Magnetic Field, Force, and Inductance Computations for an Axially Symmetric Solenoid

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April 2001
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Preface

In the late 1990s research was being performed at the Kennedy Space Center to develop *in situ* resource utilization technology for Mars. One study topic was the generation and storage of liquid oxygen (LOX) obtained from the atmosphere or regolith, but the transfer of this commodity was of concern. Mechanical LOX pumps were deemed potentially too heavy and unreliable for an autonomous mission to Mars, and alternatives were sought. One option was to use the paramagnetic property of LOX, which is significant enough that small quantities of LOX can be suspended against earth gravity with a rare earth magnet. With this application in mind, a small, internally funded project was initiated at the Kennedy Space Center in 2000 to study the use of pulsed magnetic fields to pump LOX.

Proof-of-concept testing verified the LOX pumping predictions and resulted in a journal publication [1]. Numerous small coils were fabricated on cryogenic flow lines and used to produce intense, short-duration magnetic fields resulting in dramatic motion of the LOX. In addition, effort was expended on modeling the paramagnetic forces in the LOX, which required modeling the magnetic field generated by the coils and the coil inductance, allowing the current versus time to be predicted and compared to experiment.

During the coil inductance modeling, we generated a method by which the inductance of an air-core solenoid could be computed efficiently. We wrote a short manuscript documenting this work, but did not proceed with publication. Instead, we chose to release our software (primarily generated by one of the authors, John Lane) with the manuscript attached as a reference document. The software and manuscript were made public through the Open Channel Foundation, an organization that publishes software from academic and research institutions. The link for this site is given below.

http://www.openchannelfoundation.org/project/view_docs.php?group_id=288

Through this link, our manuscript has been downloaded multiple times, has been used by several researchers, but to our knowledge, has not been cited. We suspect that this may be due to the lack of a clear citation and have decided that one should be provided. We consider it too late to seek peer-reviewed publication of this work and instead have decided to issue a NASA Technical Memorandum, with the same title and authors as the original manuscript. This will allow our work to be located by the search engines and will provide a reliable, long-term source and citation. The original manuscript, reformatted, but not in any other way altered, appears on the subsequent pages of this memorandum.

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Contents

1 INTRODUCTION .................................................................................................................. 1
2 B FIELD EVALUATION ........................................................................................................ 1
3 FORCE FIELD EVALUATION .............................................................................................. 2
4 INDUCTANCE FORMULAS .................................................................................................. 3
  4.1 Stacked Single Layer ....................................................................................................... 3
  4.2 Stacked Multiple Layers .................................................................................................. 4
  4.3 Inductance From the Stored Energy .................................................................................. 5
5 EXPERIMENTAL RESULTS AND DISCUSSION ............................................................... 7
APPENDIX A. COMPONENTS OF MATRIX P ....................................................................... 15
APPENDIX B. COMPONENTS OF MATRIX Q ........................................................................ 16

Figures

Figure 1. Circular current loop....................................................................................................... 2
Figure 2. Circular current loop of radius \(a\) with finite cross-sectional radius \(b\). .................. 3
Figure 3. \(M\) stacked current loops ............................................................................................ 4
Figure 4. Stacked multiple layer coil .......................................................................................... 5
Figure 5. Solenoid model for general inductance calculation ...................................................... 6
Figure 6a. Solenoid B-field plots from Equations (2) with dimensions \(a = 20\) mm, \(b = 0.5\) mm, \(d_x = d_z = 1\) mm, \(M_x = 20\), and \(M_z = 31\) ....................................................... 8
Figure 6b. Solenoid F-field plots from Equation (4) with dimensions \(a = 20\) mm, \(b = 0.5\) mm, \(d_x = d_z = 1\) mm, \(M_x = 20\), and \(M_z = 31\) ................................................................. 9
Figure 6c. Solenoid B-field plots from Equations (2) with dimensions \(a = 32.1\) mm, \(b = 0.362\) mm, \(d_x = 0.683\), \(d_z = 0.757\) mm, \(M_x = 8\), and \(M_z = 83\) ...................... 10
Figure 6d. Solenoid F-field plots from Equation (4) with dimensions \(a = 32.1\) mm, \(b = 0.362\) mm, \(d_x = 0.683\), \(d_z = 0.757\) mm, \(M_x = 8\), and \(M_z = 83\) ...................... 11
Figure 7. Self- and mutual-inductance vs. \(M_z\) for \(M_x = 1\). ................................................. 12
Figure 8. Self- and mutual-inductance vs. \(M_x\) for \(M_z = 1\). .................................................. 13
Figure 9. Measured vs. calculated inductance ................................................................. 13
Figure 10. Monte Carlos test of inductance formulas ......................................................... 14

Tables

Table 1. Dimensions of test coils [mm].................................................................................... 12
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1 INTRODUCTION

The pumping of liquid oxygen (LOX) by magnetic fields (B field), using an array of electromagnets, is a current topic of research and development at Kennedy Space Center, FL. Oxygen is paramagnetic so that LOX, like a ferrofluid, can be forced in the direction of a B field gradient. It is well known that liquid oxygen has a sufficient magnetic susceptibility that a strong magnetic gradient can lift it in the earth’s gravitational field. It has been proposed that this phenomenon can be utilized in transporting (i.e., pumping) LOX not only on earth, but on Mars and in the weightlessness of space.

In order to design and evaluate such a magnetic pumping system, it is essential to compute the magnetic and force fields, as well as inductance, of various types of electromagnets (solenoids). In this application, it is assumed that the solenoids are air wrapped, and that the current is essentially time independent.

2 B FIELD EVALUATION

The magnetic field B due to a single current loop of radius \( a \), as shown in Figure 1, can be computed by evaluating the curl of the magnetic vector potential \( \mathbf{A} \). With the current loop located in and symmetric with respect to the \( x-y \) plane, \( \mathbf{A} \) is given by [1]:

\[
A_r(r, \theta) = \frac{\mu_0 i a}{4\pi} \int_0^{2\pi} \frac{\cos \phi' \, d\phi'}{\sqrt{a^2 + r^2 - 2ar \sin \theta \cos \phi'}}
\]  

where \( r, \theta, \) and \( \phi' \) are spherical coordinates, \( i \) is the loop current, and \( \mu_0 \) is the permeability of free space. The curl of Equation (1) can be simplified in cylindrical coordinates [2], or in Cartesian coordinates as follows [3]:

\[
B_x(x, y, z) = \frac{C x z}{2\alpha^2 \beta \rho^2} \left( (a^2 + R^2) E(k^2) - \alpha^2 K(k^2) \right) 
\]  

\[
B_y(x, y, z) = \frac{C y z}{2\alpha^2 \beta \rho^2} \left( (a^2 + R^2) E(k^2) - \alpha^2 K(k^2) \right) 
\]  

\[
B_z(x, y, z) = \frac{C}{2\alpha^2 \beta} \left( (a^2 - R^2) E(k^2) + \alpha^2 K(k^2) \right) 
\]

where \( \rho^2 = x^2 + y^2, \, R^2 = \rho^2 + z^2, \, \alpha^2 = a^2 + R^2 - 2\alpha \rho, \, \beta^2 = a^2 + R^2 + 2\alpha \rho, \, k^2 = 1 - \alpha^2 / \beta^2 \), and \( C = \mu_0 i/\pi \). \( E(k^2) \) and \( K(k^2) \) are the complete elliptic integrals of the first and second kind, respectively.
3 FORCE FIELD EVALUATION

In order to estimate the force on a paramagnetic volume of LOX, it is necessary to evaluate the force equation [4]:

$$F = \frac{1}{\mu_0} \left( \frac{\chi}{1+\chi} \right) \mathbf{B} \cdot \nabla \times \mathbf{B}$$  \hspace{1cm} (3)

where again $\mu_0$ is the permeability of free space and $\chi$ is the magnetic susceptibility of the medium. Equation (3) can be rewritten in matrix form as:

$$F = \frac{1}{\mu_0} \left( \frac{\chi}{1+\chi} \right) \mathbf{D} \mathbf{B}$$  \hspace{1cm} (4)

where

$$\mathbf{D} = \begin{pmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial z} \end{pmatrix}$$  \hspace{1cm} (5)

It can be shown that $\mathbf{D}$ can be written as:

$$\mathbf{D} = \mathbf{P} \mathbf{E} (k^2) + \mathbf{Q} \alpha^2 \mathbf{K} (k^2)$$  \hspace{1cm} (6)

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & \frac{\gamma}{\alpha} p_3 \\ p_3 & \frac{\gamma}{\alpha} p_2 & p_5 \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_4 & \frac{\gamma}{\alpha} q_3 \\ q_3 & \frac{\gamma}{\alpha} q_2 & q_5 \end{pmatrix}$$  \hspace{1cm} (7)
where \( P \) and \( Q \) are symmetric matrices. The five independent components of \( P \) and \( Q \) are given in Appendix A and Appendix B, in terms of the \( x, y, z \) field coordinates and loop radius \( a \), where \( \gamma^2 = x^2 - y^2 \) [3].

4 INDUCTANCE FORMULAS

A common method of computing inductance of a solenoid in many engineering applications is to identify a handbook formula that most closely resembles the particular solenoid geometry at hand. Afterwards, a fine-tuning can be performed when building the coil by adding or subtracting loops of wire.

For example, an estimate of the total inductance \( L \) of a single circular current loop is given by:

\[
L = a \mu_0 \left[ \ln\left(\frac{8a}{b}\right) - \frac{1}{4} \right]
\]  

where \( a \) is the loop radius and \( b \) the wire cross-sectional radius, as shown in Figure 2. This estimate of self-inductance is good as long as \( b/a << 1 \).

![Figure 2. Circular current loop of radius \( a \) with finite cross-sectional radius \( b \).](image)

4.1 Stacked Single Layer

Figure 3 diagrams a stack of \( M \) current loops, as would be found in a single layered solenoid. The handbook formula [5] for approximating inductance of this configuration is:

\[
L = \frac{\pi \mu_0 a^2 M^2}{\sqrt{4a^2 + (M-1)^2 d^2}}
\]  

(9)
4.2 Stacked Multiple Layers

Figure 4 depicts a stack of $M_z$ current loops, configured as $M_z$ layers, as would be found in a multi-layered solenoid. The handbook formula [6] for approximating inductance of this configuration is:

$$L = \frac{\pi \mu_0 r_0^3 (M_z M_x)^2}{r_0 I + 0.9 r_0^2 + 0.32 l t + 0.84 r_0 t}$$  \hspace{1cm} (10a)$$

where:

$$l = (M_z - 1) d, \quad t = 2 M_x b$$

$$r_0 = a + (M_z - 1) b$$  \hspace{1cm} (10b)$$
4.3 Inductance From the Stored Energy

Inductance $L$ can be derived from the energy stored in a magnetic field due to a current carrying wire. In this case, $L$ is equal to the integral of the product of current density $J$ and magnetic vector potential $A$, divided by current squared [1]:

$$L = \frac{1}{I^2} \int J \cdot A \, dx^3$$

(11)

For a single wire loop of wire diameter $b$ and loop diameter $a$, the magnetic vector potential inside the wire can be approximated by [1]:

$$A = \frac{\mu_0}{4\pi} \left[ \left(1 - \frac{b^2}{a^2}\right) + 2 \left(\ln\frac{8a}{b} - 2\right) \right] \mathbb{e}_\phi$$

(12)

with the assumption that $b/a << 1$. Assuming also that the current density is uniform (no high frequency skin effects, etc.), then $J = i/\pi b^2 \mathbb{e}_\phi$. Integrating $J \cdot A$ over the wire cross-section and
total length, as described by Equation (11), and dividing by \( i^2 \) leads to an approximation of self-inductance given by Equation (8). Note that due to symmetry, the volume integral becomes a line integral, \( \int dx^3 \to 4\pi^2 a \int_0^b \rho \, d\rho \).

Equation (8) describes an approximation to the total inductance of a single circular wire loop, which becomes exact as \( b/a \) approaches zero. When multiple loops are considered, as would be found in a solenoid, the total inductance is the sum of two parts, self-inductance and mutual-inductance. Using Figure 4 as a general solenoid model, composed of \( M_z \) stacks and \( M_x \) layers, such that the total number of loops is equal to \( M_x M_z \).

The total solenoid inductance can be computed by summing the self-inductance and mutual inductance contributions of each \( m^{th}, n^{th} \) loop (refer to Figure 5):

\[
L = \sum_{m=-(M_z-1)/2}^{(M_z-1)/2} \sum_{n=0}^{M_z-1} (L_{mn} + M_{mn})
\]

where the self-inductance of the \( m^{th}, n^{th} \) loop is approximated by Equation (1):

\[
L_{mn} = \mu_0 a_n \left[ \ln \left( \frac{8a_n}{b} \right) - \frac{2}{3} \right]
\]

where \( a \) is replaced with \( a_n = a + nd_x \).

The mutual inductance component \( M_{mn} \) of each wire loop is determined by considering the magnetic vector potential of a single loop centered at the origin, in spherical coordinates:

\[
A = \frac{\mu_0 ja}{\pi \sqrt{a^2 + r^2 + 2ar \sin \theta}} \left[ (2 - \phi) K(\phi) - 2E(\phi) \right] \hat{\phi}
\]

\[
\phi = \frac{4ar \sin \theta}{a^2 + r^2 + 2ar \sin \theta}
\]

Figure 5. Solenoid model for general inductance calculation.
where $K(\phi)$ and $E(\phi)$ are again the complete elliptic integrals of the first and second kind, respectively.

Equations (15) can be used to describe the magnetic vector potential from each $m^{th}$, $n^{th}$ wire loop, simply by translating the coordinate system. The $r$ coordinate in Equations (15) describes the field at the $m^{th}$, $n^{th}$ wire loop due to the $j^{th}$, $k^{th}$ wire loop, excluding the self-potential where $j = m$ and $k = n$. To quantify this relationship, it can be shown that by following definitions given Figure 5, $r$ in Equations (15) becomes:

$$r_{jkmn} = \sqrt{a_n^2 + (m-j)^2 d_z^2}$$  \hspace{1cm} (16a)

leading to the associated modification of Equation (15b):

$$\varphi_{jkmn} = \frac{4a_n a_j}{(a_k + a_n)^2 + (m-j)^2 d_z^2}$$  \hspace{1cm} (16b)

where $a_k = a + kd_z$. The magnetic vector potential from Equation (8a) for each $m^{th}$, $n^{th}$ wire loop now becomes:

$$\left( A_\varphi \right)_{jkmn} = \left( \frac{\mu_0 j \sqrt{(a_k + a_n)^2 + (m-j)^2 d_z^2}}{\pi a_n} \right) \left( (2-\phi_{jkmn}) K(\phi_{jkmn}) - 2E(\phi_{jkmn}) \right)$$  \hspace{1cm} (17)

Finally, to express the mutual inductance of the $m^{th}$, $n^{th}$ wire loop, the integral of Equation (11) can be approximated by a simple product of the wire cross-section with a summation of Equation (17) over $j$ and $k$, excluding $j = m$ and $k = n$:

$$M_{mn} = \frac{\mu_0}{2} \sum_{j=-(M_z-1)/2}^{(M_z-1)/2} \sum_{k=0}^{M_x-1} \sqrt{(a_k + a_n)^2 + (m-j)^2 d_z^2} \cdot \left\{ (2-k_{majk}) K(k_{majk}) - 2E(k_{majk}) \right\}$$  \hspace{1cm} (18)

5 EXPERIMENTAL RESULTS AND DISCUSSION

Magnetic field and force plots using two example coils are generated by employing Equations (2) and (4). The first coil has dimensions of $a = 20$ mm, $b = 0.5$ mm, $d_x = d_z = 1$ mm, $M_x = 20$, and $M_z = 31$. The B and force field plots are shown in Figures 6a and 6b. Figures 6c and 6d are the B and force field plots of coil #7 in Table 1.
\[ B(x,0,z) \]

\[ B_x(x,0,z) \]

\[ |B(x,0,z)| \]

Figure 6a. Solenoid B-field plots from Equations (2) with dimensions 
\( a = 20 \text{ mm}, b = 0.5 \text{ mm}, d_x = d_z = 1 \text{ mm}, M_x = 20, \text{ and } M_z = 31. \)
Figure 6b. Solenoid F-field plots from Equation (4) with dimensions $a = 20 \text{ mm}$, $b = 0.5 \text{ mm}$, $d_x = d_z = 1 \text{ mm}$, $M_x = 20$, and $M_z = 31$. 
Figure 6c. Solenoid B-field plots from Equations (2) with dimensions

\( a = 32.1 \, \text{mm}, \quad b = 0.362 \, \text{mm}, \quad d_x = 0.683, \quad d_z = 0.757 \, \text{mm}, \quad M_x = 8, \text{ and } M_z = 83. \)
Figure 6d. Solenoid F-field plots from Equation (4) with dimensions $a = 32.1 \text{ mm}, b = 0.362 \text{ mm}, d_1 = 0.683, d_2 = 0.757 \text{ mm}, M_x = 8$, and $M_z = 83$. 

$F_x(x,0,z)$

$F_y(x,0,z)$

$|F(x,0,z)|$
Table 1. Dimensions of test coils [mm].

<table>
<thead>
<tr>
<th>Coil #</th>
<th>a</th>
<th>b</th>
<th>$d_x$</th>
<th>$d_z$</th>
<th>$M_x$</th>
<th>$M_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.41</td>
<td>0.051</td>
<td>0.011</td>
<td>0.013</td>
<td>4</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td>4.57</td>
<td>0.127</td>
<td>0.240</td>
<td>0.277</td>
<td>27</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>4.70</td>
<td>0.255</td>
<td>0.470</td>
<td>0.540</td>
<td>21</td>
<td>33</td>
</tr>
<tr>
<td>4</td>
<td>4.70</td>
<td>0.255</td>
<td>0.540</td>
<td>0.540</td>
<td>27</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>4.60</td>
<td>0.140</td>
<td>0.267</td>
<td>0.280</td>
<td>31</td>
<td>19</td>
</tr>
<tr>
<td>6</td>
<td>1.67</td>
<td>0.080</td>
<td>0.170</td>
<td>0.178</td>
<td>17</td>
<td>19</td>
</tr>
<tr>
<td>7</td>
<td>32.1</td>
<td>0.362</td>
<td>0.683</td>
<td>0.757</td>
<td>8</td>
<td>83/84</td>
</tr>
</tbody>
</table>

Equations (13), (14), and (18) make up a complete description of the total inductance of a solenoid described by Figure 5. In order to get some qualitative insight into the relative accuracy of this description of inductance, it is helpful to plot the self-inductance and mutual-inductance contributions, as shown in Figures 7 and 8. For these particular examples, the mutual inductance contribution quickly dominates as the number of turns increases. This implies that error resulting from the simplifying assumption of $b/a \ll 1$, used to compute the self-inductance term in Equation (14), will in most cases not be the primary source of error. Therefore, it can be concluded that accuracy of this method for computing solenoid inductance is dominated by the error introduced by allowing the integral in Equation (11) to be calculated as a product of the magnetic potential at the center of the wire cross-section times the cross-sectional area.

![Figure 7. Self- and mutual-inductance vs. $M_z$ for $M_x = 1$.](image-url)
Figure 8. Self- and mutual-inductance vs. $M_x$ for $M_z = 1$.

Figure 9 compares the calculated to the measured inductance of seven coils with dimensions given in Table 1. “$L$ Model #2” is equivalent to Equation (9); “$L$ Model #3” is described by Equations (10); and “$L$ Model #4” is represented by Equations (13), (14), and (18). Note that the simple approximation from $L$ Model #3 seems to consistently track the more accurate calculation of $L$ Model #4. Figure 10 is similar to Figure 9, but was generated using a Monte Carlo approach, comparing all inductance formula calculations to Model #4 results.

Figure 9. Measured vs. calculated inductance.
Even though these computational methods may seem “brute force” by traditional standards, the computational power of today’s desktop computers makes this approach very practical.

References


### APPENDIX A. COMPONENTS OF MATRIX P

<table>
<thead>
<tr>
<th>i</th>
<th>$p_i$</th>
</tr>
</thead>
</table>
| 1 | \[
\frac{C z}{2\alpha^4 \beta^3 \rho^4} \left[ a^4 \left( \gamma^2 (3z^2 - a^2) + \rho^2 (8x^2 - y^2) \right) - a^2 \left( \rho^4 (5x^2 + y^2) - 2\rho^2 z^2 (2x^2 + y^2) + 3z^4 \gamma^2 \right) - R^4 \left( 2x^4 + \gamma^2 (y^2 + z^2) \right) \right]
\] |
| 2 | \[
\frac{C x y z}{2\alpha^4 \beta^3 \rho^4} \left[ 3a^4 (3\rho^2 - 2z^2) - R^4 (2R^2 + \rho^2) - 2a^6 - 2a^2 (2\rho^4 - \rho^2 z^2 + 3z^4) \right]
\] |
| 3 | \[
\frac{C x}{2\alpha^4 \beta^3 \rho^2} \left[ (\rho^2 - a^2)^2 (\rho^2 + a^2) + 2z^2 (a^4 - 6a^2 \rho^2 + \rho^4) + z^4 (a^2 + \rho^2) \right]
\] |
| 4 | \[
\frac{C z}{2\alpha^4 \beta^3 \rho^4} \left[ a^6 \gamma^2 + a^4 \left( 3\gamma^2 z^2 - \rho^2 (x^2 - 8y^2) \right) + a^2 \left( 2\rho^2 z^2 (x^2 + 2y^2) - \rho^4 (x^2 + 5y^2) + 3\gamma^2 z^4 \right) + R^4 \left( \gamma^2 (x^2 + z^2) - 2y^4 \right) \right]
\] |
| 5 | \[
\frac{C z}{2\alpha^4 \beta^3} \left[ 6a^2 (\rho^2 - z^2) - 7a^4 + R^4 \right]
\] |
### COMPONENTS OF MATRIX Q

<table>
<thead>
<tr>
<th>( j )</th>
<th>( q_j )</th>
</tr>
</thead>
</table>
| 1     | \[
\frac{C z}{2\alpha^4 \beta^3 \rho^4} \left[ a^2 \left( \gamma^2 (a^2 + 2z^2) - \rho^2 (3x^2 - 2y^2) \right) + R^2 (2x^4 + \gamma^2 (y^2 + z^2)) \right] \]
| 2     | \[
\frac{C x y z}{2\alpha^4 \beta^3 \rho^4} \left[ R^2 (2R^2 + \rho^2) - a^2 (5\rho^2 - 4z^2) + 2a^4 \right] \]
| 3     | \[
\frac{C x}{2\alpha^4 \beta^3 \rho^2} \left[ (\rho^2 - a^2)^2 + z^2 (\rho^2 + a^2) \right] \]
| 4     | \[
- \frac{C z}{2\alpha^4 \beta^3 \rho^4} \left[ a^4 \gamma^2 + a^2 \left( 2\gamma^2 z^2 - \rho^2 (2x^2 - 3y^2) \right) + R^2 (\gamma^2 (y^2 + z^2) - 2y^4) \right] \]
| 5     | \[
\frac{C z}{2\alpha^4 \beta^3} \left[ a^2 - R^2 \right] \]