Analytic couple modeling introducing device design factor, fin factor, thermal diffusivity factor, and inductance factor

Jon Mackey
Mechanical Engineering, University of Akron

Alp Sehirlioglu
Materials Science and Engineering, Case Western Reserve University

Fred Dynys
NASA Glenn Research Center

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Objectives

• Investigate couple configurations analytically:
  • Rectangular
  • Cylindrical

• Investigate additional physics from classic case:
  • Thermal resistance of shoe material
  • Lateral heat transfer
  • Variable material properties
  • Transient operation

• Establish a set of simple design guidelines, for lab couple demonstration purposes
  • Applicable to automotive, power, electronic, and other industries
Analytic Couple Modeling

Classic Model

\[ T_H \cdot \varphi_H \]

\[ \begin{align*}
A_A, \sigma_A, \\
S_A, k_A, \\
A_B, \sigma_B, \\
S_B, k_B,
\end{align*} \]

Thermal-

\[ x \]

\[ L_A \]

\[ I \]

\[ R \]

\[ T_C, \varphi_{C_{A,B}} \]

\[ \frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + I_{A,B} \tau_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}} = 0 \]

Electrical-

\[ \frac{d\varphi_{A,B}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}} \]

System-

\[ \varphi_B(L_B) - \varphi_A(L_A) = IR \]

Classic Parameters

Geometric-

\[ X = \frac{A_B L_A}{A_A L_B} \]

Load-

\[ Y = \frac{R}{L_B \sigma_B A_B + L_A \sigma_A A_A} \]

Materials-

\[ Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X} \right) (k_A + k_B X)} \]

Classic Solution

\[ \eta_{opt} = \frac{\eta_c Y_{opt}}{T_h Z(X_{opt})} + (1 + Y_{opt}) - \frac{1}{2} \eta_c \]

\[ X_{opt} = \frac{k_A \sigma_A}{\sqrt{k_B \sigma_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}} \]

\[ Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sqrt{\sigma_A}} + \frac{k_B}{\sqrt{\sigma_B}} \right)^2} \]
**Analytic Couple Modeling**

### Classic Model

**Four Assumptions**

- **Thermal-**
  \[
  \frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + \frac{l_{A,B} \tau_{A,B}}{A_{A,B}} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{A,B}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

### Classic Parameters

#### Geometric-
\[
X = \frac{A_B l_A}{A_A l_B}
\]

#### Load-
\[
Y = \frac{R}{\sigma_B A_B + \frac{l_A}{\sigma_A A_A}}
\]

#### Materials-
\[
Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X} \right) (k_A + k_B X)}
\]

### Classic Solution

\[
\eta_{opt} = \frac{\eta_c Y_{opt}}{\left( 1 + Y_{opt} \right)^2 + \left( 1 + Y_{opt} \right) - \frac{1}{2} \eta_c}
\]

\[
X_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
\]

\[
Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B} \right)^2}
\]
**Classic Model**

- **Thermal-**
  \[
  \frac{d}{dx} \left[ -k_{AB} \frac{dT_{AB}}{dx} \right] + \frac{l_{AB} \tau_{AB}}{A_{A,B}} \frac{dT_{AB}}{dx} - \frac{l_{AB}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{AB}}{dx} = -S_{AB} \frac{dT_{AB}}{dx} - \frac{l_{AB}}{A_{A,B} \sigma_{A,B}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

**Classic Parameters**

- **Geometric-**
  \[
  X = \frac{A_B L_A}{A_A L_B}
  \]

- **Load-**
  \[
  Y = \frac{R}{L_B \sigma_B A_B + \frac{L_A}{\sigma_A A_A}}
  \]

- **Materials-**
  \[
  Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X} \right) (k_A + k_B X)}
  \]

**Classic Solution**

- **Load Efficiency**
  \[
  \eta_{opt} = \frac{\eta_c Y_{opt}}{\frac{(1 + Y_{opt})^2}{T_h Z(X_{opt})} + (1 + Y_{opt}) - \frac{1}{2} \eta_c}
  \]

- **Optimal Load**
  \[
  X_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B}
  \]

- **Optimal Power**
  \[
  Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{av}}
  \]

- **System Efficiency**
  \[
  Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B} \right)^2}
  \]
Analytic Couple Modeling

**Classic Model**

- **Thermal-**
  \[
  \frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + \frac{I_{A,B} \tau_{A,B} dT_{A,B}}{A_{A,B}} - \frac{I_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{AB}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{I_{A,B}}{A_{A,B} \sigma_{A,B}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

**Classic Parameters**

- **Geometric-**
  \[
  X = \frac{A_B L_A}{A_A L_B}
  \]

- **Load-**
  \[
  Y = \frac{R L_B}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}
  \]

- **Materials-**
  \[
  Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X} \right) (k_A + k_B X)}
  \]

**Classic Solution**

- \[
  \eta_{opt} = \frac{\eta_c Y_{opt}}{T_h Z(X_{opt})} \left( 1 + Y_{opt} \right) + \frac{1}{2} \eta_c
  \]

- \[
  X_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B}
  \]

- \[
  Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
  \]

- \[
  Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B} \right)^2}
  \]
**Classic Model**

\[
T_H \cdot \varphi_H
\]

**Thermal-**

\[
\frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + \frac{l_{A,B} \tau_{A,B}}{A_{A,B}} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
\]

**Electrical-**

\[
\frac{d\varphi_{AB}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}}
\]

**System-**

\[
\varphi_B(L_B) - \varphi_A(L_A) = IR
\]

**Classic Parameters**

**Geometric-**

\[
X = \frac{A_B L_A}{A_A L_B}
\]

**Load-**

\[
Y = \frac{R}{L_B \sigma_B A_B + L_A \sigma_A A_A}
\]

**Materials-**

\[
Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X} \right)} (k_A + k_B X)
\]

**Classic Solution**

\[
\eta_{opt} = \frac{\eta_c Y_{opt}}{\left( 1 + Y_{opt} \right)^2} + \left( 1 + Y_{opt} \right) - \frac{1}{2} \eta_c
\]

\[
X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
\]

\[
Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sigma_A} + \frac{k_B}{\sqrt{\sigma_B}} \right)^2}
\]
**Analytic Couple Modeling**

**Classic Model**

- **Thermal-**
  \[
  \frac{d}{dx} \left[-k_{AB} \frac{dT_{AB}}{dx} \right] + \frac{I_{AB} \tau_{AB}}{A_{AB}} \frac{dT_{AB}}{dx} - \frac{I_{AB}}{A_{AB} \sigma_{AB}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{AB}}{dx} = -S_{AB} \frac{dT_{AB}}{dx} - \frac{I_{AB}}{A_{AB} \sigma_{AB}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

**Classic Parameters**

- **Geometric-**
  \[
  X = \frac{A_B L_A}{A_A L_B}
  \]

- **Load-**
  \[
  Y = \frac{R}{\sigma_B A_B + \frac{L_A}{\sigma_A A_A}}
  \]

- **Materials-**
  \[
  Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B}X\right)(k_A + k_B X)}
  \]

**Classic Solution**

- **Optimal Efficiency**
  \[
  \eta_{opt} = \frac{\eta_c Y_{opt}}{\left(1 + Y_{opt}\right)^2 + \left(1 + Y_{opt}\right) - \frac{1}{2} \eta_c}
  \]

- **Optimal Length**
  \[
  X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}}
  \]

- **Optimal Load**
  \[
  Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
  \]

- **Optimal Material Index**
  \[
  Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B}\right)^2}
  \]
**Introduction**

**B.C.**  
**Fin**  
**Variable**  
**Transient**

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**Classic Model**

- **Thermal-**
  \[
  \frac{d}{dx} \left[-k_{A,B} \frac{dT_{A,B}}{dx} \right] + I_{A,B} r_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{A,B}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

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**Classic Parameters**

- **Geometric-**
  \[
  X = \frac{A_B L_A}{A_A L_B}
  \]

- **Load-**
  \[
  Y = \frac{R}{L_B \sigma_B A_B + L_A \sigma_A A_A}
  \]

- **Materials-**
  \[
  Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B} X\right) (k_A + k_B X)}
  \]

---

**Classic Solution**

- **η_{opt}**
  \[
  \eta_{opt} = \frac{\eta_c Y_{opt}}{T_h Z(X_{opt})} + \left(1 + Y_{opt}\right) - \frac{1}{2} \eta_c
  \]

- **X_{opt}**
  \[
  X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}}
  \]

- **Y_{opt}**
  \[
  Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
  \]

- **Z(X_{opt})**
  \[
  Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B}\right)^2}
  \]

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*Analytic Couple Modeling*
## Classic Model

### Thermal-
\[
\frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + \frac{l_{A,B} \tau_{A,B}}{A_{A,B}} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}^2}{A_{A,B} \sigma_{A,B}} = 0
\]

### Electrical-
\[
\frac{d\varphi_{A,B}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}}
\]

### System-
\[
\varphi_{B}(L_{B}) - \varphi_{A}(L_{A}) = IR
\]
**Analytic Couple Modeling**

**Classic Model**

- **Thermal-**
  \[
  \frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + \frac{l_{A,B} \tau_{A,B} dT_{A,B}}{A_{A,B}} = \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{AB}}{dx} = -S_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

**Classic Parameters**

- **Geometric-**
  \[
  X = \frac{A_B L_A}{A_A L_B}
  \]

- **Load-**
  \[
  Y = \frac{R}{\frac{L_B}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}}
  \]

- **Materials-**
  \[
  Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B}X\right)(k_A + k_B X)}
  \]

**Classic Solution**

- **\( \eta_{opt} \)**
  \[
  \eta_{opt} = \frac{\eta_c Y_{opt}}{T_h Z(X_{opt})} + (1 + Y_{opt}) - \frac{1}{2} \eta_c
  \]

- **\( X_{opt} \)**
  \[
  X_{opt} = \frac{\sqrt{k_A \sigma_A}}{\sqrt{k_B \sigma_B}}
  \]

- **\( Y_{opt} \)**
  \[
  Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
  \]

- **\( Z(X_{opt}) \)**
  \[
  Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B}\right)^2}
  \]

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**Introduction B.C. Fin Variable Transient**

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### Classic Model

- **Thermal-**
  \[
  \frac{d}{dx} \left[ -k_{A,B} \frac{dT_{A,B}}{dx} \right] + \frac{l_{A,B} T_{A,B}}{A_{A,B}} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
  \]

- **Electrical-**
  \[
  \frac{d\varphi_{A,B}}{dx} = -s_{A,B} \frac{dT_{A,B}}{dx} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}}
  \]

- **System-**
  \[
  \varphi_B(L_B) - \varphi_A(L_A) = IR
  \]

### Classic Parameters

- **Geometric-**
  \[
  X = \frac{A_B L_A}{A_A L_B}
  \]

- **Load-**
  \[
  Y = \frac{R}{L_B \sigma_B A_B + L_A \sigma_A A_A}
  \]

- **Materials-**
  \[
  Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B} \right) (k_A + k_B X)}
  \]

### Classic Solution

- **Optimal Efficiency:**
  \[
  \eta_{opt} = \frac{\eta_c Y_{opt}}{T_h Z(X_{opt})} + \left( 1 + Y_{opt} \right) - \frac{1}{2} \eta_c
  \]

- **Optimal Geometry:**
  \[
  X_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B}
  \]

- **Optimal Load:**
  \[
  Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
  \]

- **Optimal System:**
  \[
  Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sqrt{\sigma_A}} + \frac{k_B}{\sqrt{\sigma_B}} \right)^2}
  \]
Cylindrical Parameters

**Geometric-**

\[ X = \frac{w_B \ln \left( \frac{r_{oA}}{r_i} \right)}{w_A \ln \left( \frac{r_{oB}}{r_i} \right)} \]

**Load-**

\[ Y = \frac{\ln \left( \frac{r_{oB}}{r_i} \right) + \ln \left( \frac{r_{oA}}{r_i} \right)}{2\pi \sigma_B w_B + 2\pi \sigma_A w_A} \]

**Materials-**

\[ Z(X) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X} \right)(k_A + k_B X)} \]

---

**Thermal-**

\[ \frac{d}{dr} \left[ -k_{AB} r \frac{dT_{AB}}{dr} \right] + \frac{l_{AB} \tau_{AB} dT_{AB}}{2\pi w_{AB} dr} - \frac{l_{AB}^2}{4\pi^2 w_{AB}^2 r \sigma_{AB}} = 0 \]

**Electrical-**

\[ \frac{d\phi_{AB}}{dr} = -S_{AB} \frac{dT_{AB}}{dr} - \frac{l_{AB}}{2\pi w_{AB} r \sigma_{AB}} \]

**System-**

\[ \phi_B(L_B) - \phi_A(L_A) = IR \]

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**Classic Solution**

\[ \eta_{opt} = \frac{\eta_c Y_{opt}}{\left( 1 + Y_{opt} \right)^2 + \left( 1 + Y_{opt} \right) - \frac{1}{2} \eta_c} \]

\[ X_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}} \]

\[ Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B} \right)^2} \]
**Introduction**

**B.C.**

**Fin Variable Transient**

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**B.C. Model**

![Diagram of B.C. Model]

Boundary Conditions (B.C.)-

\[-k_{A,B} \frac{dT_{A,B}(0)}{dx} + \frac{l_{A,B}S_{A,B}}{A_{A,B}} T_{A,B}(0) = h_h \left( T_{\infty h} - T_{A,B}(0) \right)\]

\[-k_{A,B} \frac{dT_{A,B}(L_{A,B})}{dx} + \frac{l_{A,B}S_{A,B}}{A_{A,B}} T_{A,B}(L_{A,B}) = h_c \left( T_{A,B}(L_{A,B}) - T_{\infty c} \right)\]

\[h_{h/c}^{-1} = h^{-1} + \sum_j \frac{l_j}{k_j} + \frac{1}{\varepsilon \sigma (T_s + T_{\infty})(T_s^2 + T_{\infty}^2)}\]

---

**B.C. Parameters**

**Device Design-**

\[D_{A,B} = \frac{1}{1 + \frac{k_{A,B}(h_h + h_c)}{L_{A,B} h_h h_c}}\]

**Geometric-**

\[X = \frac{A_B L_A}{A_A L_B}\]

**Load-**

\[Y = \frac{R}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}\]

**Materials-**

\[Z(X) = \frac{(D_BS_B - D_AS_A)^2}{(\frac{1}{\sigma_A} + \frac{1}{\sigma_B}) (D_A k_A + D_B k_B X)}\]

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**B.C. Solution**

\[X_{opt} = \sqrt{\frac{k_A \sigma_A D_A}{k_B \sigma_B D_B}}\]

\[Y_{opt} = \sqrt{1 + Z(X_{opt}) \left[ T_{\infty i} \frac{S_B - S_A}{D_BS_B - D_AS_A} \left( 1 - D_{avg} \right) - \frac{\Delta T_{\infty}}{2} \right]}\]

\[Z(X_{opt}, D_A, D_B) = \left( \frac{D_BS_B - D_AS_A}{\left( \frac{k_A D_A}{\sigma_A} + \frac{k_B D_B}{\sigma_B} \right)^2} \right)\]
**Boundary Conditions (B.C.)**

\[-k_{A,B} \frac{dT_{A,B}(0)}{dx} + \frac{I_{A,B}S_{A,B}}{A_{A,B}} T_{A,B}(0) = h_h \left( T_{\infty} - T_{A,B}(0) \right)\]

\[-k_{A,B} \frac{dT_{A,B}(L_{A,B})}{dx} + \frac{I_{A,B}S_{A,B}}{A_{A,B}} T_{A,B}(L_{A,B}) = h_c \left( T_{A,B}(L_{A,B}) - T_{\infty} \right)\]

\[h_h^{-1} = h^{-1} + \sum_{j} L_j k_j + \frac{1}{\varepsilon \sigma (T_s + T_{\infty}) (T_s^2 + T_{\infty}^2)}\]

**B.C. Parameters**

**Device Design-**

\[D_{A,B} = \frac{1}{1 + \frac{k_{A,B}(h_h + h_c)}{L_{A,B} h_h h_c}}\]

**Geometric-**

\[X = \frac{A_B L_A}{A_A L_B}\]

**Load-**

\[Y = \frac{R}{\sigma_B A_B + \frac{L_A}{\sigma_A A_A}}\]

**Materials-**

\[Z(X) = \frac{(D_B S_B - D_A S_A)^2}{(\frac{1}{\sigma_A} + \frac{1}{\sigma_B} X) (D_A k_A + D_B k_B X)}\]

**B.C. Solution**

\[X_{opt} = \sqrt{\frac{k_A \sigma_A D_A}{k_B \sigma_B D_B}}\]

\[Y_{opt} = \sqrt{1 + Z(X_{opt}) \left[ T_{\infty} \frac{S_B - S_A}{D_B S_B - D_A S_A} (1 - D_{avg}) - \frac{\Delta T_{\infty}}{2} \right]}\]

\[Z(X_{opt}, D_A, D_B) = \frac{(D_B S_B - D_A S_A)^2}{\left( \frac{k_A D_A}{\sigma_A} + \frac{k_B D_B}{\sigma_B} \right)^2}\]
### Analytic Couple Modeling

**Boundary Conditions (B.C.)**-

- **Geometric-**
  \[ X = \frac{A_B L_A}{A_A L_B} \]
- **Load-**
  \[ Y = \frac{R}{\frac{L_B}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}} \]
- **Materials-**
  \[ Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X}\right)(k_A + k_B X)} \]

---

**B.C. Solution**

- **Device Design-**
  \[ D_{A,B} = \frac{1}{1 + \frac{k_{A,B}(h_h + h_c)}{L_{A,B} h_h h_c}} \]
- **Geometric-**
  \[ X = \frac{A_B L_A}{A_A L_B} \]
- **Load-**
  \[ Y = \frac{R}{\frac{L_B}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}} \]
- **Materials-**
  \[ Z(X) = \frac{(D_B S_B - D_A S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X}\right)(D_A k_A + D_B k_B X)} \]
**Classic Parameters**

**Geometric-**

\[ X = \frac{A_B L_A}{A_L B} \]

**Load-**

\[ Y = \frac{R}{L_B \sigma_B A_B + L_A \sigma_A A_A} \]

**Materials-**

\[ Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X}\right)(k_A + k_B X)} \]

**B.C. Parameters**

**Device Design-**

\[ D_{A,B} = \frac{1}{1 + \frac{k_{A,B}}{L_{A,B} h_h h_c}} \]

**Geometric-**

\[ X = \frac{A_B L_A}{A_L B} \]

**Load-**

\[ Y = \frac{R}{L_B \sigma_B A_B + L_A \sigma_A A_A} \]

**Materials-**

\[ Z(X) = \frac{(D_B S_B - D_A S_A)^2}{\left(\frac{1}{D_A k_A} + \frac{1}{D_B k_B}\right)(D_A k_A + D_B k_B X)} \]

**B.C. Solution**

\[ X_{opt} = \frac{k_A \sigma_A D_A}{\sqrt{k_B \sigma_B D_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}) \left[ \frac{S_B - S_A}{D_B S_B - D_A S_A} \left(1 - D_{avg}\right) \frac{\Delta T_{\omega}}{2} \right]} \]

\[ Z(X_{opt}, D_A, D_B) = \frac{(D_B S_B - D_A S_A)^2}{\left(\frac{k_A D_A}{\sigma_A} + \sqrt{\frac{k_B D_B}{\sigma_B}}\right)^2} \]
**B.C. Model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ambient Temperature</td>
<td>$T_\infty$</td>
<td></td>
</tr>
<tr>
<td>Outer Surface Temperature</td>
<td>$T_H$</td>
<td></td>
</tr>
<tr>
<td>Length</td>
<td>$L_A$, $L_B$</td>
<td></td>
</tr>
<tr>
<td>Area</td>
<td>$A_A$, $A_B$</td>
<td></td>
</tr>
<tr>
<td>Conductivity</td>
<td>$k_A$, $k_B$</td>
<td></td>
</tr>
<tr>
<td>Thermal Conductivity</td>
<td>$h_A$, $h_B$</td>
<td></td>
</tr>
<tr>
<td>フィルメン</td>
<td>$\phi$</td>
<td></td>
</tr>
</tbody>
</table>

**B.C. Parameters**

**Device Design**

- $D_{A,B} = \frac{1}{1 + \frac{k_{A,B}(h_h + h_c)}{L_{A,B}h_hh_c}}$

**Geometric**

- $X = \frac{A_BL_A}{A_AL_B}$

**Load**

- $Y = \frac{R}{\frac{L_B}{\sigma_BA_B} + \frac{L_A}{\sigma_AA_A}}$

---

**Classic Solution**

- $\eta_{opt} = \frac{\eta_cY_{opt}}{(1 + Y_{opt})^2 + (1 + Y_{opt}) - \frac{1}{2} \eta_c}$

- $X_{opt} = \sqrt{\frac{k_A\sigma_A}{k_B\sigma_B}}$

- $Y_{opt} = \sqrt{1 + Z(X_{opt})T_{avg}}$

- $Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\sqrt{\frac{k_A}{\sigma_A}} + \sqrt{\frac{k_B}{\sigma_B}}\right)^2}$

---

**B.C. Solution**

- $\eta = \frac{\eta_{c\infty}Y_{opt}}{(1 + Y_{opt})^2 + (1 + Y_{opt})(S_B - S_A)\left[1 - \frac{\eta_{c\infty}}{2}(1 - D_{avg})\right] - \frac{1}{2} \eta_{c\infty}}$

- $X_{opt} = \sqrt{\frac{k_A\sigma_A}{k_B\sigma_B}D_A}$

- $Y_{opt} = \sqrt{1 + Z(X_{opt})\left[\frac{S_B - S_A}{D_BS_B - D_AS_A}(1 - D_{avg}) - \frac{\Delta T_\infty}{2}\right]}$

- $Z(X_{opt}, D_A, D_B) = \frac{(D_BS_B - D_AS_A)^2}{\left(\sqrt{\frac{k_A}{\sigma_A}} + \sqrt{\frac{k_B}{\sigma_B}}\right)^2}$
**Example Calculation**

<table>
<thead>
<tr>
<th>Convection (W/m-K)</th>
<th>Design Factor</th>
<th>Max Efficiency (%)</th>
<th>Max Power Density (W/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>∞</td>
<td>1.00</td>
<td>6.15</td>
<td>17,733</td>
</tr>
<tr>
<td>50,000</td>
<td>0.98</td>
<td>6.05</td>
<td>17,118</td>
</tr>
<tr>
<td>500</td>
<td>0.38</td>
<td>2.28</td>
<td>2,300</td>
</tr>
</tbody>
</table>

**Classic Solution**

\[
\eta_{opt} = \frac{\eta_c Y_{opt}}{\frac{(1 + Y_{opt})^2}{T_h Z(X_{opt})} + (1 + Y_{opt}) - \frac{1}{2} \eta_c}
\]

\[
X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
\]

\[
Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B}\right)^2}
\]

**B.C. Solution**

\[
\eta = \frac{\eta_{c\infty} Y_{opt}}{T_{\infty H} Z(X_{opt}, D_B, D_A) + \frac{(1 + Y_{opt}) (S_B - S_A)}{(D_B S_B - D_A S_A) \left[1 - \eta_{c\infty} \left(1 - D_{avg}\right)\right]} - \frac{1}{2} \eta_{c\infty}}
\]

\[
X_{opt} = \sqrt{\frac{k_A \sigma_A D_A}{k_B \sigma_B D_B}}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}) \left[T_{\infty H} \left\{\frac{S_B - S_A}{D_B S_B - D_A S_A} \left(1 - D_{avg}\right) - \frac{\Delta T_{\infty}}{2}\right\}\right]}
\]

\[
Z(X_{opt}, D_A, D_B) = \frac{(D_B S_B - D_A S_A)^2}{\left(\frac{k_A D_A}{\sigma_A} + \frac{k_B D_B}{\sigma_B}\right)^2}
\]
Design Guideline

\[ L_D \geq \frac{D(h_H + h_C)k}{(1 - D)h_h h_c} \]

\[ L_{99\%} = \frac{99(h_H + h_C)k}{h_h h_c} \]

Classic Solution

\[ \eta_{opt} = \frac{\eta_c Y_{opt}}{\left(1 + Y_{opt}\right)^2 + (1 + Y_{opt}) - \frac{1}{2} \eta_c} \]

\[ X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}} \]

\[ Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B}\right)^2} \]

B.C. Solution

\[ \eta = \frac{\eta_{c\infty} Y_{opt}}{T_{\infty B} Z(X_{opt}, D_B, D_A) + \left(1 + Y_{opt}\right) (S_B - S_A) \left[1 - \frac{\eta_{c\infty}}{2} \left(1 - D_{avg}\right)\right] - \frac{1}{2} \eta_{c\infty}} \]

\[ X_{opt} = \sqrt{\frac{k_A \sigma_A D_A}{k_B \sigma_B D_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}) \left[T_{\infty B} \frac{S_B - S_A}{D_B S_B - D_A S_A} \left(1 - D_{avg}\right) - \frac{\Delta T_{\infty}}{2}\right]} \]

\[ Z(X_{opt}, D_A, D_B) = \frac{(D_B S_B - D_A S_A)^2}{\left(\frac{k_A D_A}{\sigma_A} + \frac{k_B D_B}{\sigma_B}\right)^2} \]
Fin Model

Thermal Governing Equation:

\[
\frac{d}{dx} \left[ -k_{A,B} \frac{d\theta_{A,B}}{dx} \right] + \frac{l_{A,B} \tau_{A,B}}{A_{A,B}} \frac{d\theta_{A,B}}{dx} + \frac{P_{A,B} h_{A,B}}{A_{A,B}} \theta_{A,B} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
\]

\[
\theta_{A,B} = T_{A,B} - T_\infty
\]

Fin Parameters

Fin Factor:

\[
F_{A,B} = L_{A,B} \sqrt{\frac{P_{A,B} h_{A,B}}{k_{A,B} A_{A,B}}}
\]

Geometric Fin:

\[
G = \sqrt{\frac{P_B A_B h_B k_A \tanh(F_A)}{P_A a_A h_A k_B \tanh(F_B)}}
\]

Load:

\[
Y = \frac{R}{\sigma_B A_B + \sigma_A A_A}
\]

Materials:

\[
Z(X,G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X}\right) (k_A + k_B G)}
\]

Fin Solution

\[
G_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}, G) T_{avg}}
\]

\[
Z(X_{opt}, G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X_{opt}}\right) (k_A + k_B G)}
\]
### Fin Model

- $T_H \cdot \varphi_H$
- $A_A, P_A, \sigma_A, S_A, k_A$
- $A_B, P_B, \sigma_B, S_B, k_B$
- $T_C \cdot \varphi_{C_{A,B}}$
- $L_A$
- $L_B$
- $R$
- $I$

### Thermal Governing Equation -

\[
\frac{d}{dx} \left[ -k_{A,B} \frac{d\theta_{A,B}}{dx} \right] + \frac{l_{A,B} \tau_{A,B}}{A_{A,B}} \frac{d\theta_{A,B}}{dx} + \frac{p_{A,B} h_{A,B}}{A_{A,B}} \theta_{A,B} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0
\]

\[
\theta_{A,B} = T_{A,B} - T_\infty
\]

### Fin Parameters

#### Fin Factor -

\[
F_{A,B} = L_{A,B} \sqrt{\frac{P_{A,B} h_{A,B}}{k_{A,B} A_{A,B}}}
\]

#### Geometric Fin -

\[
G = \frac{P_B A_B h_B k_A \tanh(F_A)}{P_A A_A h_A k_B \tanh(F_B)}
\]

#### Load -

\[
Y = \frac{L_B}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}
\]

#### Materials -

\[
Z(X, G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X_{opt}}\right) (k_A + k_B G)}
\]

### Fin Solution

#### $G_{opt}$

\[
G_{opt} = \frac{k_A \sigma_A}{k_B \sigma_B}
\]

#### $Y_{opt}$

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}, G) T_{avg}}
\]

#### $Z(X_{opt}, G)$

\[
Z(X_{opt}, G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X_{opt}}\right) (k_A + k_B G)}
\]

---

**Analytic Couple Modeling**
Analytic Couple Modeling

**Classical Parameters**

**Geometric-**

\[ X = \frac{A_B L_A}{A_A L_B} \]

**Load-**

\[ Y = \frac{R}{\sigma_B A_B + \frac{L_A}{\sigma_A A_A}} \]

**Materials-**

\[ Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X}\right)(k_A + k_B X)} \]

**Thermal Governing Equation-**

\[ \frac{d}{dx} \left[ -k_{A,B} \frac{d\theta_{A,B}}{dx} \right] + \frac{l_{A,B} \iota_{A,B}}{A_{A,B}} \frac{d\theta_{A,B}}{dx} + \frac{P_{A,B} h_{A,B}}{A_{A,B}} \theta_{A,B} - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = 0 \]

\[ \theta_{A,B} = T_{A,B} - T_{\infty} \]

**Fin Parameters**

**Fin Factor-**

\[ F_{A,B} = L_{A,B} \sqrt{\frac{p_{A,B} h_{A,B}}{k_{A,B} A_{A,B}}} \]

**Geometric Fin-**

\[ G = \sqrt{\frac{p_B A_B h_B k_A \tanh(F_A)}{p_A A_A h_A k_B \tanh(F_B)}} \]

**Load-**

\[ Y = \frac{R}{\frac{L_B}{\sigma_B A_B} + \frac{L_A}{\sigma_A A_A}} \]

**Materials-**

\[ Z(X,G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X}\right)(k_A + k_B G)} \]

**Fin Solution**

\[ G_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}, G) \Gamma_{avg}} \]

\[ Z(X_{opt}, G) = \frac{(S_y - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_g X_{opt}}\right)(k_A + k_B G)} \]
**Classic Parameters**

**Geometric-**

\[ X = \frac{A_BL_A}{A_LA_B} \]

**Load-**

\[ Y = \frac{R}{\frac{L_B}{A_B} + \frac{L_A}{A_A}} \]

**Materials-**

\[ Z(X) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_BX}\right)(k_A + k_BX)} \]

**Fin Parameters**

**Fin Factor-**

\[ F_{A,B} = L_{A,B} \sqrt{\frac{P_{A,B}h_{A,B}}{k_{A,B}A_{A,B}}} \]

**Geometric Fin-**

\[ G = \sqrt{\frac{P_B A_B h_B k_A}{P_A A_A h_A k_B}} \tanh(F_A) \]

**Load-**

\[ Y = \frac{R}{\frac{L_B}{A_B} + \frac{L_A}{A_A}} \]

**Materials-**

\[ Z(X,G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_BX}ight)(k_A + k_BG)} \]
### Fin Model

- **Fin Parameters**
  - Fin Factor-
    \[ F_{A,B} = L_{A,B} \sqrt{\frac{p_{A,B}}{k_{A,B}} A_{A,B}} \]
  - Geometric Fin-
    \[ G = \sqrt{\frac{p_{y} A_{B} h_{B} k_{A}}{p_{A} A_{A} h_{A} k_{B}} \tanh(F_{A})} \]
  - Load-
    \[ Y = \frac{R}{L_{B}} + \frac{L_{A}}{\sigma_{A} A_{A}} \]

### Classic Solution

- \( \eta_{opt} = \eta_{c} Y_{opt} \)
- \( X_{opt} = \sqrt{\frac{k_{A} \sigma_{A}}{k_{B} \sigma_{B}}} \)
- \( Y_{opt} = \frac{1 + Z(X_{opt}) T_{avg}}{2} \)
- \( Z(X_{opt}) = \frac{(S_{B} - S_{A})^{2}}{\sqrt{\frac{k_{A}}{\sigma_{A}} + \frac{k_{B}}{\sigma_{B}}}^{2}} \)

### Fin Solution

- \( \eta_{opt} = \frac{\eta_{c} Y_{opt}}{F_{A} \left(1 + Y_{opt} \right)^{2}} \sqrt{\frac{1}{\sigma_{A} G(F/2)^{2}} + \left(1 + Y_{opt} \right) - \eta_{c} \frac{\tanh \left(F_{A}^{2} \right)}{F_{A}}} \)
- \( X_{opt} = \sqrt{\frac{k_{A} \sigma_{A}}{k_{B} \sigma_{B}}} \)
- \( Y_{opt} = \sqrt{1 + Z(X_{opt}, G) T_{avg}} \)
- \( Z(X_{opt}, G) = \frac{(S_{B} - S_{A})^{2}}{\left(1 + \frac{1}{\sigma_{A} X_{opt}} \right) \left(\frac{1}{\sigma_{A}} + \frac{1}{\sigma_{B} G} \right)} \)
Example Calculation

<table>
<thead>
<tr>
<th>Convection (W/m-K)</th>
<th>Fin Factor</th>
<th>Max Efficiency (%)</th>
<th>Max Power Density (W/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>6.15</td>
<td>17,733</td>
</tr>
<tr>
<td>5</td>
<td>0.32</td>
<td>6.05</td>
<td>17,733</td>
</tr>
<tr>
<td>500</td>
<td>0.38</td>
<td>2.70</td>
<td>17,733</td>
</tr>
</tbody>
</table>

Classic Solution

\[ \eta_{opt} = \eta_c Y_{opt} \]

\[ X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt})T_{avg}} \]

\[ Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left(\frac{k_A}{\sqrt{\sigma_A}} + \frac{k_B}{\sqrt{\sigma_B}}\right)^2} \]

Fin Solution

\[ \eta_{opt} = \frac{\eta_c Y_{opt}}{F_A \left(1 + Y_{opt}\right)^2 \tanh(F_A T_h Z(X_{opt}, G)) + (1 + Y_{opt}) - \frac{1}{2} \eta_c} \]

\[ G_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}} \]

\[ Y_{opt} = \sqrt{1 + Z(X_{opt}, G)T_{avg}} \]

\[ Z(X_{opt}, G) = \frac{(S_B - S_A)^2}{\left(\frac{1}{\sigma_A} + \frac{1}{\sigma_B X_{opt}}\right)\left(k_A + k_B G\right)} \]
Design Guideline

\[
\left( \frac{P}{A_f} \right)_F \leq \frac{F^2 k}{L^2 h}
\]

\[
\left( \frac{P}{A} \right)_{5\%} = 0.05^2 k \frac{1}{L^{29.9} h}
\]

Classic Solution

\[
\eta_{opt} = \eta_c Y_{opt} = \frac{(1 + Y_{opt})^2}{T_h Z(X_{opt})} + (1 + Y_{opt}) - \frac{1}{2} \eta_c
\]

\[
X_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}) T_{avg}}
\]

\[
Z(X_{opt}) = \frac{(S_B - S_A)^2}{\left( \frac{k_A}{\sigma_A} + \frac{k_B}{\sigma_B} \right)^2}
\]

Fin Solution

\[
\eta_{opt} = \frac{\eta_c Y_{opt}}{F_A (1 + Y_{opt})^2 \tanh(F_A) T_h Z(X_{opt}, G) + (1 + Y_{opt}) - \eta_c} \left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B G(F/2)} \right) \frac{\tanh \left( \frac{F_A}{2} \right) \left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B G(F/2)} \right)}{F_A \left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X_{opt}} \right)}
\]

\[
G_{opt} = \sqrt{\frac{k_A \sigma_A}{k_B \sigma_B}}
\]

\[
Y_{opt} = \sqrt{1 + Z(X_{opt}, G) T_{avg}}
\]

\[
Z(X_{opt}, G) = \frac{(S_B - S_A)^2}{\left( \frac{1}{\sigma_A} + \frac{1}{\sigma_B X_{opt}} \right)} \left( k_A + k_B G \right)
\]
 Variable Model

\[ T_H, \varphi_H \]

Material Properties by Asymptotic Expansion -

\[ \sigma(T) = \hat{\sigma} \frac{\sigma(T)}{\hat{\sigma}} = \hat{\sigma}(\sigma_0 + \epsilon \sigma_1 T) \]

\[ S(T) = \hat{S} \frac{S(T)}{\hat{S}} = \hat{S}(S_0 + \epsilon S_1 T) \]

\[ k(T) = \hat{k} \frac{k(T)}{\hat{k}} = \hat{k}(k_0 + \epsilon k_1 T) \]

Asymptotic Expansion

\[ \hat{\tau} = \frac{T}{\Delta T} \quad \hat{\varphi} = \frac{\varphi}{\Delta S \Delta T} \quad \hat{i} = \frac{IR}{\Delta S \Delta T} \]

\[ \hat{\tau} = T_0 + \epsilon T_1 \quad \hat{\varphi} = \varphi_0 + \epsilon \varphi_1 \]

Variable Solution

Max Conversion Efficiency [%] with Fixed Average Seebeck

Analytic Couple Modeling
Variable Model

Asymptotic Expansion

\[
\hat{T} = \frac{T}{\Delta T} \quad \hat{\phi} = \frac{\phi}{\Delta S \Delta T} \quad \hat{I} = \frac{I R}{\Delta S \Delta T}
\]

\[
\hat{T} = T_0 + \epsilon T_1 \\
\hat{\phi} = \varphi_0 + \epsilon \varphi_1
\]

Material Properties by Asymptotic Expansion-

\[
\sigma(T) = \tilde{\sigma} \left( \frac{\sigma(T)}{\tilde{\sigma}} \right) = \tilde{\sigma}(\sigma_0 + \epsilon \sigma_1 T)
\]

\[
S(T) = \tilde{S} \left( \frac{S(T)}{\tilde{S}} \right) = \tilde{S}(S_0 + \epsilon S_1 T)
\]

\[
k(T) = \tilde{k} \left( \frac{k(T)}{\tilde{k}} \right) = \tilde{k}(k_0 + \epsilon k_1 T)
\]

Variable Solution

Max Conversion Efficiency [%] with Fixed Average Seebeck
Variable Solution

Max Conversion Efficiency [%] with Fixed Average Seebeck

P-type $S_P$, $[1/K]$

N-type $S_N$, $[1/K]$

Analytic Couple Modeling

Material Properties by Asymptotic Expansion

Conv. Efficiency 6.00%

Conv. Efficiency 6.15%

Conv. Efficiency 6.30%
**Transient Model**

\[ T_C(t), \varphi_{CA,B} \]

\[ T_H(t), \varphi_H \]

\[ R \text{ (Ohm)} \]
\[ H \text{ (Henry)} \]

**Thermal** -
\[ \frac{\partial}{\partial x} \left[ -k_{AB} \frac{\partial T_{AB}}{\partial x} \right] + \frac{l_{AB}\tau_{AB}}{A_{AB}} \frac{\partial T_{AB}}{\partial x} - \frac{l_{AB}^2}{A_{AB}^2 \sigma_{AB}} = \frac{\rho_{AB}c_{p,AB}}{\partial t} \]

**Electrical** -
\[ \frac{\partial \varphi_{AB}}{\partial x} = -S_{AB} \frac{\partial T_{AB}}{\partial x} - \frac{l_{AB}}{A_{AB} \sigma_{AB}} \]

**System** -
\[ \varphi_B(L_B) - \varphi_A(L_A) = IR + H \frac{dl}{dt} \]

**Green’s Function Solution**

\[ \mathcal{L}u(x) = f(x) \]

\[ \mathcal{L}^* G(x, \xi) = \delta(x - \xi) \]

\[ u(x) = \int G(x, \xi)f(\xi)d\xi \]

**Transient Parameters**

**Thermal diffusivity factor** -
\[ \Gamma_{AB} = \frac{\alpha_{avg} L_{AB}^2}{\alpha_{AB} L_{avg}^2} \]

**Inductance factor** -
\[ \beta = \frac{H \alpha_{avg}}{RL_{avg}^2} \]
**Transient Model**

\[ T_H(t), \varphi_H \]

\[ T_C(t), \varphi_{C_{A,B}} \]

\[ T(t), \varphi_t \]

\[ x, t \]

\[ L_A \]

\[ A_A, \sigma_A, \]

\[ S_A, k_A, \]

\[ \rho_A, C_P_A \]

\[ A_B, \sigma_B, \]

\[ S_B, k_B, \]

\[ \rho_B, C_P_B \]

\[ R \ (Ohm) \]

\[ H \ (Henry) \]

\[ I \]

**Thermal-**

\[ \frac{\partial}{\partial x} \left[ -k_{A,B} \frac{\partial T_{A,B}}{\partial x} \right] + \frac{l_{A,B}\tau_{A,B} \partial T_{A,B}}{A_{A,B}} \frac{\partial T_{A,B}}{\partial x} = - \frac{l_{A,B}^2}{A_{A,B}^2 \sigma_{A,B}} = \rho_{A,B}c_{p_{A,B}} \frac{\partial T_{A,B}}{\partial t} \]

**Electrical-**

\[ \frac{\partial \varphi_{A,B}}{\partial x} = -S_{A,B} \frac{\partial T_{A,B}}{\partial x} - \frac{l_{A,B}}{A_{A,B} \sigma_{A,B}} \]

**System-**

\[ \varphi_B(L_B) - \varphi_A(L_A) = IR + H \frac{dl}{dt} \]

**Green’s Function Solution**

\[ Lu(x) = f(x) \]

\[ L^* G(x, \xi) = \delta(x - \xi) \]

\[ u(x) = \int G(x, \xi)f(\xi)d\xi \]

**Transient Parameters**

**Thermal diffusivity factor-**

\[ \Gamma_{A,B} = \frac{\alpha_{avg}L_{A,B}^2}{\alpha_{A,B}L_{avg}^2} \]

**Inductance factor-**

\[ \beta = \frac{H \alpha_{avg}}{RL_{avg}^2} \]
**Periodic On/Off Operation**

- Hot Shoe Temperature (K)
- Cycle Efficiency (%)

**Design Guideline**

\[
\frac{L_A}{L_B} = \frac{\sqrt{2a} + 1}{2a - 1}
\]

\[
a = 1 + \frac{\alpha_B}{\alpha_A}
\]

\[
u(x) = \int G(x, \xi)f(\xi)d\xi
\]

**Transient Parameters**

**Thermal diffusivity factor**

\[
\Gamma_{A,B} = \frac{\alpha_{avg} L_{A,B}^2}{\alpha_{A,B} l_{avg}^2}
\]

**Inductance factor**

\[
\beta = \frac{H\alpha_{avg}}{RL_{avg}^2}
\]
Conclusion

• Several new design factors can have a large influence on couple behavior
  • Device Design Factor
  • Fin Factor
  • Thermal Diffusivity Factor
  • Inductance Factor
• The introduced design guidelines must be considered in couple design

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Appendix
Sinusoidal Operation

\[ u(x) = \int G(x, \xi)f(\xi)d\xi \]

Green’s Function Solution

Transient Parameters

Thermal diffusivity factor-

\[ \Gamma_{A,B} = \frac{\alpha_{avg} L_{A,B}^2}{\alpha_{B,A} L_{avg}^2} \]

Inductance factor-

\[ \beta = \frac{H\alpha_{avg}}{RL_{avg}^2} \]