A Step-Wise Approach to Elicit Triangular Distributions

Presented by:

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“It is better to be approximately right rather than precisely wrong.

Warren Buffett
Outline

• Purpose of Presentation
• Background
  – The Uncertainty Spectrum
  – Expert Judgment Elicitation (EE)
  – Continuous Distributions
    • More details on Triangular, Beta & Beta-PERT Distributions
• Five Expert Elicitation (EE) Phases
• Example: Estimate Morning Commute Time
  – Expert Elicitation (EE) to create a Triangular Distribution
    • With emphasis on Phase 4’s Q&A with Expert (2 iterations)
  – Convert Triangular Distribution into a Beta-PERT
• Conclusion & Potential Improvements
Adapt / combine known methods to demonstrate an expert judgment elicitation process that …

1. Models expert’s inputs as a triangular distribution
   – 12 questions to elicit required parameters for a bounded distribution
   – Not too complex to be impractical; not too simple to be too subjective

2. Incorporates techniques to account for expert bias
   – A repeatable Q&A process that is iterative & includes visual aids
   – Convert Triangular to Beta-PERT (if overconfidence was addressed)

3. Is structured in a way to help justify expert’s inputs
   – Expert must provide rationale for each of his/her responses
   – Using Risk Breakdown Structure, expert specifies each risk factor’s relative contribution to a given uncertainty (of cost, duration, reqt, etc.)

This paper will show one way of “extracting” expert opinion for estimating purposes. Nevertheless, as with most subjective methods, there are many ways to do this.
The Uncertainty Spectrum

No Estimate Required

Total Certainty = Complete information

Specific Uncertainty

Objective Probabilities

Partial information

Known unknowns

General Uncertainty

Subjective Probabilities

No information

Unknown unknowns

Total Uncertainty = No information

No Estimate Possible

Reference: Project Management Consulting by AEW Services, 2001

Expert opinion is useful when little information is available for system requirements, system characteristics, durations & cost
Expert Judgment Elicitation (EE)

Source: Making Hard Decisions, An Introduction to Decision Analysis by R.T. Clemen
Triangular Distribution

- Used in situations were there is little or no data
  - Just requires the lowest \((L)\), highest \((H)\) and most likely values \((M)\)

Each \(x\)-value has a respective \(f(x)\), sometimes called “Intensity” that forms the following PDF:

\[
f(x) = \begin{cases} 
\frac{2(x - L)}{(M - L)(H - L)}, & L \leq x < M \\
\frac{2(H - x)}{(H - M)(H - L)}, & M \leq x < H \\
0, & \text{otherwise}
\end{cases}
\]

\(L, M \& H\) are all that’s needed to calculate the Mean and Standard Deviation:

\[
\mu = \frac{(L + M + H)}{3}
\]

\[
\sigma = \sqrt{\frac{(L^2 + M^2 + H^2 - L*M - L*H - M*H)}{18}}
\]
**Beta Distribution**

Bounded on \([0,1]\) interval, scale to any interval & very flexible shape

\[
f(x) = \frac{1}{H-L} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{x-L}{H-L} \right)^{\alpha-1} \left( \frac{H-x}{H-L} \right)^{\beta-1} \quad L < x < H
\]

\[= 0 \text{ otherwise}
\]

Shape Parameters: \(\alpha > 0, \beta > 0\)

\[\beta > \alpha > 1, \text{ distribution is right skewed}\]

Calculated Gamma values using Excel’s \text{GAMMALN} function:

\[
\Gamma(\alpha) = \text{EXP}[\text{GAMMAIN}(\alpha)]
\]

\[
\Gamma(\beta) = \text{EXP}[\text{GAMMAIN}(\beta)]
\]

Most schedule or cost estimates follow right skewed pattern. But how do we know \(\alpha\) and \(\beta\)? Answer: Beta-PERT Distribution.

Sources:
2. LaserLight Networks, Inc, “Beta Modeled PERT Schedules”
Beta-PERT Distribution

Requires lowest \((L)\), highest \((H)\) & most likely values \((M)\)

\[\mu = \frac{L + \lambda \cdot M + H}{\lambda + 2}\]
\[\sigma = \frac{H - L}{6}\]

Use \(L, M\) and \(H\) to calculate mean \((\mu)\) and standard deviation \((\sigma)\):

\[\alpha = \frac{(\mu - L)}{(H - L)} \cdot \frac{(\mu - L)(H - \mu)}{\sigma^2} - 1\]

\[\beta = \frac{(H - \mu)}{(\mu - L)} \cdot \alpha\]

where \(\alpha > 0, \beta > 0\)

To calculate shape parameters, \(\alpha\) & \(\beta\):

\[f(x) = \frac{1}{(H - L)} \cdot \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \cdot \left(\frac{x - L}{H - L}\right)^{\alpha - 1} \cdot \left(\frac{H - x}{H - L}\right)^{\beta - 1}\]

\(L < x < H\)

\[\Gamma(\alpha + \beta) = \text{EXP}[\text{GAMMALN}(\alpha + \beta)]\]
\[\Gamma(\alpha) = \text{EXP}[\text{GAMMALN}(\alpha)]\]
\[\Gamma(\beta) = \text{EXP}[\text{GAMMALN}(\beta)]\]

Sources:
2. LaserLight Networks, Inc, “Beta Modeled PERT Schedules”
**Expert Elicitation (EE) Phases**

Expert Elicitation consists of five phases:  
*(note that Phases 4 & 5 are iterative)*

1. Motivating the expert
2. Training (conditioning) the expert
3. Structuring objective, assumptions & process
4. Assessing (encoding) expert’s responses
   - Q&A – Expert’s technical opinion is elicited
   - Quantitative results w/ documented rationale
5. Verifying encoded values & documentation

*Our Example will emphasize the Phase 4 Q&A*
Example: Estimate Commute Time

• Why this example?
  – Fairly easy to find a subject matter expert
  – It is a parameter that is measurable
  – Most experts can estimate a most likely time
  – Factors that drive uncertainty can be readily identified
  – People generally care about their morning commute time!

Let’s begin with Phase 1 … Motivating the Expert:

1. Motivating the expert
   • Explain the importance & reasons for collecting the data
   • Explore stake in decision & potential for motivational bias
EE Phase 2: Commute Time

2. Structuring objective, assumptions & process

- Be explicit about what you want to know & why you need to know it
  - Clearly define variable & avoid ambiguity and explain data values that are required (e.g. hours, dollars, %, etc)

The Interviewer should have worked with you to develop the Objective and up to 5 Major Assumptions in the table below

- Please resolve any questions or concerns about the Objective and/or Major Assumptions prior to continuing to "Instructions".

Objective: Develop uncertainty distribution associated with time (minutes) it will take for your morning commute starting 1 October 2014.

Assumption 1: Your commute estimate includes only MORNING driving time
Assumption 2: The commute will be analogous to the one you've been doing
Assumption 3: Period of commute will be from 1 Oct 2014 thru 30 Sep 2015
Assumption 4: Do not try to account for extremely rare & unusual scenarios
Assumption 5: Unless you prefer otherwise, time will be measured in minutes
3. Training (conditioning) the expert
   • Go over instructions for Q&A process
   • Emphasize benefits of time constraints & 2 iterations

Instructions: This interview is intended to be conducted in two Iterations. Each iteration should take no longer than 30 minutes.

A. Based on your experience, answer the 12 question sets below.
B. Once you've completed the questions, review them & take a 15 minute break.
C. Using the triangular graphic to assist you, answer all of the questions again.

Notes:

A. The 2nd iteration is intended to be a refinement of your 1st round answers.
B. Use lessons-learned from the 1st iteration to assist you in the 2nd iteration.
C. Your interviewer is here to assist you at any point in the interview process.
For 2 Questions, you’ll need to provide your assessment of likelihood:

<table>
<thead>
<tr>
<th>Descriptor</th>
<th>Explanation</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolutely Impossible</td>
<td>No possibility of occurrence</td>
<td>0.0%</td>
</tr>
<tr>
<td>Extremely Unlikely</td>
<td>Nearly impossible to occur; very rare</td>
<td>1.0%</td>
</tr>
<tr>
<td>Very Unlikely</td>
<td>Highly unlikely to occur; not common</td>
<td>10.0%</td>
</tr>
<tr>
<td><em>Indifferent between&quot;Very Unlikely&quot; &amp; &quot;Even chance&quot;</em></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Even Chance</td>
<td>50/50 chance of being higher or lower</td>
<td>50.0%</td>
</tr>
<tr>
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<td></td>
<td></td>
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<td>90.0%</td>
</tr>
<tr>
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<td>99.0%</td>
</tr>
<tr>
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<td>100.0%</td>
</tr>
</tbody>
</table>

Example: Assume you estimated a "LOWEST" commute time of 20 minutes.
Your place a value = 10.0% as the probability associated with "Very Unlikely."

Therefore:
- a) You believe it’s "VERY UNLIKELY" your commute time will be less than 20 minutes, and
- b) This is equal to a 10.0% chance that your commute time would be less than 20 min.
EE Phase 4: **Commute Time** (iteration 1)

Question 1: Expert creates “value-scale” tailored his/her bias …

*What probability would you assign to a value that's "Very Unlikely"*

*What probability would you assign to a value that's "Extremely Unlikely"*

**Available Selection of Values to the Expert** (shaded cells were selected by expert):

<table>
<thead>
<tr>
<th>VERY LIKELY</th>
<th>VERY UNLIKELY</th>
<th>EXTREMELY LIKELY</th>
<th>EXTREMELY UNLIKELY</th>
</tr>
</thead>
<tbody>
<tr>
<td>80.0%</td>
<td>20.0%</td>
<td>96.0%</td>
<td>4.0%</td>
</tr>
<tr>
<td>82.5%</td>
<td>17.5%</td>
<td>97.0%</td>
<td>3.0%</td>
</tr>
<tr>
<td>85.0%</td>
<td>15.0%</td>
<td>98.0%</td>
<td>2.0%</td>
</tr>
<tr>
<td>87.5%</td>
<td>12.5%</td>
<td>98.5%</td>
<td>1.5%</td>
</tr>
<tr>
<td>90.0%</td>
<td>10.0%</td>
<td>99.0%</td>
<td>1.0%</td>
</tr>
<tr>
<td>92.5%</td>
<td>7.5%</td>
<td>99.5%</td>
<td>0.5%</td>
</tr>
<tr>
<td>95.0%</td>
<td>5.0%</td>
<td>99.9%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>
Question 1: Expert creates “value-scale” tailored his/her bias …

What probability would you assign to a value that's "Very Unlikely"?

What probability would you assign to a value that's "Extremely Unlikely"?

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<td></td>
<td>30.0%</td>
</tr>
<tr>
<td>Even Chance</td>
<td>50/50 chance of being higher or lower</td>
<td>50.0%</td>
</tr>
<tr>
<td>Indifferent between &quot;Very Likely&quot; &amp; &quot;Even chance&quot;</td>
<td></td>
<td>70.0%</td>
</tr>
<tr>
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<td>Highly likely to occur; common occurrence</td>
<td>90.0%</td>
</tr>
<tr>
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<td>99.0%</td>
</tr>
<tr>
<td>Absolutely Certain</td>
<td>100% Likelihood</td>
<td>100.0%</td>
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</table>

Only 2 probabilities needed to be elicited in order to create a Value-Scale that has 9 categories!
Given the objective and assumptions …

2. Describe input parameter (WBS 4): Commute (in minutes) as Triangular Dist’n

3. What’s the Most Likely value, M? 50

4. Adjust M (if applicable) 55

5. What’s the chance actual value could exceed M? N/A

6. What’s the Lowest value, L 42

7. What’s the chance actual value could be less than L? Indifferent-Low

8. What’s the Highest value, H 80

9. What’s the chance actual value could be higher than H? Very Unlikely

This 1st iteration tends to result in anchoring bias on M, over-confidence on L and H, and poor rationale
4. Assessing expert’s responses (Q&A)

Given from Expert: $L=42$, $M=55$, $H=80$, $p(x<L)=0.30$ and $p(x>H)=0.10$

Calculation of ‘true’ $L$ and $H$: $L = 1.56$ and $H = 101.15$ … Do these #’s appear reasonable?

(a) Method to solve for $L$ and $H$ presented in “Beyond Beta,” Ch1 (The Triangular Distribution)
EE Phase 4: *Commute Time* (iteration 1)

**Question 10:** Expert & Interviewer brainstorm risk factors …

*What risk factors contributed to the uncertainty in your estimate?*

<table>
<thead>
<tr>
<th>Objective</th>
<th>Means</th>
<th>Barriers / Risks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize Average Speed</td>
<td>Avoid Dense Traffic</td>
<td>Weather</td>
</tr>
<tr>
<td></td>
<td>Avoid stops</td>
<td>Accident(s)</td>
</tr>
<tr>
<td></td>
<td>Optimize driving</td>
<td>Road Construction</td>
</tr>
</tbody>
</table>

Create Risk Breakdown Structure (RBS)

- **Weather**
- **Accident(s)**
- **Road Construction**
- **Departure Time**
- **Red Lights**
- **Emergency vehicles**
- **School buses**
- **Not feeling well**
- **Inexperienced driver**
- **Unfamiliar with route**

**Question 11:** Expert selects top 6 risk factors …

*What are the top 6 risk factors that contributed to your estimate uncertainty?*

**User Input**

**Examples or Justification:**

<table>
<thead>
<tr>
<th>Risk Factor</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weather</td>
<td>Rain, snow &amp; especially ice, have caused major delays in the past; I expect similar impacts in 2014.</td>
</tr>
<tr>
<td>Accident(s)</td>
<td>Accidents occasionally occur. In some cases, these have added 60 minutes to my commute!</td>
</tr>
<tr>
<td>Road Construction</td>
<td>Sometimes road crews shut down 1 or 2 lanes; typically adding 10 - 20 minutes to my commute.</td>
</tr>
<tr>
<td>Departure Time</td>
<td>I try to leave 1 hour before rush hour. Leaving later can add 10-15 minutes to my commute.</td>
</tr>
<tr>
<td>Not Feeling Well</td>
<td>If I'm not feeling well, I'll drive more slowly or even make a wrong turn! Can add 5 min to commute.</td>
</tr>
<tr>
<td>Red Lights</td>
<td>I tend to &quot;catch&quot; the same lights every day so this factor could add 1-2 minutes to my commute.</td>
</tr>
</tbody>
</table>
EE Phase 4: *Commute Time* (iteration 1)

Question 12: Expert scores each risk factor’s contribution to uncertainty …

Score each risk factor a value based upon the following instruction:

<table>
<thead>
<tr>
<th>If the specified risk factor: *</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>is the largest contributor to uncertainty (e.g. biggest driver of H)</td>
<td>5.0</td>
</tr>
<tr>
<td><strong>Indifference</strong></td>
<td>4.5</td>
</tr>
<tr>
<td>is a significant contributor to uncertainty (e.g. big driver of H)</td>
<td>4.0</td>
</tr>
<tr>
<td><strong>Indifference</strong></td>
<td>3.5</td>
</tr>
<tr>
<td>has a moderate effect on uncertainty (e.g. nominal impact on H)</td>
<td>3.0</td>
</tr>
<tr>
<td><strong>Indifference</strong></td>
<td>2.5</td>
</tr>
<tr>
<td>has a small effect on uncertainty (e.g. not a big driver of H)</td>
<td>2.0</td>
</tr>
<tr>
<td><strong>Indifference</strong></td>
<td>1.5</td>
</tr>
<tr>
<td>is the smallest contributor to uncertainty (e.g. smallest driver of H)</td>
<td>1.0</td>
</tr>
</tbody>
</table>

* Note: You can have 2 or more risk factors with a score of 5 (or score of 1).

<table>
<thead>
<tr>
<th>Risk Factor</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weather</td>
<td>5.0</td>
</tr>
<tr>
<td>Accident(s)</td>
<td>5.0</td>
</tr>
<tr>
<td>Road Construction</td>
<td>2.0</td>
</tr>
<tr>
<td>Departure Time</td>
<td>4.0</td>
</tr>
<tr>
<td>Not Feeling Well</td>
<td>1.0</td>
</tr>
<tr>
<td>Red Lights</td>
<td>1.5</td>
</tr>
</tbody>
</table>

*Expert provides a score for each risk factor (rationale not shown).*

The 1st iteration of Q&A is complete. Recommend the expert take a 15 minute break before re-starting Q&A.
EE Phase 4: *Commute Time* (Iteration 2)

4. **Assessing expert’s responses (Q&A)**

Given the objective, assumptions & input parameter (WBS4):

1. Do you need to modify the probability value scale?  
   - No

2. Do you need to re-characterize the input parameter?  
   - No

3. Do you want to adjust your Most Likely Value, $M$?  
   - No

4. What’s the chance the actual value could exceed $M$?  
   - N/A

   Assuming best case: weather, accidents, road const, departure time, etc.  
   - Document Scenario

5. What’s the Lowest value, $L$?  
   - 40

6. What’s the chance actual value could be less than $L$?  
   - Extremely Unlikely

   Assuming worst case: weather, accidents, road const, departure time, etc.  
   - Document Scenario

7. What’s the Highest value, $H$?  
   - 90

8. What’s the chance actual value could be higher than $H$?  
   - Indifferent-Low
EE Phase 4: Commute Time (iteration 2)

4. Assessing expert’s responses (Q&A)

User-Provided Distribution for Commute Time

Red dot depicts unadjusted point estimate. Dashed lines depict unadjusted lowest & highest

Given from Expert: $L = 40, M = 55, H = 90$, $p(x < L) = 0.10$ and $p(x > H) = 0.30$

Calculation of ‘true’ $L$ and $H$ (a): $L = 35.44$ and $H = 143.92$ ... Do these #’s appear reasonable?

2\textsuperscript{nd} iteration helps “condition” expert to reduce anchoring bias on $M$, counter over-confidence on $L$ & $H$, calibrate ‘values’ & improve rationale.

(a) Method to solve for $L$ and $H$ presented in “Beyond Beta,” Ch1 (The Triangular Distribution)
EE Phase 5: **Commute Time** (iteration 2)

5. Verifying encoded values & documentation

The 2\textsuperscript{nd} iteration helped elicit an $L$ that seems feasible and an $H$ that accounts for worst-case risk factors.
• **In most cases, Beta-PERT is preferred (vs triangular)**
  – Beta-PERT’s mean is only slightly greater than its mode

• **However, triangular would be preferred (vs Beta-PERT) if elicited data seems to depict over-confidence (e.g. H value is optimistic)**
  – Triangular PDF compensates for this by ‘exaggerating’ the mean value
We provided an expert elicitation overview that …

1. Demonstrated a way to model expert opinion as a triangular distribution
   – A process that **does not “over-burden”** the subject matter expert

2. Incorporated techniques to address expert bias
   – Iterative Q&A process that includes use of visual aids
   – Relied on at least a **2nd iteration** to help minimize inaccuracy & bias
   – Convert Triangular to Beta-PERT (if overconfidence was addressed)

3. Structured the process to help justify expert’s inputs
   – Rationale required for each response
   – RBS to help identify what **risk factors** contribute to uncertainty
   – **Weight risk factors** to gain insight as each risk factor’s relative contribution to uncertainty (cost, schedule, etc.,)
Potential Improvements

- More upfront work on “training” the expert
- Criteria when to elicit mean or median (vs mode)
- Add 2 questions to create Modified Beta-PERT
- Improve scaling tables that depict expert’s judgment
- Create “starter” Risk Breakdown Structures”
  - Facilitates brainstorming process of possible risk factors
- Convert best case & worst case scenarios to probabilities
- Improve method of weighting risk factors
- Explore other distributions, e.g. Weibull & LogNormal
- Incorporate methods to combine expert judgments

So … hopefully … this adds to the conversation on how best to leverage expert judgment in the cost community.
Intuition versus Analysis

Quickly answer the question:

“A bat and a ball cost $1.10 in total. The bat costs $1 more than the ball. How much does the ball cost?”


A Step-Wise Approach to Elicit Triangular Distributions

Formerly entitled “An Elicitation Method to Generate Minimum-Bias Probability Distributions”

Questions?

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**Probability Distributions**

**Bounded**
- Triangular & Uniform
- Histogram
- Discrete & Cumulative
- Beta & Beta-PERT

**Unbounded**
- Normal & Student-t
- Logistic

**Left bounded**
- Lognormal
- Weibull & Gamma
- Exponential
- Chi-square

**Non-Parametric Distributions:** Mathematics defined by the shape that is required. Empirical, intuitive and easy to understand.

**Parametric Distributions:** Shape is born of the mathematics describing theoretical problem. Model-based. Not usually intuitive.

Of the many probability distributions out there, Triangular & Beta-PERT are among the most popular used for expert elicitation.
Reasons For & Against Conducting EE

Reasons for Conducting an Expert Elicitation
- The problem is complex and more technical than political
- Adequate data (of suitable quality and relevance) are unavailable or unobtainable in the decision time framework
- Reliable evidence or legitimate models are in conflict
- Qualified experts are available & EE can be completed within decision timeframe
- Finances and expertise are sufficient to conduct a robust & defensible EE

Reasons Against Conducting an Expert Elicitation
- The problem is more political than technical
- A large body of empirical data exists with a high degree of consensus
- Findings of an EE will not be considered legitimate or acceptable by stakeholders
- Information that EE could provide is not critical to the assessment or decision
- Cost of obtaining EE info is not commensurate with its value in decision-making
- Finances and/or expertise are insufficient to conduct a robust & defensible EE
- Other acceptable methods or approaches are available for obtaining the needed information that are less intensive and expensive
Some Common Cognitive Biases

• Availability
  – Base judgments on outcomes that are more easily remembered

• Representativeness
  – Base judgments on similar yet limited data and experience. Not fully considering other relevant, accessible and/or newer evidence

• Anchoring and adjustment
  – Fixate on particular value in a range and making insufficient adjustments away from it in constructing an uncertainty estimate

• Overconfidence (sometimes referred to as Optimistic bias)
  – Strong tendency to be more certain about one’s judgments and conclusions than one has reason. Tends to produce optimistic bias.

• Control (or “Illusion of Control”)
  – SME believes he/she can control or had control over outcomes related to an issue at hand; tendency of people to act as if they can influence a situation over which they actually have no control.