Indirect M-MRAC for Systems with Time Varying Parameters and Bounded Disturbances

Vahram Stepanyan and Kalmanje Krishnakumar

Abstract—The paper presents a prediction-identification model based adaptive control method for uncertain systems with time varying parameters in the presence of bounded external disturbances. The method guarantees desired tracking performance for the system’s state and input signals. This is achieved by feeding back the state prediction error to the identification model. It is shown that the desired closed-loop properties are obtained with fast adaptation when the error feedback gain is selected proportional to the square root of the adaptation rate. The theoretical findings are confirmed via a simulation example.

Index Terms—Time-varying parameters, guaranteed transient performance, disturbance rejection;

I. INTRODUCTION

Adaptive control has been considered as a promising technology to improve stability and performance of uncertain systems. However, limitations of conventional adaptive methods (see for example recent results in [2], [4], [5], [9] for the systems with time varying uncertainties in the system can be compensated for.

During past two decades majority of the efforts have been directed to improving the transient of the tracking error (see for example recent results in [2], [4], [5], [9] for the systems with time varying uncertainties), but not the control signal, the behavior of which significantly contributes to the aforementioned limitations.

These limitations have been addressed in the indirect adaptive control schemes, which is the reason to name the approach indirect M-MRAC. However, our prediction model differs from the conventional ones by a prediction error feedback term, which turns out to play the same role as the tracking error feedback term plays in the direct M-MRAC approach. Hence, the desired closed-loop behavior can be achieved with fast adaptation by selecting a proper error feedback gain, which also separates the time scale of the adaptive estimation from that of the system’s dynamics.

The rest of the paper outlines the properties of the proposed indirect M-MRAC control architecture and demonstrates the benefits of it in a simulation example.

II. PROBLEM STATEMENT

Consider the system
\[ \dot{x}(t) = Ax(t) + BA(t) [u(t) + K(t)g(x(t)) + d(t)] \]
with \( x(0) = x_0 \), where \( x \in \mathbb{R}^n \) is the state of the system, \( u \in \mathbb{R}^q \) is the control, \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times q} \) are given constant matrices with \( A \) being Hurwitz and \((A,B)\) controllable, \( g : \mathbb{R}^n \rightarrow \mathbb{R}^p \) is continuously differentiable, \( \Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n} \) is positive definite with bounded and piecewise continuous unknown entries, which have bounded derivatives, \( K : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times q} \) is an unknown parameter matrix with bounded and piecewise continuous entries, which have bounded derivatives, and \( d : \mathbb{R}^+ \rightarrow \mathbb{R}^q \) is bounded and piecewise continuous disturbance with a bounded derivative.

The control objective is to design a control input such that the system (1) tracks the reference model.

\[ \dot{x}_m(t) = Ax_m(t) + Br(t) \]
with \( x_m(0) = x_{m0} \), where \( r : \mathbb{R}^+ \rightarrow \mathbb{R}^q \) is bounded and piecewise continuous command with a bounded derivative.

We notice that the system (1) can be represented in the form
\[ \dot{x} = A \dot{x} + Br(t) + BA(t) [u(t) + \Theta(t)f(x,r) + d(t)] \] ,
where \( \Theta(t)f(x,r) = K(t)g(x(t)) - \Lambda^{-1}(t)r(t) \). Let
\[ \|A(t)\|_{\infty} \leq \lambda_*^a, \quad \|\Theta(t)\|_{\infty} \leq \vartheta_\ast, \quad \|d(t)\|_{\infty} \leq d_\ast \]
\[ \|\dot{\Lambda}(t)\|_{\infty} \leq \lambda_*^d, \quad \|\dot{\Theta}(t)\|_{\infty} \leq \vartheta_d^\ast, \quad \|d(t)\|_{\infty} \leq d_d^\ast \]

In this paper, we extend the approach to the systems with time varying parameters using a prediction (or identification) model based approach. The parameters are assumed to be bounded and vary with bounded derivatives, no matter how large these bounds are. The parameter estimates are generated using the state prediction error as in the case of conventional indirect adaptive control schemes, which is the reason to name the approach indirect M-MRAC. However, our prediction model differs from the conventional ones by a prediction error feedback term, which turns out to play the same role as the tracking error feedback term plays in the direct M-MRAC approach. Hence, the desired closed-loop behavior can be achieved with fast adaptation by selecting a proper error feedback gain, which also separates the time scale of the adaptive estimation from that of the system’s dynamics.

The rest of the paper outlines the properties of the proposed indirect M-MRAC control architecture and demonstrates the benefits of it in a simulation example.

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III. Prediction Model

We introduce the following adaptive prediction model

\[
\dot{x}(t) = A\dot{x}(t) + Br(t) + BΛ(t)[u(t) + \hat{θ}(t)f(x, r) + \dot{d}(t)] + k\tilde{x}(t)
\]

with \( \tilde{x}(0) = \tilde{x}_0 \), where \( \tilde{x}(t) = x(t) - \hat{x}(t) \) is the prediction error, \( k > 0 \) is a design parameter, \( Λ(t), \hat{θ}(t) \) and \( \dot{d}(t) \) are the estimates of the unknown quantities, generated according to adaptive laws

\[
\begin{align*}
\dot{\hat{θ}}(t) &= \gamma Pr \left( \hat{θ}(t), B^T P\tilde{x}(t) f^T (x, r) \right), \\
\dot{Λ}(t) &= \gamma Pr \left( Λ(t), B^T P\tilde{x}(t) [u(t) + \hat{θ}(t) f(x, r)]^T \right), \\
\dot{\dot{d}}(t) &= γ Pr \left( \dot{d}(t), B^T P\tilde{x}(t) \right),
\end{align*}
\]

where \( γ > 0 \) is the adaptation rate, \( P = P^T > 0 \) is the solution of the Lyapunov equation \( A^T P + PA = -Q \) for some \( Q = Q^T > 0 \), and \( Pr(\cdot, \cdot) \) denotes the projection operator [7], which is defined as \( Pr(\hat{θ}, y) = [I - G(\hat{θ})]y \), where

\[
G(\hat{θ}) = \begin{cases} 
0, & \text{if } \varphi(\hat{θ}) < 0, \\
0, & \text{if } \varphi(\hat{θ}) ≥ 0, \nabla \varphi^T(\hat{θ}) y ≤ 0, \\
\frac{\nabla \varphi(\hat{θ})}{∥\nabla \varphi(\hat{θ})∥}, & \text{if } \varphi(\hat{θ}) ≥ 0, \nabla \varphi^T(\hat{θ}) y > 0.
\end{cases}
\]

with the notation \( \nabla \varphi(\hat{θ}) = \frac{∂ \varphi(\hat{θ})}{∂ θ} \), and the smooth convex functions \( \varphi(\hat{θ}) \) is given by \( \varphi(\hat{θ}) = \frac{\nabla \varphi(\hat{θ})}{\max{\nabla \varphi(\hat{θ})}} \) \( \text{with } \theta_{\text{max}} \text{ denotes the norm bound imposed on the parameter matrix } \hat{θ} \text{ and } \epsilon_0 \text{ denotes the convergence tolerance.} \)

The projection operator has the following properties

Lemma 3.1: [7] Let \( θ₀ ∈ Ω₀ = \{ \hat{θ} ∈ ℜⁿ | \varphi(\hat{θ}) ≤ 0 \} \), and let the parameter \( θ(t) \) evolve according to the dynamics

\[
\dot{θ}(t) = Pr(\hat{θ}(t), y), \quad \hat{θ}(t₀) ∈ Ω.
\]

Then \( θ(t) ∈ Ω₁ = \{ θ ∈ ℜⁿ | \varphi(θ) ≤ 1 \} \) or \( ∥θ(t)∥ ≤ θ^* \) for all \( t ≥ t₀ \), where \( θ^* = \sqrt{A + \epsilon_0 θ_{\text{max}}} \), 2) \( [\hat{θ}(t) - θ₀]^T Pr(θ(t), y) - y] ≤ 0 \) for all \( t ≥ t₀ \).

It is straightforward to verify that

\[
\dot{x}(t) = (A - kI)\hat{x}(t) + A\hat{θ}(t)f(x, r) + \dot{d}(t) + BΛ(t)[u(t) + \hat{θ}(t)f(x, r) + \dot{d}(t)] + k\tilde{x}(t),
\]

where \( \hat{θ}(t) = θ(t) - Ω(t) = Λ(t) - \dot{Λ}(t) \) and \( \dot{d}(t) = d(t) - \dot{d}(t) \) are the estimation errors.

Lemma 3.2: If \( \hat{x}(t) = x₀ \), then the prediction error \( \tilde{x}(t) \) satisfies the bound

\[
∥\tilde{x}(t)∥ ≤ \frac{c}{\sqrt{λ_{\text{min}}(P)}} e^{-kt} \sqrt{2\gamma},
\]

where \( c = c₁ + c₂ \), \( c₁ = 4λ^2 d₁^2 + 4λ^2 θ^2 + 4λ^2, \text{and } c₂ = 4λ^2 θ^2 d₂ + 4λ^2 d² + 4λ^2 θ^2 d^2 \).

Proof: The derivative of the candidate Lyapunov function

\[
V(t) = \tilde{x}^T(t)P\tilde{x}(t) + γ^{-1}d^T(t)Λ(t)\tilde{d}(t) + γ^{-1}tr(\hat{θ}^T(t)Λ(t)\dot{θ}(t) + \hat{λ}^T(t)Λ(t)\dot{λ}(t)),
\]

computed along the trajectories of the prediction error dynamics (8) and the adaptive laws (6), satisfies the inequality

\[
\dot{V}(t) ≤ -x^T(t)Q\tilde{x}(t) - 2kd^T(t)P\tilde{x}(t) + 2γ^{-1}tr(\hat{θ}^T(t)Λ(t)\dot{θ}(t) + \hat{λ}^T(t)Λ(t)\dot{λ}(t))
\]

\[
+ γ^{-1}d^T(t)Λ(t)\tilde{d}(t) + γ^{-1}tr(\hat{θ}^T(t)Λ(t)\dot{θ}(t)).
\]

The projection operator in the adaptive laws (6) guarantees the inequalities \( ∥θ(t)∥ ≤ θ^*, ∥Λ(t)∥ ≤ Λ^* \) and \( ∥d(t)∥ ≤ d^* \). Therefore \( ∥θ(t)∥ ≤ 2θ^*, ∥Λ(t)∥ ≤ 2Λ^*, ∥d(t)∥ ≤ 2d^* \) and \( 2tr(\hat{θ}^T(t)Λ(t)\dot{θ}(t)) + 2d^T(t)Λ(t)\tilde{d}(t) + d^T(t)Λ(t)\tilde{d}(t) + γ^{-1}tr(\hat{θ}^T(t)Λ(t)\dot{θ}(t)) ≤ c₂ \). That is

\[
V(t) ≤ -x^T(t)Q\tilde{x}(t) - 2kd^T(t)P\tilde{x}(t) + γ^{-1}c₂.
\]

On the other hand we have \( d^T(t)Λ(t)\tilde{d}(t) = tr(\hat{θ}^T(t)Λ(t)\dot{θ}(t) + \hat{λ}^T(t)Λ(t)\dot{λ}(t)) ≤ c₁ \). It follows that \( V(t) ≤ x^T(t)P\tilde{x}(t) + γ^{-1}c₁ \). Therefore, if \( V(t) > γ^{-1}c₁ \), for some \( τ \) then \( x^T(t)P\tilde{x}(t) = (2kγ^2)\tilde{c}_2 \), which implies that \( V(t) < 0 \). Since \( \tilde{x}(0) = 0 \) it follows that \( V(0) ≤ γ^{-1}c₁ < γ^{-1}c \). Therefore \( V(t) ≤ γ^{-1}c \) for all \( t ≥ 0 \).

Since \( ∥\tilde{x}(t)∥^2 ≤ x^T(t)P\tilde{x}(t)/λ_{\text{min}}(P) ≤ V(t)/λ_{\text{min}}(P) \), the inequality (9) follows.

It can be observed from Lemma 3.2 that the state prediction error can be decreased as desired by increasing the adaptation rate \( γ \), when the prediction model is precisely initialized. The next lemma shows that the initialization error results in an additive exponentially decaying term.

Lemma 3.3: If \( \hat{x}_₀ ≠ x₀ \), then \( x(t) \) satisfies the bound

\[
∥\tilde{x}(t)∥ ≤ \sqrt{\frac{c_₃}{λ_{\text{min}}(P)}} e^{-kt} + \sqrt{\frac{c}{γλ_{\text{min}}(P)}},
\]

where \( c_₃ = |V(0) - \bar{V}|, \text{and } V(t) \text{ is defined by (10).} \)

Proof: Using the same \( V(t) \) as in Lemma 3.2 and following the same steps one can arrive to the inequality

\[
V(t) ≤ -2k[V(t) - γ^{-1}c₁] + γ^{-1}c₂,
\]

integration of which results in

\[
V(t) ≤ \left[V(0) - \frac{c}{γ}\right] e^{-2kt} + \frac{c}{γ} ≤ c₃ e^{-2kt} + \frac{c}{γ}.
\]

Recalling that \( ∥\tilde{x}(t)∥^2 ≤ V(t)/λ_{\text{min}}(P) \), we readily obtain

\[
∥\tilde{x}(t)∥ ≤ \frac{1}{λ_{\text{min}}(P)} \sqrt{c₃} e^{-kt} + \frac{c}{γ}.
\]

Taking into account the inequality \( √a + b ≤ √a + √b \) for any \( a ≥ 0, b ≥ 0 \), the bound (13) is concluded.
IV. CONTROL DESIGN

Since the reference model is designed to satisfy the robustness and performance specifications, one would naturally select the control signal

$$u(t) = -\Theta(t)f(x,r) - d(t).$$

(17)

to achieve the control objective, if the system (3) were completely known. Obviously, \(u(t)\) is not implementable, therefore its adaptive version

$$\hat{u}(t) = -\hat{\Theta}(t)f(x,r) - \hat{d}(t)$$

(18)
is used. When this control signal is applied, the prediction model (5) reduces to the modified reference model introduced in the M-MRAC architecture, that is

$$\dot{x}(t) = A\hat{x}(t) + B\hat{r}(t) + k\hat{x}(t),$$

(19)

Lemma 4.1: Let the system (3) be controlled by the adaptive control \(\hat{u}(t)\) (18). Then closed loop signals are bounded.

**Proof:** Under the action of the adaptive control \(\hat{u}(t)\), the error between the prediction model and reference model \(e_m(t) = \hat{x}(t) - x_m(t)\) satisfies the equation

$$\dot{e}_m(t) = A e_m(t) + \lambda \hat{x}(t),$$

(20)

Since \(A\) is Hurwitz, and \(\hat{x}(t)\) is bounded according to Lemma 3.2, it follows that \(e_m(t)\) is bounded. Since the input \(\hat{r}(t)\) is bounded, the reference model’s state \(x_m(t)\) is bounded, therefore the predicted state \(\hat{x}(t)\) is bounded as well. Then, it follows that the system’s state \(x(t)\) is bounded. The parameter estimates are guaranteed to be bounded by the projection operator, therefore \(\hat{u}(t)\) is also bounded.

Lemma 4.2: Let the system (3) be controlled by the controller (18), which is defined by the prediction model (5) and the adaptive law (6). Then

$$\|\hat{u}(t)\| \leq \beta_1 e^{-\nu t} + \beta_2 \gamma^t + \frac{\alpha_2}{2},$$

(21)

where \(\hat{u}(t) = u(t) - \hat{u}(t)\), and \(\nu, \beta_1, \beta_2\) and \(\alpha_2\) are positive constants to be specified in the proof.

**Proof:** It is easy to show that \(\hat{u}(t)\) satisfies the equation

$$
\begin{bmatrix}
\dot{u}(t) \\
\dot{\hat{u}}(t)
\end{bmatrix}
= 

\begin{bmatrix}
0 & I_{q \times q} \\
-\gamma F(t)L(t) & -k_{\lambda}I_{q \times q}
\end{bmatrix}
\begin{bmatrix}
u(t) \\
\hat{u}(t)
\end{bmatrix}

+ 

\begin{bmatrix}
0 \\
\gamma
\end{bmatrix}
\begin{bmatrix}z_1(t) \\
1
\end{bmatrix}
\begin{bmatrix}z_2(t)
\end{bmatrix},

(22)

where we denote \(\rho(t) = f^T(x,r)f(x,r) + 1, F(t) = \rho(t)I_{q \times q} - H(t), \hat{H}(t) = G(\Theta)\hat{f}f(x,r) + G(d), L = B^TP\Lambda(t)(L(t) is positive definite), z_1(t) = \hat{\rho}(t)B_0^TP + \rho(t)B_0^TPA_m\hat{x}(t), \) and \(z_2(t) = -\Theta^T(t)f(x,r) - \hat{d}(t).\) Since \(x(t)\) is bounded, it follows that \(\rho(t)\) is bounded. That is, there exists a positive constant \(\alpha_1\) such that \(1 \leq \rho(t) \leq \alpha_1\) for all \(0 \leq t < \infty.\) On the other hand, it follows from the definition of the projection operator that \(\|G(\Theta)\| \leq 1\) and \(\|G(d)\| \leq 1.\) Therefore \(F(t)\) is bounded. Further, it follows from the dynamics (3) that \(\dot{x}(t)\) is bounded. Therefore \(\rho(t)\) and \(z_2(t)\) are bounded. That is, there exist positive constants \(\alpha_2\) and \(\alpha_3\) such that \(|\rho(t)| \leq \alpha_2\) and \(\|z_2(t)\| \leq \alpha_3\) for all \(0 \leq t < \infty.\) Hence, (22) can be considered as a second order linear system in \(\hat{u}(t)\) with time varying coefficients, where the adaptation rate \(\gamma\) determines the frequency of \(\hat{u}(t)\) and the feedback gain \(k\) determines the damping ratio. We notice that selection of the initial parameter estimates inside the convex sets defined by the projection operator results in \(\hat{H}(t) = 0\) on some initial interval \([0, t_1].\) Therefore, \(F(t) = \rho(t)I_q\) on \([0, t_1].\) Let \(a_0 = \alpha_1^2 + \lambda\), where \(\lambda = \max_{t \geq 0} \lambda(L(t))\) and \(\lambda_0 = \min_{t \geq 0} \lambda(L(t)).\) Denoting \(E(t) = a_0\hat{z}_q - \rho(t)L(t),\) we can write

$$
\begin{bmatrix}
\dot{u}(t) \\
\dot{\hat{u}}(t)
\end{bmatrix}
= 

\begin{bmatrix}
0 & I_{q \times q} \\
-\gamma a_0I_{q \times q} & -k_{\lambda}I_{q \times q}
\end{bmatrix}
\begin{bmatrix}
u(t) \\
\hat{u}(t)
\end{bmatrix}

+ 

\begin{bmatrix}
0 \\
\gamma
\end{bmatrix}
\begin{bmatrix}z_1(t) \\
1
\end{bmatrix}
\begin{bmatrix}z_2(t)
\end{bmatrix},

(23)

the solution of which has an equivalent integral form

$$\begin{bmatrix}
u(t) \\
\hat{u}(t)
\end{bmatrix}
= 

\begin{bmatrix}
u(0) \\
\hat{u}(0)
\end{bmatrix} + 

\gamma \int_{0}^{t} \psi_2(t - \tau)|z_1(\tau) + E(\tau)\hat{u}(\tau)|d\tau + \int_{0}^{t} [\psi_1(t - \tau) + k\psi_2(t - \tau)]z_2(\tau) d\tau,$$

(24)
where \( \psi(t) = [\psi_1(t), \psi_2(t)] \) is the first row of the transition matrix of the LTI part of system (23). Following [8], we select
\[
k = 2\sqrt{\gamma a_0},
\]
which results in the minimum norm \( \|\psi_2(t)\|_{L_1} = (\gamma a_0)^{-1} \). For the same \( k \), we have \( \|\psi_1(t) + k\psi_2(t)\|_{L_1} \leq 4(\gamma a_0)^{-1/2} \). Since \( \|E(t)\|_{L_\infty} = a_0 - \lambda_0 \), we obtain \( 1 - \gamma\|\psi_2(t)\|_{L_1} \|E(t)\|_{L_\infty} = 2\lambda_0 a_0^{-1} \). Then, according to [8], it follows from the expression (24) that
\[
\|
\tilde{u}(t)
\| \leq b_1(\|\tilde{u}(0)\| + \|\tilde{u}(0)\|e^{-\nu_1 t}) + \frac{1}{2\lambda_0} \|z_1(t)\|_{L_\infty} + \frac{4\sqrt{a_0}}{2\lambda_0} \|z_2(t)\|_{L_\infty},
\]
where \( b_1 \) is a positive constant and
\[
\nu_1 = -\frac{\gamma}{2} \left( \sqrt{a_0} - \sqrt{a_0 - \lambda_0} \right).
\]

From the definition of \( z_1(t) \) and Lemma 3.2 we have
\[
\|z_1(t)\|_{L_\infty} \leq (\alpha_2\|B^TP||\alpha_1\|B^TPA||e^{-\nu_1 t}).
\]
Then, it is straightforward to obtain the bound (21) with
\[
\beta_1 = \frac{b_1(\|\tilde{u}(0)\| + \|\tilde{u}(0)\|)}{2\lambda_0},
\]
\[
\beta_2 = \frac{4\alpha_3\sqrt{a_0}}{2\lambda_0} + \frac{\alpha_2\|B^TP||\alpha_1\|B^TPA||e^{-\nu_1 t}}{2\lambda_0\sqrt{\gamma a_0}}.
\]
This concludes the proof.

V. Tracking Error

In this section we derive a norm bound for the tracking error \( e(t) = x(t) - x_m(t) \), which is given by the following lemma.

\textbf{Lemma 5.1:} Let the system (1) be controlled by the controller (18), which is defined by the prediction model (5) and the adaptive law (6). Then
\[
\|e(t)\| \leq \beta_3 e^{-\nu_1 t} + \beta_4 \gamma^{-\frac{1}{2}},
\]
\( \beta_3 \) and \( \beta_4 \) are positive constants to be specified in the proof.\n\textit{Proof:} It is straightforward to obtain the tracking error dynamics in the form
\[
\dot{e}(t) = Ae(t) - B\Lambda(t)\tilde{u}(t).
\]
Since \( A \) is Hurwitz, it follows that there exist positive constants \( b_2 \) and \( \nu_2 \) such that \( \|e^{At}\| \leq b_2 e^{-\nu_2 t}. \) Therefore the following bound can be obtained
\[
\|e(t)\| \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau \leq b_2\|e(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|e^{-\nu_1 t} \int_0^t e^{\nu_2(t-\tau)} d\tau } \]
which can be expressed in the form of (28) with
\[
\beta_3 = b_2\|e(0)\| + \frac{b_2\beta_1}{\nu_2 - \nu_1} \|B\Lambda(t)\|_{L_\infty},
\]
\[
\beta_4 = \frac{1}{\nu_1} b_2\beta_2 \|B\Lambda(t)\|_{L_\infty}.
\]
closed-loop dynamics, which is not achievable by conventional adaptive methods [1].

VI. ILLUSTRATIVE EXAMPLE

In this section, the advantages of the proposed indirect M-MRAC architecture are demonstrated in simulations for a dynamic model that represents the lateral-directional motion of a generic transport aircraft (GTM) [6]. The nominal model is the linearized lateral-directional dynamics of GTM at the altitude of 30,000 ft and speed of 0.8M and is given by

\[ \dot{x}(t) = A_n x(t) + B_n u(t), \]

where \( x = [\beta \ r \ \phi]^\top \) is the lateral-directional state vector, in which \( \beta \) is the sideslip angle, \( r \) is the yaw rate, \( p \) is the roll rate, \( \phi \) is the bank angle, and \( u = [\delta_a \ \delta_r]^\top \) is the control signal that includes the aileron deflection \( \delta_a \) and the rudder deflection \( \delta_r \), and the numerical values for \( A_n \) and \( B_n \) are

\[
A_n = \begin{bmatrix}
-0.1578 & -0.9907 & 0.0475 & 0.0404 \\
2.7698 & -0.3842 & 0.0240 & 0 \\
-10.1076 & 0.5000 & -1.7520 & 0 \\
0 & 0.0506 & 1.0000 & 0
\end{bmatrix},
\]

\[
B_n = \begin{bmatrix}
0.0042 & 0.0476 \\
0.0351 & -2.2464 \\
6.3300 & 1.7350 \\
0 & 0
\end{bmatrix}.
\]

The reference model is selected from the perspective of improving the performance characteristics of the nominal dynamics and is given by the equation (2), where \( A = A_n - B_n K \) and \( B = B_n N \), with the feedback and feedforward matrices

\[
K = \begin{bmatrix}
0 & 0 & 0.43 & 0.55 \\
1.92 & -1.5 & 0 & 0
\end{bmatrix},
N = \begin{bmatrix}
1.26 & 0.65 \\
3.33 & -0.07
\end{bmatrix}.
\]

The reference model is driven by a command, which is chosen to be a series of coordinated turn maneuvers. That is, sideslip angle command is set to zero and the bank angle command is chosen to be a series of coordinated turn maneuvers. That and

\[
\Theta(t) = \begin{cases}
\begin{bmatrix}
-0.1820 & 0.0149 & -0.1049 & 0 \\
0.0807 & -0.0109 & 0.0168 & 0
\end{bmatrix}, & t \leq 20 \\
-0.2268 & 0.0209 & -0.1053 & 0 \\
-0.8514 & 0.0692 & 0.0003 & 0
\end{cases},
\]

\[
\Lambda(t) = \begin{cases}
\begin{bmatrix}
0.5401 & 0.0167 \\
-0.0632 & 1.0524
\end{bmatrix}, & t \leq 20 \\
0.5413 & -0.0492 \\
0.0408 & 0.4225
\end{cases},
\]

The uncertain model of GTM roughly corresponds to 28% loss of left wing tip at \( t = 0 \) sec, and 55% loss of rudder surface and vertical tail at \( t = 20 \) sec. Its dynamics are in the form of the equation (3) with piecewise constant \( \Theta(t) \) and \( \Lambda(t) \), and \( f(x) = x \). The corresponding numerical values are

The external disturbance is chosen to be a sinusoid of amplitude 0.1 and frequency \( 2\pi/3 \) rad/sec in the yaw channel and a square wave of amplitude 0.15 and frequency \( \pi/3 \) rad/sec in the roll channel. The disturbance magnitude corresponds to 8.6 degrees of aileron deflection and 5.7 degrees of rudder deflection. In the definition of the projection operator the conservative bounds \( \lambda^* = \theta^* = d^* = 10 \) are used.

First, a simulation is run with \( \gamma = 1000 \), \( Q = I_4 \) and \( k \) is computed according to (25), where we used conservative bounds \( \lambda_0 = 0.2 \) and \( \lambda^* = 2 \). Figures 1 and 2 display the tracking performance of the states. Clearly good tracking is achieved with the chosen controller gains, for which the control time history is presented in Figure 3. It can be observed that the adaptive control signal closely follows the reference signal given by (17). Small spikes in the control signal are attributed to the discontinuities of the disturbance. For the comparison purposes we also present the conventional MRAC performance with the same setup. It can be observed from the Figure 4 that MRAC achieves output tracking with small oscillations in sideslip angle. However, the rates and the control surface deflection commands are experiencing unacceptable...
oscillations (see Figures 5 and 6).

Next we increase the adaptation rate 10 fold. As it can be viewed from Figures 7 and 8, the output and input tracking performances are substantially improved as predicted. Computations show that the tracking error is decreased more than $\sqrt{10}$ fold, implying the the derived bounds are conservative. Further increase of adaptation rate to $\gamma = 100000$ further improves the system’s input and output performance (see Figures 9 and 10), which verifies the theoretical derivations.

VII. CONCLUSIONS

We have presented indirect modified reference model MRAC (M-MRAC) approach to uncertain systems with time varying parameters and bounded external disturbances without imposing "slow variation" restriction on the system’s parameters. The method uses a prediction error feedback term to speed up the adaptive estimation process, which results in predictable transient behavior for both state and input variables of the system. It has been shown that the unwanted high frequency effects of the fast adaptation in the control signal can be regulated by the proper choice of the feedback gain.

REFERENCES