 13 and 16 can be applied to a wide range of physical situations. However, the motivation for deriving the solution was to gain insight into the thermal response of an insulated structure of an aerospace vehicle subjected to a transient aerodynamic heating pulse. Therefore a numerical example was chosen to represent a typical location on the Space Shuttle Orbiter at which the aluminum structure is protected from aerodynamic heating by an LI-900 ceramic tile. The material properties, obtained from Reference [4], are listed in Table 1 along with dimensions, heating duration and the corresponding nondimensional parameters. Thermal properties are temperature dependent, so the properties for the aluminum structure are for 200\( ^\circ F \) and the properties for the LI-900 tile are for 1250\( ^\circ F \). The LI-900 thermal conductivity is also a strong function of pressure, so the conductivity is chosen for a pressure of 0.01 atm (an arbitrary value in the mid-range of a typical ambient pressure history for an atmospheric entry trajectory).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_s )</td>
<td>0.125 (0.003175)</td>
<td>( \text{in(}m\text{)} )</td>
</tr>
<tr>
<td>( \rho_s )</td>
<td>175 (2800)</td>
<td>( \text{lbm/}ft^3(\text{kg/m}^3) )</td>
</tr>
<tr>
<td>( c_{ps} )</td>
<td>0.216 (904)</td>
<td>( Btu/\text{lbm}/^\circ F(\text{J/kg/K}) )</td>
</tr>
<tr>
<td>( d_e )</td>
<td>2,3,4 (0.051,0.076, 0.102)</td>
<td>( \text{in(m)} )</td>
</tr>
<tr>
<td>( \rho_e )</td>
<td>9 (144)</td>
<td>( \text{lbm/}ft^3(\text{kg/m}^3) )</td>
</tr>
<tr>
<td>( c_{pe} )</td>
<td>0.296 (1238)</td>
<td>( Btu/\text{lbm}/^\circ F(\text{J/kg/K}) )</td>
</tr>
<tr>
<td>( k_e )</td>
<td>0.0492 (0.0851)</td>
<td>( Btu/\text{ft/hr}/^\circ F(\text{W/m/K}) )</td>
</tr>
<tr>
<td>( t_h )</td>
<td>25 (1500)</td>
<td>( \text{min(s)} )</td>
</tr>
</tbody>
</table>

A computer program was written using Version 2.7 of the Python programming language to implement the series solution described by Eqs. 13 and 16. Routines from Version 0.9 of the SciPy programming library [5] were used for numerical solutions of nonlinear equations and fitting coefficients to nonlinear equations. For times just after an instantaneous change in surface temperature, many terms of the series solution were required for an accurate solution. However, for the results shown in Figs. 2 through 5, eight terms were used to calculate the coefficients \( b_n \) and three terms to calculate \( c_m \). Use of additional terms produced negligible changes in the results. To generate results for the wide range of parameter values shown in Figs. 6, 7, 9, and 10, twelve terms each were used to calculate \( b_n \) and \( c_m \) to reduce the chance of truncation error.
Three different tile thicknesses, \(d_e\) in Table 1, are considered to gain understanding of the solution described by Eqs. 13 and 16. The nondimensional temperature distributions through the thickness of the insulator at several times are shown in Figs. 2, 3, and 4 for LI-900 tile thicknesses of 2, 3, and 4 in. (0.051, 0.076, 0.102 m) respectively. In each figure, the solid blue line represents the temperature distribution halfway through the heat pulse and the solid green line represents the temperature at the end of the heat pulse, \(\tau = \tau_h\). The dashed lines represent temperature distributions after the heat pulse has ended, \(\tau > \tau_h\). As expected the tile interior and underlying structure heats up more quickly and reaches a higher temperature for thinner tiles. Fig. 2 shows that the structure at \(x = 1\) has already started heating up midway through the heating pulse, \(\tau = \frac{\tau_h}{2} = 0.14\) for a tile thickness of 2 inches. In contrast, Fig. 4 shows that the structure has not even started to heat up at the end of the heat pulse, \(\tau = \tau_h = 0.07\) for a tile thickness of 4 in. This behavior is expected because it should take much longer for the heat to diffuse through twice the thickness of tile.

![Figure 2. Temperature distributions for 2 inch thick tile](image)

After the tile surface temperature drops back to the initial temperature, the heat stored in the tile interior begins to be conducted back out of the cooled outer tile surface. However, the underlying structure, \(x = 1\), continues to increase in temperature until its temperature matches that of the tile material in contact with it, then it begins to cool. This behavior is consistent with the assumption that the structure is perfectly insulated on its inner surface.

For sizing of thermal protection systems, the temperature of the underlying structure is of primary concern. Further insight into the solution can be obtained by calculating the structural temperature as a function of time. Figure 5 shows a
number of temperature histories for different combinations of the governing nondimensional parameters. The ranges of the two parameters were chosen to bound the values calculated for the numerical examples illustrated in Figs. 2 through 4. In Fig. 5, the blue lines represent the structural temperature histories for \( \tau_h = 0.05 \), the
green for $\tau_h = 0.15$, and the red for $\tau_h = 0.3$. The vertical dotted lines represent the end of the heat pulse for the respective values of $\tau_h$. The solid, dashed and dash-dot lines represent values of $\gamma$ of 1, 2, and 3 respectively. Because the nondimensionalized time axis makes the results more difficult to interpret physically, it is helpful to consider the case of fixed material properties and heating duration. The remaining free parameters would be the insulator thickness, $d_e$, and the structural thickness, $d_s$. The parameter $\tau_h$ varies as $\frac{1}{d_e^2}$, so smaller values of $\tau_h$ imply larger values of $d_e$. Choosing a value of $\tau_h$ fixes the insulator thickness, so for a specified value of $\tau_h$, $\gamma$ can only be varied by changing the structural thickness. The parameter $\gamma$ varies as $\frac{1}{d_s}$ so larger values of $\gamma$ imply smaller structural thicknesses. In Fig. 5 the structural temperature histories for $\tau_h = 0.05$ stay much cooler, which is consistent with the expected behavior for thicker insulation. Conversely the structural temperatures reach much higher values for $\tau_h = 0.3$, which is consistent with the expected behavior for thinner insulation. For each value of $\tau_h$ the curve for $\gamma = 1$ reaches the lowest maximum temperature, as expected for the correspondingly highest structural heat capacity. Higher $\gamma$’s result in higher maximum structural temperatures as expected for the associated reduction in structural heat capacity. All of the structural temperature history curves in Fig. 5 exhibit similar behavior. The temperature of the structure continues to rise well after completion of the heating pulse, reaches a maximum value, and then begins to decrease. The maximum temperature is a primary design driver for sizing of thermal protection systems, so it is desirable to be able to readily calculate the maximum structural temperature.
5 Maximum Structural Temperature

All of the temperature histories, away from the heated surface, generated using Eq. 16 would appear to heat up, reach a maximum temperature, and then begin to cool. The time at which the maximum temperature occurs can be found by differentiating Eq. 16 with respect to time, setting it equal to zero, and solving for time. This results in the following equation to be solved for $\tau_m$

\[
0 = -\sum_{n=1}^{\infty} \lambda_n^2 b_n \sin(\lambda_n x) e^{-\lambda_n^2 (\tau_m - \tau_h)}
\]  

Equation 18 can be solved for the time of the maximum temperature for any location through the thickness of the insulator. However, for the current problem, the maximum structural temperature at $x = 1$ is of greatest interest. The equation therefore becomes

\[
0 = -\sum_{n=1}^{\infty} \lambda_n^2 b_n \sin(\lambda_n) e^{-\lambda_n^2 (\tau_m - \tau_h)}
\]  

Although Eq. 19 cannot be readily solved for $\tau_m$ in closed form, it can be solved numerically. The nondimensional times at which the maximum structural temperature occurs for a range of the governing parameters are shown in Fig. 6.

![Figure 6. Times at which maximum structural temperatures occur](image)

The resulting $\tau_m$ can be substituted into Eq. 16 to obtain the maximum structural temperature rise. Following this procedure, the maximum structural temperature rise was calculated over a range of the governing nondimensional parameters.
(0.2 ≤ γ ≤ 5 and 0.02 ≤ τ_h ≤ 0.5). Figure 7 shows a surface plot of the maximum structural temperature rises calculated over a 49 by 49 point grid. The color contours from dark blue to red indicate increasing maximum structural temperature rises. To gain insight into the physical implications of this plot it is again helpful to consider the case of fixed material properties. For fixed properties, small values of τ_h imply a short heat pulse and/or thick insulator and large values imply a long heat pulse and/or thin insulator. It is logical to expect that as the heat pulse duration goes to zero, τ_h → 0, the maximum structural temperature rise will also go to zero. The parameter, γ, is the ratio of insulation to structural heat capacity per unit area, so small values imply that the structural heat capacity per unit area is much larger than that of the insulation and large values imply a relatively small amount of structural heat capacity. Therefore, as the structural heat sink increases towards infinity, γ will approach 0 and, for a finite heating duration, the maximum structural temperature rise will also go to zero. Also, for this problem, the maximum structural temperature cannot exceed the applied surface temperature, so \( \frac{T_m}{T_h} \leq 1. \)

![Figure 7. Maximum structural temperatures](image)

The maximum structural temperature rise plot in Fig. 7 is useful to illustrate how the maximum structural temperature rise varies with the two governing parameters, but it was generated using a complicated series of numerical solutions. For quick calculations, it would be much more useful to have a relatively simple algebraic equation to approximate this surface. The approximate solution should go to zero if either of the governing parameters goes to zero, and it should approach 1 as either parameter goes to infinity. A number of candidate equations were evaluated and the following approximate solution was chosen as a good compromise between simplicity and accuracy.
\[
\left( \frac{T_m}{T_h} \right)_a = 1 - e^{(a_1 \gamma a_2 \tau_h^{2a_2})}
\] (20)

where the subscript \( a \) indicates approximate.

A “least squares” routine was used to find the values of the coefficients in Eq. 20 that best approximate the surface shown in Fig. 7. The coefficient values are given in Table 2.

Table 2. Coefficient values for Eq. 20

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>-0.72058</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0.53649</td>
</tr>
</tbody>
</table>

Inspection of Table 2 reveals that \( a_1 \approx -\frac{1}{\sqrt{2}} \) and \( a_2 \approx \frac{1}{2} \). Substituting these coefficients into Eq. 20 simplifies the equation to

\[
\left( \frac{T_m}{T_h} \right)_a = 1 - e^{-\tau_h \sqrt{\frac{\gamma}{2}}}
\] (21)

The approximate maximum structural temperature rises calculated using Eq. 20 with the coefficient values listed in Table 2 are shown in Fig. 8. The plot is similar to that in Fig. 7.

![Figure 8. Approximate maximum structural temperatures](image-url)
The values from Fig. 7 were subtracted from those of Fig. 8 to calculate the absolute errors resulting from the approximation.

\[ \epsilon_a = \left( \frac{T_m}{T_h} \right)_a - \frac{T_m}{T_h} \]  

(22)

Figure 9 shows the distribution of error over the previously chosen range of governing parameters. Over much of the surface errors are within ±1% of the applied surface temperature rise, \( T_h \). Highest errors occur for small values of \( \gamma \) and large values of \( \tau_h \).

**Figure 9. Errors using approximate maximum structural temperature equation**

It may be more instructive to consider the relative error distribution. The relative errors between the approximation of Eq. 20 and the series solution were calculated using the following equation

\[ \epsilon_r = \frac{\left( \frac{T_m}{T_h} \right)_a - \frac{T_m}{T_h}}{\frac{T_m}{T_h}} \]  

(23)

Figure 10 shows the distribution of the relative error over the previously chosen range of governing parameters. Over most of the parameter space, the relative error is within ±10%. For small values of \( \gamma \), however, the relative error rises precipitately. Therefore it would be prudent to check the values of \( \gamma \) for any calculations made using Eq. 20 to avoid the inaccurate region. This inaccurate region of the approximate solution corresponds to a physical situation where the structural skin is much heavier than the insulator.
6 Comparison of Analytical and Numerical Solutions

The problem illustrated in Fig. 1 is considerably simpler than a typical numerical simulation of aerodynamic heating of an insulated aerospace vehicle structure. Material properties, which can vary with temperature and pressure, are treated as constant. The applied surface temperature is a simple square pulse, rather than a more realistic transient profile. A numerical model was developed to: 1) demonstrate that the series solution, Eqs. 13 and 16, produces the expected results, 2) verify that the governing nondimensional parameters ($\gamma$ and $\tau_h$), rather than individual material property values, determine the calculated temperatures, 3) develop techniques for best using the analytical solution to approximate a realistic numerical simulation, and 4) estimate typical errors involved in using the analytical solution to approximate a realistic numerical simulation.

6.1 Numerical Model

A one-dimensional finite element model was developed using the DOLFIN [6] finite element library for the Python programming language. Using the DOLFIN library, linear one dimensional elements were used to discretize the spatial dimension and an implicit Crank-Nicolson time marching scheme was used to solve the weak formulation of the diffusion equation. The model consisted of a layer of insulator material in perfect contact with a layer of structural material similar to the configuration shown in Fig. 1, except that the structure in the finite element model was
not treated as a lumped heat capacitance. For the results shown in this paper, the model consisted of 50 elements through the thickness of the insulator and 2 elements through the thickness of the structure. Although a careful convergence study was not performed, a model with half the number of elements was shown to produce nearly identical results. The boundary condition on the outer surface of the insulator consisted of an imposed surface temperature that could be varied arbitrarily with time and updated at each time step of the solution. The inner surface of the structure was adiabatic. The material properties could be arbitrary functions of temperature and ambient pressure. Material property values for each finite element were updated at each time step of the analysis for the average temperature of each element from the previous time step. Property values could be different for each element, but did not vary spatially within an element. Time steps between 1 and 5 seconds were used to calculate the results presented in this paper.

When comparing results from the finite element and analytical solutions it is important to understand and correctly handle a subtle difference in how the two solutions treat temperature. Because material properties are constant, the analytical solution deals strictly with temperature rise from a uniform initial temperature (assumed to be zero for simplicity in the derivation). Therefore, \( T_h \) and \( T_m \) have been defined for the analytical solution as maximum temperature rise for the insulator surface and structure, respectively. For the numerical solutions, however, absolute temperatures must be used so that the solution is able to look up the correct temperature dependent material properties. Therefore, it is helpful to define two more variables to clarify this distinction in subsequent discussions.

\[
T_{ma} = T_m + T_i \\
T_{ha} = T_h + T_i
\]  

(24)

### 6.2 Simplified Problem

A finely discretized finite element model of the problem illustrated in Fig. 1 should give exactly the same answer as the analytical series solution defined by Eqs. 13 and 16. Both solution techniques solve the same differential equation with the same boundary and initial conditions.

A numerical example, using the properties, dimensions, and heating values given in Table 1, was studied with the previously described finite element model and the series solution. A thickness of 3 inches for the LI-900 insulation, an initial temperature of 60°F and an applied surface temperature, \( T_{ha} \), of 2000°F were chosen to complete the problem definition. Calculated temperature distributions at several times are shown in Fig. 11. This solution is the same as that shown in nondimensional form in Fig. 3. The symbols in Fig. 11 represent the temperatures calculated at each node of the finite element model. The solid lines represent the series solution defined by Eqs. 13 and 16. The blue and green circles represent distributions halfway through and at the end of the heating pulse, respectively. The triangular symbols represent temperature distributions at several times after the heat pulse has
ended. The finite element and series solutions are essentially in exact agreement, as expected.

Figure 11. Temperature distributions from finite element model vs series solution

The primary concern for thermal protection system sizing is the maximum structural temperature. Figure 12 shows the calculated temperature history of the structure. Again the finite element analysis (temperature gradients in the structure were negligible) and the series solution are essentially in exact agreement, as expected. The red square shown in Fig. 12 was obtained by solving Eq. 19 for the time of maximum structural temperature and substituting that value into Eq. 16 to calculate the maximum structural temperature. The green circle represents the approximate maximum structural temperature calculated using Eq. 20, which is a temperature rise 6% lower than the exact answer.

As shown earlier in the mathematical formulation of the problem, any solution should depend only on the values of the nondimensional governing parameters and not the individual parameter values. A parametric study was performed using the finite element model to confirm that the numerical solution also depends only on the value of the nondimensional parameters. Results of the study are shown in Table 3. Again, the baseline properties are from Table 1 and were also used to generate Figs. 11 and 12. Numerical values of the governing nondimensional parameters and the maximum structural temperatures shown in Fig. 12 are presented in Table 3. For the parametric study, selected combinations of material properties, dimensions, and/or heating time were scaled so that the nondimensional parameter values remain unchanged. Even though individual parameter values were doubled or halved, the calculated maximum structural temperature was the same within 6 significant digits because the nondimensional parameter values were unchanged.
6.3 Material Property Variation

Constant material properties were assumed to obtain the analytical solution. However, the material properties can vary significantly during a transient heating pulse. The specific heat capacities of the structure and insulator, $c_{ps}$ and $c_{pe}$, can vary significantly with temperature. The thermal conductivity of the insulator, $k_e$, can vary greatly with both temperature and ambient pressure. To investigate the effect of variable material properties, the temperature dependent specific heats of the aluminum structure and LI-900 insulation and the temperature and pressure dependent thermal conductivity of LI-900, Tables B1-B3 in Appendix B, were incorporated into the one-dimensional finite element model. In addition, the thermal model for heat transfer through flexible fibrous insulation from Reference [7] was incorporated into a Python computer program and used to predict the effective thermal conductivity of Saffil® insulation in air. The objective of the finite element analysis was to develop a rationale for choosing equivalent constant material properties for the analytical solution. The equivalent properties are chosen to approximate the effect of temperature- and pressure-dependent properties on the calculated maximum structural temperature. The effect of each variable material property was investigated separately with all other material properties held constant.

6.3.1 Structural Specific Heat Capacity

The structural specific heat capacity, $c_{ps}$, was investigated first because it was expected to be the easiest to understand. The structure undergoes the smallest
temperature change, is most isolated from the sharp temperature transients on the surface of the insulator, and undergoes relatively moderate changes in specific heat capacity as temperature varies. The obvious approach would be to use the specific heat capacity at the average of the maximum and initial structural temperatures. The previously described one-dimensional finite element model was used to investigate the effect of temperature dependent $c_{ps}$ on the calculated maximum structural temperature. The finite element analysis was the same as that used for the baseline results in Table 3, except that the constant value of $c_{ps}$ was replaced by the temperature dependent values from Table B1. The results of the finite element analysis with temperature dependent $c_{ps}$ are shown in Table 4. The second column shows the initial temperature, the average structural temperature, and the maximum structural temperature from the finite element analysis. The third column shows the value of the temperature dependent structural specific heat capacity for the initial, average, and maximum structural temperatures, respectively. The corresponding $\gamma$ and $\tau_h$ are shown for each value of $c_{ps}$. These values of $\gamma$ and $\tau_h$ were used to calculate the associated maximum structural temperatures using the series (Eq. 16) and approximate (Eq. 20) solutions. Comparing the quantities in bold faced type in Table 4, it is obvious that using $c_{ps}$ for the average structural temperature in the series solution gives almost exactly (within 0.1%) the same answer as the finite element analysis with temperature dependent $c_{ps}$.

### 6.3.2 Insulator Specific Heat Capacity

The effect of temperature dependency of the insulator specific heat capacity, $c_{pe}$, on maximum structural temperature is a little more complicated. The insulator undergoes much larger temperature excursions than the structure and experiences large, transient temperature gradients whereas the structure has negligible gradients. It is not obvious which value of $c_{pe}$ should be used in the analytical solutions to best match the finite element solution with temperature dependent $c_{pe}$. Again,
the same finite element model was used as that for the baseline results in Table 3, except that the constant value of $c_{pe}$ was replaced by the temperature dependent values from Table B2. Finite element simulations were performed for two different values of maximum surface temperature and three different values of heating time. Results are presented in Table 5. For each simulation $c_{pe}$ values for the insulator were calculated at three different temperatures: initial temperature, maximum structural temperature, and the average between the initial and maximum insulator temperatures. The associated values of the governing nondimensional parameters are shown for each value of $c_{pe}$ along with the corresponding maximum structural temperatures calculated using the series and approximate solutions. Using the $c_{pe}$ values at the maximum structural temperature in the analytical solutions most closely matched the finite element result, as indicated by the bold faced temperatures shown in Table 5 for each simulation. The series solution results are within $\pm 8\%$ of the structural temperature rise predicted by the finite element analysis of this limited set of cases. Therefore, using $c_{pe}(T_{ma})$ as a constant value in the analytical solutions may be an acceptable approach for approximating the effect of temperature dependent $c_{pe}$ on maximum structural temperature.

### 6.3.3 Insulator Thermal Conductivity

Thermal conductivity of the insulator can be a strong function of both temperature and ambient pressure. For simplicity, the effects of pressure and temperature were considered separately.

High aerodynamic heating is generally associated with high speed flight at high altitudes with correspondingly low ambient pressure. Pressure increases rapidly to one atmosphere as the vehicle slows and descends for landing. For an adiabatic structure, the maximum temperature may occur long after the heating pulse is completed. Therefore, a simplified pressure history was investigated. The pressure was assumed to have a uniform value, $P_1$, during the heating pulse and a different uniform value, $P_2$, after the heating pulse. The one-dimensional finite element model was used to investigate the effect of this simplified pressure profile on the calculated maximum structural temperature. Again, the material property values and heating parameters were the same as those used for the baseline results in Table 3, except
Table 5. Effect of temperature dependent insulator specific heat capacity

<table>
<thead>
<tr>
<th>Insulator</th>
<th>Finite Element</th>
<th>$c_{pe}(T)$</th>
<th>$\gamma$</th>
<th>$\tau_h$</th>
<th>$T_{ma\text{ (Series)}}$</th>
<th>$T_{ma\text{ (Eq. 20)}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>$T_i$</td>
<td>288.71</td>
<td>688.0</td>
<td>0.9393</td>
<td>0.22191</td>
<td>426.96</td>
</tr>
<tr>
<td></td>
<td>$T_{ma}$</td>
<td><strong>418.44</strong></td>
<td>909.6</td>
<td>1.2419</td>
<td>0.16784</td>
<td><strong>414.04</strong></td>
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<tr>
<td></td>
<td>$T_{ha+T_1}$</td>
<td>827.59</td>
<td>1209.8</td>
<td>1.6518</td>
<td>0.12620</td>
<td>400.15</td>
</tr>
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</tr>
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<td>$T_i$</td>
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<td>$T_{ma}$</td>
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<td>0.09537</td>
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<tr>
<td></td>
<td>$T_{ha+T_1}$</td>
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<td>1209.8</td>
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<td>0.9393</td>
<td>0.44382</td>
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<td></td>
<td>$T_{ma}$</td>
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<td>0.29016</td>
<td><strong>518.86</strong></td>
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<tr>
<td></td>
<td>$T_{ha+T_1}$</td>
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<td>1209.8</td>
<td>1.6518</td>
<td>0.25239</td>
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<tr>
<td></td>
<td>$T_i$</td>
<td>288.71</td>
<td>688.0</td>
<td>0.9393</td>
<td>0.22191</td>
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<td>$T_{ma}$</td>
<td><strong>355.42</strong></td>
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<td>1.1041</td>
<td>0.14238</td>
<td><strong>347.37</strong></td>
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<td>$T_{ha+T_1}$</td>
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that the constant value of $k_e$ was replaced by the temperature and pressure dependent values from Table B3. The values of $P_1$ and $P_2$ were systematically varied between values of 0.0001, 0.001, 0.01, 0.1, and 1.0 atm. The resulting structural temperature histories are shown in Fig. 13. Different color lines represent different values of $P_1$. Different line patterns represent different values of $P_2$: solid is 1.0 atm, dashed is 0.1 atm, and dotted is 0.01 atm. The curves in Fig. 13 exhibit a consistent pattern. The maximum structural temperature depends only on $P_1$. Different values of $P_2$, cause the time of maximum temperature to vary, but the maximum temperature itself varies less than 0.2K over this wide range of $P_2$ values. Similar results were obtained for Saffil insulation in air.

Thermal conductivity of the insulator is also generally a strong function of temperature. A simple approach for choosing an effective constant value of $k_e$ would be to use the conductivity for a pressure, $P_1$, at a temperature, $T_{ke}$, that is the average of $T_{ha}$ and $T_i$. A slightly more general approach would be to choose a thermal conductivity at a temperature defined by

$$T_{ke} = T_i + f_{ke}T_h$$

(25)

where $f_{ke}$ is the fraction (0 to 1) of the insulator temperature range. A value of 0.5 would result in a simple average of $T_{ha}$ and $T_i$.

A parametric numerical study was performed to determine if there was a sin-
A range of parameters were varied, including \( P_1, T_h, t_h, d_e, \) and \( \rho_e \). All other material properties, except \( k_e \), were held constant. In Table 7 the default value of \( d_e \) is 3 in. (0.0762 m) and the default value of \( \rho_e \) is 4.5 lbm/ft\(^3\) (72 kg/m\(^3\)). For each combination of parameters, the one-dimensional finite element model, with \( k_e(P_1, T_h) \), was used to calculate the maximum structural temperature (column 2 of Tables 6 and 7). The series solution, Eqs. 16 and 19, was solved iteratively to find the fixed value of \( k_e \) that would produce the same maximum structural temperature as the finite element solution. The associated values of \( T_{ke} \) (column 3 of Tables 6 and 7) and \( f_{ke} \) (column 4 of Tables 6 and 7) were also calculated. For the majority of the results shown, \( f_{ke} \) ranged between 0.57 and 0.62. A value of 0.60 was chosen for \( f_{ke} \), and the corresponding values of \( T_{ke}, k_e(P_1, T_{ke}), \) and \( \tau_h \) (column 5 of Tables 6 and 7) were calculated. The resulting approximate maximum structural temperatures were calculated using the series solution (column 6 of Tables 6 and 7) and the approximate solution (column 7 of Tables 6 and 7). For \( f_{ke} = 0.6 \), Tables 6 and 7 show good agreement between the finite element and series solutions for maximum structural temperature. Over most of the parameter ranges the approximate solution is close to the series solution, but starts to diverge for small values of \( \gamma \) (Table 7), as expected from the errors shown in Figs. 9 and 10.
Table 6. Maximum structural temperature calculated using effective LI-900 thermal conductivity

<table>
<thead>
<tr>
<th>$P_1$ atm</th>
<th>Finite Element $T_{ma}, K$</th>
<th>$T_{ke}, K$</th>
<th>$f_{ke}$</th>
<th>$\tau_h$</th>
<th>$T_{ma}$ (Series), $K$</th>
<th>$T_{ma}$ (Eq. 20), $K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>358.72</td>
<td>954.0</td>
<td>0.617</td>
<td>0.07525</td>
<td>356.48</td>
<td>351.14</td>
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<tr>
<td>0.001</td>
<td>368.45</td>
<td>955.9</td>
<td>0.619</td>
<td>0.08577</td>
<td>365.86</td>
<td>360.21</td>
</tr>
<tr>
<td>0.01</td>
<td>399.02</td>
<td>950.6</td>
<td>0.614</td>
<td>0.12071</td>
<td>396.79</td>
<td>390.27</td>
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<td>0.1</td>
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</table>
Table 7. Maximum structural temperature calculated using effective Saffil thermal conductivity \( (t_h = 1500s) \)

<table>
<thead>
<tr>
<th>( P_1 ) atm</th>
<th>Finite Element ( T_{ke}, K )</th>
<th>( f_{ke} )</th>
<th>( \tau_h )</th>
<th>( T_{ma} ) (Series) ( T_{ma} ) (Eq. 20)</th>
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</thead>
<tbody>
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<td>( \rho_e = 72 \ \frac{kg}{m^3} ), ( d_e = 0.0762m ), ( \gamma = 0.8452 ), ( T_{ha} = 1367K )</td>
<td>( f_{ke} = 0.6 ), ( T_{ke} = 935.6K )</td>
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<td>( f_{ke} = 0.6 ), ( T_{ke} = 768.9K )</td>
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<td>( f_{ke} = 0.6 ), ( T_{ke} = 768.9K )</td>
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</table>

\[ \rho_e = 72 \ \frac{kg}{m^3}, \ d_e = 0.0762m, \ \gamma = 0.8452, \ T_{ha} = 1367K \] 
\[ \rho_e = 72 \ \frac{kg}{m^3}, \ d_e = 0.0762m, \ \gamma = 0.8452, \ T_{ha} = 1089K \] 
\[ \rho_e = 72 \ \frac{kg}{m^3}, \ d_e = 0.0762m, \ \gamma = 0.5634, \ T_{ha} = 1089K \] 
\[ \rho_e = 48 \ \frac{kg}{m^3}, \ d_e = 0.0762m, \ \gamma = 0.5634, \ T_{ha} = 1089K \]
6.4 Heating History Profile

For hypersonic aerospace vehicles, surface heating histories can vary greatly with the vehicle mission, trajectory, configuration, and location on the vehicle. Surface heating histories are often provided to a TPS designer in the form of radiation equilibrium heating rates, radiation equilibrium temperatures, convective heating coefficients with total enthalpies or recovery temperatures, or a series of heating rates at constant wall temperatures that bracket the expected surface temperature range. For simplicity, the current study will only consider radiation equilibrium temperature histories, which are suitable for well insulated surfaces that have relatively low thermal mass at the surface. Radiation equilibrium temperature histories can also be readily calculated from the other forms of heating histories.

For the current study, surface temperature histories were chosen from Space Shuttle Orbiter entry flight data [8, 9] and from previous studies [10, 11] of single stage-to-orbit reusable launch vehicles (RLV’s). A total of seven surface temperature histories are shown in Fig. 14. Surface temperature histories from four different body points (BP9740, BP9678, BP9591, and BP9489) during the same atmospheric entry flight of the Shuttle Orbiter were chosen to illustrate a range of different profile shapes. The flight temperature data was smoothed to reduce noise from the crude discretization produced by the flight instrumentation. Two heating histories from an earlier TPS parametric weight study [10], body point A on the windward center line of a winged cylindrical vehicle for the Access to Space study (ATSpA), and body point A on windward centerline of a lifting body reusable launch vehicle (RLVpA) are also shown in Fig. 14. The seventh surface temperature history, also studied in
Figure 15. Ambient pressure histories

Reference [11], was for a point on the windward center line of a slightly different variation of the lifting body vehicle (RLV3c). The corresponding ambient pressure histories are shown in Fig. 15.

The challenge was to find an equivalent square heating pulse for each of the heating histories that would enable the analytical solution to calculate an accurate estimate of the maximum structural temperature. The first step in idealizing the heating histories was to simplify the temperature histories in Fig. 14 by truncating the variable low temperature portions of each history at the beginning and end. The histories were truncated by eliminating any portions below a threshold surface temperature defined by

\[
T_{thr} = T_i + f_{thr}(T_{mx} - T_i)
\]  

(26)

where \(T_{mx}\) is the maximum surface temperature for a surface temperature history, and \(f_{thr}\) is a fraction of the surface temperature range.

Several quantities, including integrated heat load, integrated absolute surface temperature, and integrated surface temperature rise from the initial temperature, were integrated over the time of the truncated temperature histories.

The one-dimensional finite element model with fixed material properties from Table 1 was used to calculate the maximum structural temperature for a structural thickness of 0.125 inch and insulator thicknesses of 1, 2, 3, and 4 inches for each heating history. The resulting maximum structural temperatures were plotted as a function of various integral values, such as integrated heat load, integrated absolute temperature, and integrated surface temperature rise, in hopes of identifying a clear correlation. A linear correlation, Fig. 16, was discovered between the maximum
structural temperature rise, \( T_m \), and the integrated surface temperature rise from the initial temperature, which is defined as

\[
I_T = \int_{t_1}^{t_2} (T - T_i) dt
\]  

(27)

where \( t_1 \) and \( t_2 \) are the beginning and ending times of the truncated temperature history.

Figure 16. Maximum structural temperature vs integrated surface temperature rise

In Fig. 16, the maximum structural temperature rise is shown as a function of the integral defined in Eq. 27. Each marker shape represents one of the seven surface heating histories from Fig. 14. The triangular symbols represent the four shuttle histories and the other symbols represent the RLV histories. The dashed lines represent the best fit line for all the maximum structural temperatures calculated using the finite element model at each insulator thickness. The lines were constrained to pass through the origin. The figure implies that for each configuration with fixed material properties, the maximum structural temperature rise is directly proportional to the integral defined in Eq. 27. For the simple square heat pulse, that integral is simply

\[
I_T = T_h t_h
\]  

(28)

To determine a reasonable equivalent square heat pulse it is helpful to bound the ranges of the two parameters that define it. The maximum possible value of \( t_h \) would be the time span of the truncated surface temperature history, \( t_2 - t_1 \), and the largest that \( T_h \) could be is the maximum surface temperature rise, \( T_{mx} - T_i \). Therefore,
the smallest that \( t_h \) could be is \( I_T / (T_{mx} - T_i) \). For fixed material properties, the series solution was used to calculate \( T_m \) by varying combinations of \( t_h \) and \( T_h \) for fixed values of \( I_T \). The calculated value of \( T_m \) was almost totally a function of \( I_T \) and showed little sensitivity to the particular combinations of \( t_h \) and \( T_h \) over their feasible ranges. However, for variable material properties, it is important to chose a combination of \( t_h \) and \( T_h \) that will enable calculation of constant effective properties and average pressure during the heating pulse, \( P_1 \). A simple approach is to use an average of the bounding values of \( t_h \).

\[
t_h = \frac{(t_2 - t_1) + \frac{I_T}{T_{mx} - T_i}}{2}
\]

(29)

Then \( T_h \) can be readily found from Eq. 28. The resulting square heat pulse can then be centered in the truncated time range between \( t_1 \) and \( t_2 \) for the purposes of calculating an average value of \( P_1 \), \( P_{avg} \).

![Figure 17. Simplified heating and pressure histories for BP9740](image)

This approach for simplifying the surface heating history of Shuttle Orbiter body point 9740 is illustrated in Fig. 17. The solid black line represents the surface temperature history for BP9470. The horizontal blue dotted line represents the threshold temperature defined in Eq. 26 for \( f_{thr} = 0.15 \). The vertical blue dotted lines, \( t_1 \) and \( t_2 \), bound the truncated heating pulse. The shaded area illustrates the integral defined by Eq. 27. The heavy black dashed line represents the equivalent square heating pulse with \( t_h \) calculated from Eq. 29. The square heat pulse has been centered between \( t_1 \) and \( t_2 \) for the purpose of calculating an average pressure to use as \( P_1 \). The ambient pressure over the duration of the square heat pulse is indicated.
by a solid green line. The heavy green dot-dash line represents a simple average of the pressure over the interval of the square heat pulse.

Selecting the threshold temperature, or more specifically $f_{\text{thr}}$, for truncating the surface temperature histories is not straightforward. The intent of truncating the temperature histories is to capture the main heating pulse without any spurious complications from variations in convective cooling effects before landing or the missing initial portion of the heating history (RLV histories). The heating history simplifications illustrated in Fig. 17 were applied to all seven heating histories for three different values of $f_{\text{thr}}$: 0.10, 0.15, and 0.20. Table 8 lists the resulting calculated quantities. The second column shows the maximum surface temperature rise. The third column shows the threshold temperature for the given value of $f_{\text{thr}}$ and the fourth and fifth columns show the times bounding the truncated heating interval. Columns 6 and 8 show the maximum and minimum possible durations for the equivalent square heat pulse and column 9 is their average. The last three columns contain the values that would be used for further calculations using the analytical solution. The integral in column 7 changes little over this range of $f_{\text{thr}}$, which indicates that there should be little effect on the calculated maximum structural temperature for fixed material properties.

Table 8. Simplified surface temperature histories

<table>
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<tr>
<th>Profile</th>
<th>$T_{mx} - T_i$</th>
<th>$T_{thr}$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_2 - t_1$</th>
<th>$T_h$</th>
<th>$T_h - T_i$</th>
<th>$I_T$</th>
<th>$I_T$</th>
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The equivalent square heat pulses, defined by the values of $t_h$ and $T_h$ in Table 8, and the fixed material properties from Table 1 were substituted into the series
solution to calculate the maximum structural temperature rise. The temperature was predicted for a structural thickness of 0.125 in. and insulator thicknesses of 1, 2, 3, and 4 in. for each equivalent square heat pulse and compared to the finite element results shown in Fig. 16. The largest errors in maximum structural temperature rise were -5.4%, -5.9%, and -6.5% for \( f_{thr} \) values of 0.10, 0.15, and 0.20 respectively. Therefore, for fixed material properties, the solution is not sensitive to the threshold temperature over the range considered. However, variable material properties are sensitive to both \( T_h \) and \( P_{avg} \), which vary somewhat with \( f_{thr} \), as shown in Table 8. With these considerations in mind, a value of \( f_{thr} = 0.15 \) was chosen for further calculations.

6.5 Approximation of Realistic Simulations

The key remaining question is: how accurately can the analytical solution, with the previously described effective material properties and simplified square heat pulse, predict the maximum structural temperature calculated using a finite element solution with variable material properties and a time-accurate surface temperature history? The one-dimensional finite element model with temperature-dependent structural properties from Table B1 and insulator properties from Tables B2 and B3 was used to calculate the maximum structural temperature for a structural thickness of 0.125 in. and LI-900 insulator thicknesses of 1, 2, 3, and 4 in. for each heating history.

For comparison, the maximum structural temperature was calculated for each surface temperature history and geometry using the series and approximate analytical solutions, with the following approximations.

\[
c_{ps} = c_{ps}(T_{cs}), \text{ where } T_{cs} = T_i + \frac{T_m}{2}, \tag{30}
\]

\[
c_{pe} = c_{pe}(T_{ce}), \text{ where } T_{ce} = T_i + T_m, \text{ and } \tag{31}
\]

\[
k_e = k_e(T_{ke}, P_{avg}), \text{ where } T_{ke} \text{ is defined in Eq. 25} \tag{32}
\]

Values for \( t_h \), \( T_h \), and \( P_{avg} \) are taken from Table 8 for \( f_{thr} = 0.15 \).

One difficulty is that the maximum structural temperature rise, \( T_m \), must be known to calculate the effective constant properties. This difficulty can be readily overcome by making an initial guess and iterating to a converged solution. The series solution and both approximate analytical solutions, Eqs. 20 and 21 were each iterated independently to arrive at their respective converged solutions.

The finite element and analytical solutions for maximum structural temperature are compared in Fig. 18. The figure shows the maximum structural temperature rise as a function of LI-900 tile thickness. The solid circles represent the finite element solutions, the solid lines represent the series solution, the dashed lines represent the approximate solution given by Eq. 20, and the dotted lines represent the approximate solution given by Eq. 21. The colors correspond to the applied surface temperature histories: black - Shuttle orbiter body point 9470, green - Shuttle
orbiter body point 9489, and red - ATS reusable launch vehicle point A (windward centerline). The series solution agrees with the finite element solution to within -0.9% and 18.7% for the predicted maximum structural temperature rise of the cases shown on Fig. 18. The close agreement between the finite element results and the series solution tends to validate the methodology used to calculate the effective material properties and to map the transient surface temperature history to an equivalent square pulse. The two approximate equations track the series solution and the finite element solution well for most of the LI-900 thickness range, but start to diverge for small thickness values.

Table 9 gives the detailed numerical comparisons between the finite element calculations and results using the equations developed in this paper. The table is divided into seven sections with each section consisting of results for four different insulator thicknesses for a surface temperature history identified in bold print. For each insulation thickness, values of the two governing nondimensional parameters (associated with the series solution) are given, based on the effective constant material properties and equivalent square heating pulse defined previously. The finite element prediction of the maximum structural temperature rise is given for variable material properties and a time-accurate surface heating history. The series and approximate solutions for the maximum structural temperature rise are also shown, along with their errors relative to the finite element solution. The best assessment of the accuracy of using the material property simplifications combined with the surface temperature history simplifications is error associated with the series solution. The approximate solution has the same errors as the series solution combined with the approximation errors illustrated in Fig. 10. The errors for the series so-
olution range from 5.6% to 21.3%. The series solution is remarkably close to the finite element solution, considering the complexity of the numerical simulation and the large variation of material properties that occurs with time. For the much simpler approximate equations, Eq. 20 has errors that range from 1.5% to 38.2% and Eq. 21 has errors that range from 9.6% to 46.0%. The errors for the approximate solutions start to get large for relatively small values of $\gamma$, as expected from the errors shown in Fig. 10. The series solution errors for the reusable launch vehicles (ATSpA, RLVpA, and RLV3c) are less than 10% for all calculated cases.

7 Mass of Insulated Structure

For aerospace vehicles reducing mass is of utmost importance. Therefore, it is helpful to develop further insight into the interplay between the mass of structure and the mass of insulator required to limit the structural temperature. The mass per unit area of this simplified, insulated structure can be written simply as

$$m = m_e + m_s = \rho_e d_e + \rho_s d_s \tag{33}$$

The relationship between $d_e$ and $d_s$ that is required to limit the maximum structural temperature to a specified value is given by Eq. 21. By substituting Eqs. 4 and 10 into Eq. 21 and rearranging terms, the following expression can be obtained.

$$d_s d_e^{\frac{3}{2}} = \frac{(k_e t_h)^2}{2(\rho_s c_{ps})(\rho_e c_{pe}) \left(-\ln\left(1 - \frac{T_m}{T_h}\right)\right)^2} \tag{34}$$

The equation can be further manipulated to give an explicit expression for $d_e$.

$$d_e = \left(\frac{(k_e t_h)^2}{2(\rho_s c_{ps})\rho_e c_{pe} \left(-\ln\left(1 - \frac{T_m}{T_h}\right)\right)^2} \right)^\frac{1}{3} \tag{35}$$

Substituting Eq. 35 into Eq. 33 produces an expression for mass per unit area as a function of structural thickness.

$$m = \left(\frac{\rho_e k_e}{\sqrt{c_{pe}}} t_h^2 \left(-\ln\left(1 - \frac{T_m}{T_h}\right)\right)^2}{2\rho_s d_s c_{ps}} \right)^\frac{1}{3} + \rho_s d_s \tag{36}$$

The first term in Eq. 36 represents the mass of insulator required to limit the structure to the specified maximum temperature and the second term represents structural mass. Eq. 36 can be written as a function of $m_s$. 
Table 9. Maximum structural temperature rise for LI900 insulation

<table>
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33
\[ m = \left( \frac{\left( \frac{\rho k}{\sqrt{cp_e}} \right)^2 t_h^2}{2c_{ps} \left( -\ln(1 - \frac{T_m}{T_h}) \right)^2} \right)^{\frac{1}{3}} \left( m_s - \frac{1}{3} m_s \right) + m_s \]  

(37)

### 7.1 Material Properties and Insulator Mass

The first term in Eq. 37 represents the mass of insulator required to limit the structure to the specified maximum temperature. Inspection of this equation illustrates the effect of key parameters on required insulator mass.

\[ m_e = \left( \frac{\left( \frac{\rho k}{\sqrt{cp_e}} \right)^2}{c_{ps} \left( -\ln(1 - \frac{T_m}{T_h}) \right)^2} \right)^{\frac{1}{3}} \left( \frac{t_h^2}{2m_s} \right)^{\frac{1}{3}} \]  

(38)

From Eq. 38, it is obvious that a longer heat pulse (increased \( t_h \)) will result in a larger required insulator mass. Similarly, a larger structural mass will provide more structural heat sink capacity and require less insulator mass. For a given structural mass and a particular transient heating pulse, required insulation mass will be determined by the thermal properties of the insulator and structure.

The specific heat capacities of insulators and structural materials vary significantly with temperature and the thermal conductivity of low density insulators is a function of both temperature and pressure. Therefore, it is not immediately obvious which material property values to use in the preceding equations to obtain results that are meaningful and useful. However, the previously derived expressions for equivalent constant thermal properties, Eqs. 30, 31, and 32, can be used to calculate values of \( c_{ps} \), \( c_{pe} \), and \( k_e \) for each surface temperature history. The quantities defining the equivalent square heating pulses, \( t_h \), \( T_h \) and \( P_{avg} \), are listed in Table 8 for the seven heating histories previously studied. The first four temperature histories were measured at four different locations on the windward surface of a Shuttle Orbiter during a single atmospheric entry flight [8, 9]. It is interesting to note the variation in the equivalent square heating pulses that occur over just the windward surface in a single flight. The last three heating histories were predicted for a single point on the windward surface of three different proposed reusable launch vehicles [10, 11].

Inspection of Eqs. 37 and 38 reveals that the thermal properties of the structure and insulator can each be grouped separately. These grouped properties have the potential to be useful for comparing the thermal effectiveness of both the structural and insulator materials.

The specific heat capacity and maximum allowable temperature of the structure directly affect the required insulation mass. However, the thermal properties of the structure are coupled to the amplitude of the equivalent square heating pulse, \( T_h \). The following expression is a candidate figure of merit for the thermal effectiveness of the structural material.
\[ \beta_s = c_{ps} \left( -\ln \left( 1 - \frac{T_m}{T_h} \right) \right)^2 \]  

(39)

A larger value of this parameter will lead to a smaller required mass of insulation. Therefore a structure with a high specific heat capacity and high maximum temperature limit will tend to require less insulation.

For steady state heat conduction, minimizing the product of insulator density and thermal conductivity, \( \rho_e k_e \), minimizes the mass of required insulator [11]. However, a similar figure of merit for minimum mass of insulation subjected to transient heating has not been available. The grouping of insulator properties in Eq. 38 suggests a candidate figure of merit for minimum mass insulation for a transient heating pulse (or at least for the simplified heat pulse in this derivation).

\[ \kappa_e = \frac{\rho_e k_e}{\sqrt{c_{pe}}} \]  

(40)

A smaller value of this parameter will lead to a smaller required mass of insulation. Therefore, a low mass insulator for the transient heat pulse should have a combination of low \( \rho_e k_e \) and high \( c_{pe} \).

### 7.2 Minimum Mass of Insulated Structure

The mass of the insulated structure, calculated using Eq. 37, is obviously a function of the mass of the structural material. Increasing the amount of structural mass linearly adds to the total mass, but results in a corresponding decrease in the mass of required insulation. Therefore, in the absence of any overriding structural requirements, there should be a structural mass for which the total mass is minimum. The structural mass for which the total mass is minimum can be readily calculated by taking the first derivative of Eq. 37 with respect to \( m_s \) and setting it equal to zero. The result can be solved for \( m_s \) to produce

\[ m_{so} = 54 \left( -\frac{1}{4} \right) \sqrt{t_h} \left( \frac{\left( \frac{\rho_e k_e}{\sqrt{c_{pe}}} \right)^2}{c_{ps} \left( -\ln \left( 1 - \frac{T_m}{T_h} \right) \right)^2} \right)^{\frac{1}{4}} \]  

(41)

Equation 41 is potentially useful for sizing insulated structure. The structure could first be sized to carry the design structural loads. The resulting equivalent structural mass can then be compared to that calculated using Eq. 41. If the result of Eq. 41 is greater, then increasing the structural mass to that value will result in the lowest combined mass of structure and insulator. However, if the result of Eq. 41 is less, then adding additional structural mass will only increase the combined mass.

Substituting Eq. 41 back into Eq. 37 produces the following expression for the minimum mass of an insulated structure.

\[ m_{opt} = (3 + 1) m_{so} \]  

(42)
\[ m_{opt} = 2 \left( \frac{2}{3} \right)^{\frac{1}{2}} \sqrt{t_h} \left( \frac{\left( \rho c k_e \right)^2}{c_{ps} \left( -\ln \left( 1 - \frac{T_m}{T_h} \right) \right)^2} \right)^{\frac{1}{4}} \]  

The first term in parentheses in Eq. 42 indicates the relative contributions of the insulation and structure, respectively, to the minimum total mass. For minimum total mass, Eq. 42 predicts that the mass of the insulation will be three times the mass of the structure. Eq. 43 defines the lower bound for the mass of an insulated structure subjected to a transient heat pulse. The magnitude of the minimum total mass is solely a function of the duration of the equivalent square heating pulse, \( t_h \), and the figures of merit defined in Eqs. 39 and 40.

8 Analytical Predictions and Numerical Simulations

Several numerical studies were performed to assess how the simple approximate solutions derived in this paper compare to more accurate numerical simulations of Earth atmospheric entry heating on reusable launch vehicles. The objectives of the numerical studies were: 1) to assess the accuracy of using Eqs. 35 and 38 to size insulation for a realistic transient heating pulse, 2) to determine the applicability of the figures of merit defined by Eqs. 39 and 40 to results of atmospheric entry simulations, and 3) to investigate the interplay between the mass of the structure and the corresponding mass of insulation required to limit the maximum structural temperature.

The finite element model and heating histories previously described were used for all of these numerical studies. However, the finite element simulation was iterated to determine the insulation thickness required to limit the structural temperature rise to the specified value. Each transient simulation was continued until the structural temperature had reached its maximum value.

8.1 Thermal Properties of Structural Material

Obviously, raising the structural temperature limit will decrease the amount of required insulation. Less commonly considered is the effect of the structural specific heat capacity on the required insulation. Equation 39 defines a candidate figure of merit that combines the effects of structural temperature limit and specific heat capacity on the amount of required insulation.

A numerical study was performed to assess how well the results of a series of full numerical simulations would correlate to the parameter \( \beta_s \). To compare the effects of the structural thermal properties, it was necessary to fix all other parameters in the problem and vary only the choice of structural material. LI-900 tile was chosen as the insulator and the structural mass per unit area was fixed at 6.10 \( \text{kg/m}^2 \) (1.25 \( \text{lbm/ft}^2 \)). Two different heating histories were considered, but results can only be directly compared for a particular heating history. Table 10 shows thermal properties for four structural
materials chosen for the study. The temperature dependent properties for these materials are tabulated in Reference [11]. The maximum allowable temperature for each material is application dependent, so an approximate upper and lower bound was chosen for each material. For each limiting temperature, the structural specific heat capacity (Eq. 30) and $\beta_s$ for each of the two heating histories are shown in Table 10.

Iterative finite element analysis was used to size insulation for 16 cases (4 structural materials, 2 temperature limits, and 2 heating histories). The resulting insulation masses are shown in Fig. 19 and indicated by the square symbols. For the ATSpA heating history, each particular structural material and temperature limit is identified in the figure. The square symbols are not labeled for the BP7490 heating history to avoid cluttering the figure. The approximate insulation masses, calculated using Eq. 38, are indicated by circles corresponding to each of the 16 cases sized by the finite element solution. For the Space Shuttle heating history, BP7490, half of the approximate solutions are within 10% of the finite element solutions and all of the approximate solutions are within 25%. For the reusable launch vehicle heating history, ATSpA, half of the approximate solutions are within 10% of the finite element solutions and all of the approximate solutions are within 20%. Because the value of $c_{pe}$ used in Eq. 38 is a function of the maximum structural temperature limit (Eq. 31), the approximate insulation masses are not solely a function of $\beta_s$.

The solid lines in Fig. 19 represent the approximate solution using a value of $c_{pe}$ that is an average of the 8 cases for each temperature history. The close agreement between the solid lines and circles indicates that, for LI-900 insulation, the variation

Table 10. Thermal Properties of Four Structural Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho_s$ (kg/m$^3$)</th>
<th>$\beta_s$ (BP7490)</th>
<th>$\beta_s$ (ATSpA)</th>
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<td>2803 (0.10 lbm/in$^3$)</td>
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<td>25 (0.006)</td>
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<td>450 (0.217)</td>
<td>41 (0.010)</td>
<td>38 (0.009)</td>
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<td>Graphite/epoxy</td>
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<td>8 (0.002)</td>
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<td>422 (0.218)</td>
<td>30 (0.007)</td>
<td>27 (0.006)</td>
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<td>Beryllium Aluminum</td>
<td>2098 (0.076 lbm/in$^3$)</td>
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<td>138 (0.033)</td>
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<td>505 (0.404)</td>
<td>267 (0.064)</td>
<td>244 (0.058)</td>
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<td>Titanium 6Al-4V</td>
<td>4437 (0.16 lbm/in$^3$)</td>
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<td>533 (0.137)</td>
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</tr>
<tr>
<td></td>
<td>700 (0.142)</td>
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</table>
Figure 19. Effect of structural thermal properties on required insulator mass

of $c_{pe}$ with $T_m$ has little effect on the mass of required insulation. The solid lines clearly indicate the effect of $\beta_s$ on the required insulation mass predicted by the approximate solution, Eq. 38. Remarkably, the finite element solutions closely follow this trend, even with four different structural materials. Therefore, the parameter $\beta_s$ appears to be a useful indicator of the thermal effectiveness of a structural material.

8.2 Thermal Properties of Insulator

To better understand the significance of the parameter defined in Eq. 40, it is helpful to consider a range of candidate insulator materials that could be used for thermal protection on a reusable launch vehicle. A numerical study was performed to investigate the relative mass efficiency of a range of insulators for two different surface temperature histories (BP7490 and ATSpA) from Table 8. Five rigid ceramic tile materials (LI-900, LI-2200, FRCI-12, AETB-8, AETB-16) and four flexible insulations (Saffil-3, Saffil-6, Qfiber-3, and Qfiber-6) were considered. The temperature and pressure dependent thermal properties of the three Space Shuttle Orbiter tile materials ((LI-900, LI-2200, and FRCI-12) were obtained from published tables [4]. The thermal properties for AETB-8 and AETB-16 were calculated using the model developed by Daryabeigi [12] and the flexible insulation properties were calculated using the models also developed by Daryabeigi [7]. For this study, aluminum with a maximum temperature limit of 450K (350°F) was chosen as the underlying structure. The structural mass per unit area was fixed at $6.10 \text{ kg/m}^2$ ($1.25 \text{ lbm/ft}^2$).

Table 11 shows the effective $c_{pe}$ (Eq. 31) and $k_e$ (Eq. 32) of each insulation for the two surface heating histories. The insulation figure of merit, calculated using

\[
\beta_s = c_{pe} \left( -\ln(1 - \frac{T_m}{T_h}) \right)^2 \left( \frac{W}{kg^2}\right)
\]
Eq. 40, is shown for each set of material properties. As shown in Table 11, the insulation effective material properties and figure of merit, $\kappa_e$, for a given insulation can be significantly different for different surface temperature histories. Therefore, insulations should be directly compared only for a given surface temperature history, structural material, and structural mass.

The iterative finite element analysis, previously described, was used to calculate the required insulation mass for each insulation and surface heating history listed in Table 11. Fig. 20 shows the required insulation masses versus the insulation figure of merit, $\kappa_e$. Each symbol shape represents the finite element solution for each of the insulations represented in Table 11 and the solid line represents the approximate solution calculated using Eq. 38. The red line and symbols correspond to the BP7490 temperature history and the black line and symbols correspond to the ATSpA temperature history. For each temperature history, the finite element results for this wide range of insulators clearly follow the trend predicted by the approximate solution.

For both surface temperature histories the flexible insulation materials are more mass efficient than the rigid tile materials. For the Shuttle Orbiter surface temperature history (BP7490), the bulk of the heating takes place at such a low pressure that gas conduction is not significant and radiation dominates the heat transfer through the insulation. Therefore the density of the flexible insulation (affects the length of gas conduction path) has little effect on the thermal performance, so that all four flexible insulations have nearly the same mass efficiency. Conversely, the bulk of the heating for the reusable launch vehicle surface temperature history (ATSpA) occurs at a higher pressure at which gas conduction through the insulation is significant. Therefore, the lower density flexible insulations are more mass efficient than the higher density flexible insulations. The lower density insulations are more efficient because at a given mass they are thicker and thus provide a longer path for gas conduction. For both surface temperature histories, lower density ceramic tiles tend to be more mass efficient than higher density tiles because heat transfer due to solid conduction becomes more significant with increased density. However, the denser LI-900 tile is more mass efficient than the AETB-8 tile.

### Table 11. Insulation figure of merit values for aerospace insulations

<table>
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<th>Temp. History</th>
<th>$c_{pe}$</th>
<th>$k_e$</th>
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<td></td>
<td></td>
</tr>
<tr>
<td>BP7490</td>
<td>975</td>
<td>0.02420</td>
<td>0.0744</td>
</tr>
<tr>
<td>ATSpA</td>
<td>975</td>
<td>0.05489</td>
<td>0.1688</td>
</tr>
<tr>
<td>Qfiber-3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>932</td>
<td>0.04106</td>
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<tr>
<td>ATSpA</td>
<td>932</td>
<td>0.07328</td>
<td>0.1152</td>
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<tr>
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</tr>
<tr>
<td>ATSpA</td>
<td>932</td>
<td>0.04975</td>
<td>0.1564</td>
</tr>
</tbody>
</table>
The required insulation masses predicted by the approximate equation are within 15% of the finite element solutions.

Figure 20. Required insulation mass for several insulators

8.3 Interplay Between Mass of Insulator and Structure

An additional numerical study was performed to gain a better understanding of the interplay between the mass of the structure and the mass of insulator required to limit the structural temperature. LI-900 ceramic tile was used as the insulator. Aluminum, with a maximum temperature limit of 450K (350°F), was chosen as the underlying structure. The structural mass per unit area was varied between 0.49 and 14.6 kg/m^2 (0.1 and 3.0 lbm/ft^2). For each of the two different surface temperature histories (BP7490 and ATSpA) from Table 8, iterative finite element analyses were performed for thirteen different values of structural mass per unit area to calculate the required insulation mass.

Results of this numerical study are shown in Fig. 21. The red symbols and lines correspond to the BP7490 heating history and the black to the ATSpA heating history. The blue line is the mass of the structure, which was the same for both heating histories. Required insulation masses calculated using the finite element analyses are represented by the solid circles. Dashed lines represent the corresponding approximate insulation masses calculated using Eq. 38 with properties defined by Eqs. 30-32. The solid squares represent the sum of the finite element solutions for insulation mass and the structural mass. The solid lines represent the sum of the approximate solution for insulation mass and the structural mass. The solid triangles represent the minimum total mass calculated using Eqs. 41 and 43.
Comparison of the required insulation masses calculated using the finite element analysis and using Eq. 38 (solid circles vs dashed lines in Fig. 21) shows remarkably good agreement. Over the middle range of structural masses studied the approximate solution is within 15% of the finite element solution for both heating histories. However, for values of structural mass below $1 \frac{kg}{m^2}$, the approximate solution begins to sharply diverge from the finite element solution. Also, at the high end of the range of structural masses, the approximate solution for the BP7490 heating history is beginning to diverge from the finite element solution. This indicates that the approximate solution may become much less accurate when the ratio of insulation mass to structural mass becomes very high or very low. The total mass is obtained by simply adding the structural mass to the insulation mass, so the accuracy comparison between the finite element and approximate solutions is similar to that for the required insulation masses.

As the structural mass increases, the required insulation mass decreases, so there is the potential for a combination of structure and insulator that produces the minimum total mass. For the finite element results, the total mass minimum occurs at a structural mass value of about $1 \frac{kg}{m^2}$ (0.2 lbm ft$^2$) for both heating histories. Equation 41 predicts a minimum at a structural mass of 2.1 $\frac{kg}{m^2}$ for the BP7490 heating history and 3.6 $\frac{kg}{m^2}$ for the ATSpA heating history. Although Eq. 41 does not accurately predict the structural mass at which the total mass is a minimum, the values of the minimum total mass predicted using Eq. 43 are within 10% of the finite element calculations for both heating histories. The three to one ratio of insulator mass to structural mass predicted to occur (Eq. 42) at the minimum mass appears to be an
artifact of error in the approximation.

The total mass curves in Fig. 21 are relatively flat over a significant range of structural masses. Therefore, the mass (and thickness) of the insulator can be reduced by increasing the structural mass, for a relatively modest total mass penalty. Thinner vehicle walls, insulator plus structure, could improve vehicle packaging efficiency, perhaps resulting in a net mass savings.

9 Conclusions

A simplified transient thermal problem was investigated in an attempt to gain basic insight into the thermal response of an insulated structure. A one-dimensional problem, consisting of a homogeneous insulator in perfect contact with an underlying, perfectly insulated structure was defined. From an initial uniform temperature, the outer surface of the insulator is instantaneously raised to an elevated temperature, held at that temperature for finite time, and then instantaneously returned to the initial temperature.

An analytical solution was derived for the transient response of this simplified transient problem. Although the solution is a rather unwieldy infinite series, the thermal response is completely governed by two nondimensional parameters with physical significance. Numerical examples were presented using properties and a heating duration representative of ceramic tiles on the Space Shuttle Orbiter.

The analytical series solution was used to calculate the maximum structural temperatures over a range of the two governing parameters. A simple function of the two governing parameters was constructed and used to approximate the maximum structural temperature over the selected range of the parameters. From this function, two approximate equations were developed for predicting the maximum structural temperature rise of an insulated structure.

Techniques were developed to choose a constant effective value for each of the temperature and pressure dependent material properties of the insulator and structural materials. A technique was also developed for defining an equivalent square heating pulse for a wide range of surface temperature histories associated with atmospheric entry. Analytical solutions for maximum structural temperature rise, using these constant effective material properties and simplified equivalent square heating pulses were compared to finite element solutions with variable material properties and time-accurate surface temperature histories for a range of insulator thicknesses. Results for the analytical series solution were typically within 10% to 20% of the finite element solutions. The approximate analytical solution had similar accuracy for many of the cases studied, but began to lose accuracy as one of the nondimensional governing parameters, the ratio of insulation heat capacity to structure heat capacity, became small.

Approximate analytical equations were developed for sizing the insulator thickness and mass required to maintain the insulated structural skin of hypersonic aerospace vehicles below a specified temperature. Manipulation of the equations revealed that the thermal properties of the insulator and of the structure could each be collected into a single term. These terms could be used as figures of merit to
indicate the effect of the choice of insulator or structural material on the required
insulator mass. An equation was also developed for the minimum total mass of an
insulated structure sized to stay below the maximum structural temperature limit,
ignoring other design considerations.

A one-dimensional, finite element, transient thermal analysis was used to perform
numerical studies for comparison with the results of approximate equations devel-
oped in this paper. The finite element analysis included the effects of temperature
and pressure dependent material properties. Time-accurate surface temperature
histories and ambient pressure histories were also incorporated into the numerical
models.

A numerical study was performed to compare the mass of insulation required
to protect four different structural materials. Two different maximum structural
temperatures were considered for each structural material. The insulation material
and the structural mass were kept the same. Calculations were performed for two
different surface temperature histories. The approximate solution predicted the
required insulation masses to within 10% of the finite element results for more than
half of the cases and to within 25% for all of the cases considered. The finite element
results exhibited a close correlation to the figure of merit, $\beta_s$, derived in this paper.

A second numerical study was performed to compare the mass efficiency of nine
different aerospace insulators, five ceramic tile materials and four flexible insula-
tions. The structural material, mass and maximum temperature limit were held
constant while the required insulation masses were calculated for the same two sur-
face temperature histories. For the insulations considered, the required insulation
masses predicted by the approximate equation were within 15% of the finite ele-
ment solutions. The finite element results were closely correlated to the insulation
efficiency figure of merit, $\kappa_e$, derived in this paper.

A third numerical study was performed to investigate the effect of structural
mass on the mass of required insulation and to determine the minimum mass for an
insulated structure designed with only thermal constraints. The structural material
and maximum temperature, as well as the insulation material, were held constant
and the required insulation masses were calculated for a range of structural masses
and for the same two surface temperature histories. For very small or very large
values of structural mass the approximate solution diverges from the finite element
solutions, however, away from these extremes the approximate solution was within
15% of the finite element solution. The approximate equation for minimum total
mass was within 10% of the finite element solution, however, the approximate equa-
tion predicted the minimum total mass at a much larger value of the structural mass
than the finite element solution.

The approximate equations developed in this paper were shown to predict the
results of much more complex finite element calculations with surprising accuracy
for the wide range of cases considered. Achieving these accurate results requires
carefully following the techniques developed for mapping the transient surface tem-
perature histories to equivalent square temperature pulses and for calculating the
effective property values to use in the approximate equations. These simple approxi-
mate equations are useful for the preliminary investigation of a wide range of design
space to identify attractive regions for more detailed study. The figures of merit
developed in this paper can also be helpful for choosing between available insulation
and structural materials as well as providing guidance for developing more efficient
materials. The successful development of constant, effective values for the temper-
ature and pressure dependent thermal properties of monolithic insulators raises the
intriguing possibility of developing effective properties for more complex sandwich
cores or composite insulations for use in the simple approximations.

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Appendix A

Derivation of Series Solution

In Reference [3] De Chant derives the solution to a problem that is similar to the second portion of the current problem, \( t > t_h \). De Chant’s solution is for a uniform initial temperature distribution with the outer surface temperature of the insulator instantaneously reduced to zero. A solution to the same problem is presented in Reference [2], but it appears to be incorrect – as noted by De Chant and verified in the current effort.

In Reference [3] De Chant defines the following problem (converted to the nomenclature of the current paper) to be solved.

\[
\frac{\partial T}{\partial t} = \beta \frac{\partial^2 T}{\partial x^2} \tag{A-1}
\]

with the boundary condition at \( x = 1 \) as

\[
\frac{\partial T(1,t)}{\partial t} = -\beta \gamma \frac{\partial T(1,t)}{\partial x} \tag{A-2}
\]

the boundary condition at \( x = 0 \) as

\[
T(0,t) = 0 \tag{A-3}
\]

and the initial condition as

\[
T(x,0) = T_{io} \tag{A-4}
\]

The dimension \( d_e \) appears, indirectly through \( \beta \), in Eqs. A-1 and A-2 as a result of nondimensionalizing the spatial variable \( x \).

De Chant then separates variables to get a first order ordinary differential equation in time that can be readily solved to obtain an exponential decay term and an eigenvalue problem in the spatial variable.

\[
\phi''(x) + \lambda^2 \phi'(x) = 0 \tag{A-5}
\]

The eigenfunctions that solve Eq. A-5 are given by

\[
\phi_n(x) = \sin(\lambda_n x) \tag{A-6}
\]

and the eigenvalues can be obtained by solving Eq. 15.

The boundary condition at \( x = 1 \) causes some difficulty in proceeding with the solution. De Chant overcomes this difficulty by developing a “weighting” function.

\[
\sigma(x) = 1 + \frac{1}{\gamma} \delta(x - 1) \tag{A-7}
\]

where \( \delta \) denotes the Dirac delta function.
The weighting function is then used to ensure orthogonality of the eigenfunctions.

\[
\int_0^1 \phi_i(x) \phi_j(x) \sigma(x) dx = 0 \quad (A-8)
\]

The next step in the solution is to satisfy the initial condition by using an eigenvalue expansion.

\[
\frac{T_o(x)}{T_h} = \sum_{n=1}^{\infty} b_n \phi_n(x) \quad (A-9)
\]

However, rather than using a uniform temperature distribution for the initial condition of the second part of the problem, the temperature distribution from Eq. 13 at time \( \tau = \tau_h \) is used as the initial condition.

\[
\frac{T_o(x)}{T_h} = 1 - \sum_{m=1}^{\infty} c_m \sin (\lambda_m x) e^{-\lambda_m^2 \tau_h} \quad (A-10)
\]

Application of the orthogonality relationship, Eq. A-8, yields

\[
b_n = \frac{\int_0^1 \frac{T_o(x)}{T_h} \phi_n(x) \sigma(x) dx}{\int_0^1 \phi_n^2(x) \sigma(x) dx} \quad (A-11)
\]

Substituting Eqs. A-6 and A-7 into Eq. A-11 and applying the Dirac delta function produces

\[
b_n = \frac{\int_0^1 \frac{T_o(x)}{T_h} \sin(\lambda_n x) dx + \frac{1}{7} \frac{T_o(1)}{T_h} \sin \lambda_n}{\int_0^1 \sin^2(\lambda_n x) dx + \frac{\sin^2 \lambda_n}{\gamma}} \quad (A-12)
\]

Further substitution of Eq. A-10 into Eq. A-12 and rearranging terms gives

\[
b_n = \left( \int_0^1 \sin(\lambda_n x) dx + \frac{\sin \lambda_n}{\gamma} \right) - \sum_{m=1}^{\infty} \left( \int_0^1 c_m \sin (\lambda_m x) \sin (\lambda_n x) dx + c_m \sin \lambda_m \sin \lambda_n \right) e^{-\lambda_n^2 \tau_h} \int_0^1 \sin^2(\lambda_n x) dx + \frac{\sin^2 \lambda_n}{\gamma} \right) \quad (A-13)
\]

Integrating Eq. A-13 produces

\[
b_n = \left( \frac{1-\cos \lambda_n}{\lambda_n} + \frac{\sin \lambda_n}{\gamma} \right) - \sum_{m=1}^{\infty} \left( \frac{\lambda_n \sin \lambda_m \cos \lambda_n - \lambda_m \sin \lambda_n \cos \lambda_m}{\lambda_n^2 - \lambda_m^2} + \frac{\sin \lambda_m \sin \lambda_n}{\gamma} \right) c_m e^{-\lambda_n^2 \tau_h} \quad (A-14)
\]

The first term in the numerator represents the solution for the uniform initial temperature problem. It can be shown to be the same as the coefficients defined in Eq. 14. Using this observation and trigonometrical identities for the product of sine and cosine, Eq. A-14 can be rewritten as

47
\[ b_n = c_n - \sum_{m=1}^{\infty} \left( \frac{\sin(\lambda_n - \lambda_m)}{\lambda_n - \lambda_m} - \frac{\sin(\lambda_n + \lambda_m)}{\lambda_n + \lambda_m} + \frac{2 \sin \lambda_m \sin \lambda_n}{\gamma} \right) c_m e^{-\lambda_m^2 \tau_h} \]  

(A-15)

Inspection of Eq. A-15 reveals a potential numerical problem when \( m = n \). The first term in the large parentheses of Eq. A-15 approaches 1 in the limit as \( \lambda_m \to \lambda_n \). Therefore, for \( n = m \) the quantity in the large parentheses becomes equivalent to the denominator and cancels out. Eq. A-15 can thus be rewritten as

\[ b_n = c_n (1 - e^{-\lambda_n^2 \tau_h}) - \sum_{m=1 \atop m \neq n}^{\infty} \left( \frac{\sin(\lambda_m - \lambda_n)}{\lambda_m - \lambda_n} - \frac{\sin(\lambda_m + \lambda_n)}{\lambda_m + \lambda_n} + \frac{2 \sin \lambda_m \sin \lambda_n}{\gamma} \right) c_m e^{-\lambda_m^2 \tau_h} \]  

(A-16)

which is the same as Eq. 17. A simple offset in the time variable completes the solution shown in Eq. 16.
Appendix B

Material Properties

Table B1. Properties for aluminum 2024 (Ref. [4])

<table>
<thead>
<tr>
<th>Temperature $^\circ R(K)$</th>
<th>$c_p$ Btu/lbm/$^\circ R(J/kg/K)$</th>
<th>$k$ Btu/ft/hr/$^\circ R(W/m/K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>160 (89)</td>
<td>0.177 (741)</td>
<td>61.2 (106)</td>
</tr>
<tr>
<td>260 (144)</td>
<td>0.147 (615)</td>
<td>69.6 (120)</td>
</tr>
<tr>
<td>460 (256)</td>
<td>0.195 (816)</td>
<td>84.0 (145)</td>
</tr>
<tr>
<td>660 (367)</td>
<td>0.216 (904)</td>
<td>95.0 (164)</td>
</tr>
<tr>
<td>760 (422)</td>
<td>0.224 (937)</td>
<td>99.0 (171)</td>
</tr>
<tr>
<td>860 (478)</td>
<td>0.233 (975)</td>
<td>102.5 (177)</td>
</tr>
<tr>
<td>1060 (589)</td>
<td>0.250 (1046)</td>
<td>104.5 (181)</td>
</tr>
</tbody>
</table>

Table B2. Specific heat capacity for LI-900 (Ref. [4])

<table>
<thead>
<tr>
<th>Temperature $^\circ R(K)$</th>
<th>$c_p$ Btu/lbm/$^\circ R(J/kg/K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>460 (256)</td>
<td>0.150 (628)</td>
</tr>
<tr>
<td>710 (394)</td>
<td>0.210 (879)</td>
</tr>
<tr>
<td>960 (533)</td>
<td>0.252 (1054)</td>
</tr>
<tr>
<td>1210 (672)</td>
<td>0.275 (1151)</td>
</tr>
<tr>
<td>1460 (811)</td>
<td>0.288 (1205)</td>
</tr>
<tr>
<td>1710 (950)</td>
<td>0.296 (1238)</td>
</tr>
<tr>
<td>1960 (1089)</td>
<td>0.300 (1255)</td>
</tr>
<tr>
<td>2210 (1228)</td>
<td>0.303 (1268)</td>
</tr>
<tr>
<td>2460 (1367)</td>
<td>0.303 (1268)</td>
</tr>
<tr>
<td>3460 (1922)</td>
<td>0.303 (1268)</td>
</tr>
</tbody>
</table>
### Table B3. Thermal conductivity for LI-900 (Ref. [4])

<table>
<thead>
<tr>
<th>T, °R</th>
<th>k, Btu/(ft hr °R)</th>
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</thead>
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<tr>
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<td>0</td>
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<tr>
<td>460</td>
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<tr>
<td>710</td>
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<td>960</td>
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<td>0.0233</td>
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<tr>
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<tr>
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<td>0.1540</td>
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<tr>
<td>3460</td>
<td>0.1900</td>
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</table>
An analytical solution was derived for the transient response of an insulated structure subjected to a simplified heat pulse. The solution is solely a function of two nondimensional parameters. Simpler functions of these two parameters were developed to approximate the maximum structural temperature over a wide range of parameter values. Techniques were developed to choose constant, effective thermal properties to represent the relevant temperature and pressure-dependent properties for the insulator and structure. A technique was also developed to map a time-varying surface temperature history to an equivalent square heat pulse. Equations were also developed for the minimum mass required to maintain the inner, unheated surface below a specified temperature. In the course of the derivation, two figures of merit were identified. Required insulation masses calculated using the approximate equation were shown to typically agree with finite element results within 10%-20% over the relevant range of parameters studied.

15. SUBJECT TERMS
Insulated structure; Insulation; Thermal analysis; Thermal protection system