Interpolation Method needed for Numerical Uncertainty Analysis of Computational Fluid Dynamics

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Problem

• Using Computational Fluid Dynamics (CFD) to predict a flow field is an approximation to the exact problem and uncertainties exist.
• There is a method to approximate the errors in CFD via Richardson’s Extrapolation.
  – This method is based off of progressive grid refinement.
• Unless using a Structured Grid with every other point, some interpolation method must be used.
Summary of Richardson’s Extrapolation

- Navier Stokes Equations
  - 2\textsuperscript{nd} order, non-homogeneous, non-linear partial differential equations
- Richardson’s Extrapolation is used to produce 4\textsuperscript{th} order accurate solution from separate 2\textsuperscript{nd} order accurate Navier Stokes Solutions
Summary of Richardson’s Extrapolation


• Assumptions
  1. Three discrete solutions are in the asymptotic range
  2. Meshes have a uniform spacing over the domain
  3. Meshes are related through systematic refinement
  4. Solutions are smooth
  5. Other sources of numerical error are small
5 Step Procedure for Uncertainty Estimation

- Step 1: Representative Grid Size

\[ h = \left[ \left( \frac{\sum_{i=1}^{N} \Delta V_i}{N} \right) \right]^{1/3} \]

where

- \( N \) = total number of cells used for the computations
- \( \Delta V_i \) = volume of the \( i^{th} \) cell [4]

\( h_1 < h_2 < h_3 \)
– Step 2: Select 3 significantly (r>1.3) different grid sizes

\[ r_{21} = \frac{h_2}{h_1} \]
\[ r_{32} = \frac{h_3}{h_2} \]

– Use CFD Simulation to analyze key variables, \( \varphi \)

\[ \varepsilon_{32} = \varphi_3 - \varphi_2 \]
\[ \varepsilon_{21} = \varphi_2 - \varphi_1 \]
Step 3: Calculate observed order, \( p \)

\[
p = \left[ \frac{1}{\ln(r_{21})} \right] \left[ \ln \left| \frac{\varepsilon_{32}}{\varepsilon_{21}} \right| + q(p) \right]
\]

\[
q(p) = \ln \left( \frac{r_{21}^p - s}{r_{32}^p - s} \right)
\]

\[
s = 1 \cdot \text{sign}(\varepsilon_{32}/\varepsilon_{21})
\]
– Step 4: Calculate extrapolated values

\[
\varphi_{\text{ext}}^{21} = \left( r_{21}^p \varphi_1 - \varphi_2 \right) / \left\| r_{21}^p - 1 \right\|
\]

\[
e_{a}^{21} = \left| \frac{\varphi_1 - \varphi_2}{\varphi_1} \right|
\]
– Step 5: Calculate Fine Grid Convergence Index & Numerical Uncertainty

\[ GCI_{fine}^{21} = \frac{Fs \cdot e_a^{21}}{r_p^{21} - 1} \]

The Factor of Safety, \( Fs = 1.25 \)

– Assumption that the distribution is Gaussian about the fine grid, 90% Confidence

\[ U_{num} = GCI / 1.65 \]
Solver Interpolation

• FLUENT
  • Includes a Mesh-to-Mesh Interpolation
  • Performs a zeroth-order (nearest neighbor) interpolation
  • Designed for initial conditions from a previous solution

• OPENFOAM
  • Mapfields fuction interpolation
  • Used for initialization of a solution from a previous model

• Using these ‘zeroth-order’ interpolation schemes is not sufficient for comparing errors from the mesh
Matlab Interpolation Schemes

- Matlab
  - High level language used for numerical computations
- CFD data is in various forms
  - 1D, 2D, 3D, uniform, non-uniform
  - Generic Scheme is sought for all CFD data

<table>
<thead>
<tr>
<th>Interpolation Method</th>
<th>Matlab Function</th>
<th>interp1</th>
<th>interp2</th>
<th>interp3</th>
</tr>
</thead>
<tbody>
<tr>
<td>'nearest'</td>
<td>Nearest neighbor interpolation</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>'linear'</td>
<td>Linear interpolation (default)</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>'spline'</td>
<td>Cubic spline interpolation</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>'pchip'</td>
<td>Piecewise cubic Hermite interpolation</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>'cubic'</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'v5cubic'</td>
<td>Cubic interpolation used in Matlab 5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>interp1</th>
<th>interp2</th>
<th>interp3</th>
</tr>
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<tr>
<td>X</td>
<td>X (uniformly-spaced only)</td>
<td>X (uniformly-spaced only)</td>
</tr>
<tr>
<td>X</td>
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<td>X</td>
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Example Problem

- Fully developed flow between parallel plates
  - Exact Solution to Navier Stokes
  - Provide a good example of errors that can be induced from interpolation

\[
\bar{V} = -\frac{1}{12\mu} \left( \frac{\delta P}{\delta x} \right) a^2
\]

\[
u = \frac{a^2}{2\mu} \left( \frac{\delta P}{\delta x} \right) \left[ \left( \frac{y}{a} \right)^2 - \left( \frac{y}{a} \right) \right]
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a (m))</td>
<td>0.1</td>
</tr>
<tr>
<td>(\rho (kg/m^3))</td>
<td>1.225</td>
</tr>
<tr>
<td>(\mu (Ns/m^2))</td>
<td>0.00001789</td>
</tr>
<tr>
<td>(\frac{dp}{dx} (N/m^3))</td>
<td>-0.004</td>
</tr>
</tbody>
</table>
Example Problem

- Constructed a CFD Model in FLUENT
  - 3 Grids
    - Coarse, 7,140 Cells
    - Medium, 14,186 Cells
    - Fine, 24,780 Cells
Example Problem

- Interpolation Direction?
  1. Interpolate Coarse and Medium Mesh -> Fine
  2. Interpolate Medium and Fine Mesh -> Coarse

\[ \varepsilon_{21} = \varphi_2 - \varphi_1 \]
\[ \varepsilon_{32} = \varphi_3 - \varphi_2 \]
Example Problem

1. Linearly Interpolate Coarse and Medium Mesh -> Fine

Max % Error Extrapolated Values
Average % Error Extrapolated Values

0.8950
0.0596
2. Linearly Interpolate Fine and Medium Mesh -> Coarse

Max % Error Extrapolated Values

Average % Error Extrapolated Values

0.0792 0.0175
Example Problem

• Interpolation Direction?

1. Interpolate Coarse and Medium Mesh -> Fine

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1. Interpolate Medium and Fine Mesh -> Coarse

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Example Problem

- Interpolating to the coarse grid was selected
- Other interpolation methods
  - "nearest" – Fluent’s Mesh-to-Mesh
  - "linear" – Matlab
    \[ y_{fi} = \text{interp1}(	ext{fine}(:,2),\text{fine}(:,1),\text{coarse}(:,2),'linear') \]
  - "cubic" – Matlab
    \[ y_{fi} = \text{interp1}(	ext{fine}(:,2),\text{fine}(:,1),\text{coarse}(:,2),'cubic') \]
Example Problem

- “nearest” – Fluent’s Mesh-to-Mesh

Zeroth Order Interpolation
Example Problem

- “linear”

Matlab

```matlab
yfi = interp1(fine(:,2),fine(:,1),coarse(:,2),'linear')
```

1st Order Interpolation
Example Problem

- "cubic" – Matlab

\[ yfi = \text{interp1}(\text{fine}(:,2), \text{fine}(:,1), \text{coarse}(:,2), 'cubic') \]

3rd Order Interpolation
Matlab Interpolation Schemes

- Extending the Interpolation Schemes to 2D and 3D
  - Interp2 and Interp3 Matlab Functions
    - Require use of MeshGrid
      - Transforms the domain of vectors into arrays
      - For Meshes in the 4 million to 8 million Cell Range
        - Error “Maximum variable size allowed by program is exceeded”
  - Griddata Function
    - Nearest, Linear, Natural, Cubic, and v4
    - Nearest, Linear, and Natural are the only options available in 2D and 3D

- The only options available for 1D, 2D, and 3D
  - Interp1 and Griddata – ‘nearest’ and ‘linear’
3D Example

- Airflow around encapsulated spacecraft
  - Matlab griddata ‘linear’ option used
  - Interpolating Fine and Medium Grid onto Coarse Grid
3D Example

• Comparing using a Line Plot
3D Example

• Comparing using a Line Plot
Conclusion / Recommendation

• By comparing the interpolation schemes in one, two, and three dimensions and investigating the options that are readily available in Matlab
  • Recommend the “linear” option be used when comparing the error or uncertainty due to the grid
    • interp1 or griddata Matlab commands
• If coarse grid has the level of detail required
  • Recommend interpolating from the fine and medium grids onto the coarse grid
    • Lower Error in the Extrapolated Solution
    • Smaller Data Set

— Future Work include higher order interpolation schemes in 3D (Radial Basis Function Interpolation, 4th order)