On spectral invariance of single scattering albedo for water droplets and ice crystals at weakly absorbing wavelengths

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A B S T R A C T

The single scattering albedo $\omega_{0\lambda}$ in atmospheric radiative transfer is the ratio of the scattering coefficient to the extinction coefficient. For cloud water droplets both the scattering and absorption coefficients, thus the single scattering albedo, are functions of wavelength $\lambda$ and droplet size $r$. This note shows that for water droplets at weakly absorbing wavelengths, the ratio $\omega_{0\lambda}(r)/\omega_{0\lambda}(r_0)$ of two single scattering albedo spectra is a linear function of $\omega_{0\lambda}(r)$. The slope and intercept of the linear function are wavelength independent and sum to unity. This relationship allows for a representation of any single scattering albedo spectrum $\omega_{0\lambda}(r)$ via one known spectrum $\omega_{0\lambda}(r_0)$. We provide a simple physical explanation of the discovered relationship. Similar linear relationships were found for the single scattering albedo spectra of non-spherical ice crystals.

In general, the single scattering albedo $\omega_{0\lambda}$ as a function of effective radius $r$ and wavelength $\lambda$, is defined as

$$\omega_{0\lambda}(r) = \frac{\sigma_{s\lambda}(r)}{\sigma_{s\lambda}(r) + \sigma_{a\lambda}(r)}$$

where $\sigma_{s\lambda}$ and $\sigma_{a\lambda}$ are the scattering and absorption coefficients, respectively. To compute the single scattering albedo a Mie code [3] was run for four wavelengths (0.86, 1.65, 2.13 and 3.75 $\mu$m) and for $r$ from 5 to 50 $\mu$m assuming a gamma size distribution with effective variance 0.1 $\mu$m [6]. When the ratio $\omega_{0\lambda}(r)/\omega_{0\lambda}(r_0)$ with $r_0=5\mu$m was plotted against $\omega_{0\lambda}(r)$ as a function of the four wavelengths $\lambda$, a remarkable linear relationship was discovered:

$$\frac{\omega_{0\lambda}(r)}{\omega_{0\lambda}(r_0)} = p_1 \omega_{0\lambda}(r) + p_2,$$

Here slope $p_1$ and intercept $p_2$ are wavelength independent parameters. Since $\omega_{0\lambda}(r_0)=1$ at $\lambda=0.86\mu$m for all effective radii $r$, the sum of $p_1$ and $p_2$ is equal to 1; so (2)
can be rewritten as
\[
\frac{\omega_{\lambda_s}(r)}{\omega_{\lambda_s}(r_0)} = p \omega_{\lambda_s}(r) + (1-p).
\] (3)

Fig. 1 illustrates Eq. (3) for \(r=10, 20\) and \(40\) \(\mu\)m and \(r_0=5\) \(\mu\)m. In the next section we provide a simple physical explanation of the spectral invariant relationship (3) and demonstrate its validity for a wide spectral region.

2. Justification of the main results for water droplets

We start with the definition of the single scattering albedo (1). For the visible and near-infrared spectral regions, the scattering efficiency \(\sigma_s\) does not vary much with either wavelength or droplet size; thus we can assume that
\[
\sigma_{\lambda_s} \approx \sigma_s < r^2 >
\] (4)

where \(\sigma_s=\text{const}\) and \(< r^2 >\) is the second moment of the droplet size distribution. To be consistent with notations, we will use symbol \(\sigma_{\lambda_s}\) for the ratio of the absorption coefficients to \(< r^2 >\) through the rest of this paper. In terms of these notations, Eq. (1) can be rewritten as
\[
\omega_{\lambda_s}(r) = \frac{\sigma_s}{\sigma_s + \sigma_{\lambda_s}(r)}.
\] (5)

The ratio of \(\omega_{\lambda_s}\) for two effective radii \(r\) and \(r_0\) \((r_0 < r)\) will be
\[
\frac{\omega_{\lambda_s}(r)}{\omega_{\lambda_s}(r_0)} = \frac{\sigma_s}{\sigma_s + \sigma_{\lambda_s}(r)}
\] (6)

Following Twomey and Bohren [12], for weakly absorbing water droplets
\[
1 - \omega_{\lambda_s}(r) \approx k_s r
\] (7)

where \(k_s\) is the bulk absorption coefficient (4\(\pi\) multiplied by the ratio of the imaginary part of the refractive index to wavelength). For wavelengths \(\lambda\) between 0.2 and 2.5 \(\mu\)m, the absorption coefficient \(10^{-4} < k_s < 10^2\) \(\text{cm}^{-1}\) ([4], Fig. 2.25, pg. 113). Coefficient \(c\) in front of \(k_s\) (equal to 0.85 in [12] for a single droplet) can be well approximated by 2/3 (Frank Evans, personal communication).

Next, since
\[
1 - \omega_{\lambda_s}(r) = \frac{\sigma_{\lambda_s}(r)}{\sigma_s + \sigma_{\lambda_s}(r)}
\] (8)

it follows from Eqs. (7) and (8) that
\[
\sigma_s \approx \frac{\omega_{\lambda_s}(r)}{1 - \omega_{\lambda_s}(r)} \approx \frac{1 - ck_s r}{ck_s r}.
\] (9)

Neglecting the second order terms \(\delta(k_s r)^2\), we get
\[
\sigma_{\lambda_s}(r) \approx \sigma_s - \sigma_s k_s r = \sigma_s (ck_s r/(ck_s r)^2 + (ck_s r)^3 + ... \approx \sigma_s c k_s r.
\] (10)

Let us now define a wavelength independent parameter \(p\) as
\[
p = 1 - r_0/r.
\] (11)

It follows that
\[
r_0 = r(1-p).
\] (12)

We now rewrite the ratio (6) as
\[
\frac{\omega_{\lambda_s}(r)}{\omega_{\lambda_s}(r_0)} = 1 - \frac{\sigma_{\lambda_s}(r_0)}{\sigma_s + \sigma_{\lambda_s}(r_0)}
\] (13)

Applying approximation (10) and relationship (12) to the second term in (13) gives
\[
\frac{\sigma_s (1-p) + \sigma_{\lambda_s}(r_0)}{\sigma_s + \sigma_{\lambda_s}(r)} \approx \frac{(1-p) + \sigma_s c k_s r_0}{\sigma_s + \sigma_s c k_s r} = \frac{(1-p) + ck_s r (1-p)}{1 + ck_s r} = 1-p.
\] (14)

Eq. (14) completes the “proof” of (3) with \(p\) defined by (11). Eq. (3) is an approximation but it can be good enough for many purposes. Rearranging Eq. (3) leads to
\[
\omega_{\lambda_s}(r) = \omega_{\lambda_s}(r_0) \frac{1 - p}{1 - p \omega_{\lambda_s}(r_0)}.
\] (15)
Eq. (15) states that the single scattering albedo \( \omega_{0}(r) \) can be expressed via one known spectrum \( \omega_{0}(r_{0}) \). If one knows \( r \) and \( r_{0} \), the slope \( p = \frac{p_{\text{app}}}{r_{0}} \) can be approximated by Eq. (11). However if \( r \) is unknown, \( p \) can be obtained directly from Eq. (3). For example, if one knows the two single scattering albedos \( \omega_{0}(r) \) and \( \omega_{0}(r_{0}) \) at least at one weakly absorbing wavelength, say, \( \lambda = 2.1 \mu \text{m} \), then \( p \) can be determined from Eq. (3) as the slope between two points, one at \( \lambda = 2.1 \mu \text{m} \) and the other at any visible wavelength since \( \omega_{0}(r) = \omega_{0}(r_{0}) = 1 \) there. Knowing \( p \), Eq. (15) provides \( \omega_{0}(r) \) for any wavelength \( \lambda \) if \( \omega_{0}(r_{0}) \) is available.

How well does Eq. (11) approximate the slope \( p \) as a function of \( r \) and \( r_{0} \)? Fig. 2 illustrates two slopes: \( p \) from the Mie calculations and its approximation \( p_{\text{app}} = 1 - r_{0}/r \) as a function of \( r \) with given \( r_{0} = 5 \mu \text{m} \). In spite of a similar shape, there is a substantial difference which is due to approximation (7) and to neglecting the second and higher order terms in (10). (In general, \( p \) is a linear function of the ratio \( r_{0}/r \).) The left panel shows the strong correlation between the two slopes.

### 3. More wavelengths

As mentioned earlier, Eq. (3) is valid for weakly absorbing wavelengths only. For each droplet effective radius \( r \) we define the weakly absorbing wavelengths as ones with \( k_{r} r < 0.1 \) [thus \( \langle k_{r}^{2} \rangle < 0.01 \)]. For example, for \( r = 10 \mu \text{m} \), the weakly absorbing wavelengths are the ones with bulk absorption coefficient \( k_{b} > 100 \text{ cm}^{-1} \). The left panel in Fig. 3 illustrates the ratio between two single scattering albedo spectra \( \omega_{0}(r)/\omega_{0}(r_{0}) \) for all wavelengths from 0.2 to 4 \( \mu \text{m} \) and droplet effective radii: \( r = 20 \mu \text{m} \), \( r_{0} = 10 \mu \text{m} \). In this plot wavelengths are divided into two spectral regions: 0.2 to 2.5 \( \mu \text{m} \) (black dots) and 2.5 to 4 \( \mu \text{m} \) (grey dots). For the first region \( k_{b} < 100 \text{ cm}^{-1} \) while for the second one \( k_{b} > 100 \text{ cm}^{-1} \) (see the right panel in Fig. 4 adapted from Fig. 2.25 of [4]).

For the weakly absorbing spectral interval (0.2–2.5 \( \mu \text{m} \)) the ratio fits a linear function because contributions from the second, \( o\langle k_{r}^{2} \rangle \), and higher order terms in (7) and (10) are negligible. The slope of the linear fit \( p = 0.47 \) is close to...
the one \( (p=0.5) \) predicted by Eq. (11) with \( r=20 \mu m \) and \( r_0=10 \mu m \). We have also added the four wavelengths used in Fig. 1 (red dots). Note that \( \lambda = 3.75 \mu m \) is just on the edge between the two spectral regions since for water \( k_{s,75} = 117 \text{ cm}^{-1} \).

### 4. Examples for ice crystals

Is Eq. (3) valid for non-spherical ice crystals? To check this we used the single scattering albedo for different types of ice crystals (bullet rosettes, hollow and solid columns, aggregates and plates) provided by Dr. Ping Yang (see [14]). The results are shown in Fig. 4. The regression coefficients \( R \) while all above 0.998, are not as good as in the case of spheres (cf. Fig. 1), but the single scattering albedo for the ice crystals exhibits the same spectral invariant behaviour as Eq. (3) predicts.

The bottom right panel of Fig. 4 illustrates that the slopes \( p \) can be well approximated by Eq. (11). To get a better linear regression the longer wavelength 3.75 \( \mu m \) was excluded from the slope calculation and only three shorter wavelengths (0.86, 1.65, and 2.13 \( \mu m \)) were used.

![Fig. 4. (upper 5 panels) The same as in Fig. 1 but for different ice crystal habits. (bottom right panel) Slope \( p \) as a function of effective radius \( r \) for the ice crystal habits from the upper five panels and the slope \( p_{appr} \) is from Eq. (11) where \( r_0=5 \mu m \). Slope \( p \) is calculated using only three wavelengths (0.86, 1.65, and 2.13 \( \mu m \)) since the regression is poorer when all 4 wavelengths are used (see text for details).](image-url)
As shown in Fig. 3b for water, the bulk absorption coefficient $k_\lambda$ at $\lambda=3.75$ μm is substantially bigger than that for shorter wavelengths; moreover, the imaginary part of the refractive index for ice at $\lambda=3.75$ μm exceeds almost twice the one for water (e.g., $k_{3.73}=225$ cm$^{-1}$ at temperature of 266 K, see [13]). This causes a poorer regression when all 4 wavelengths were used.

The Moderate Resolution Imaging Spectroradiometer (MODIS) ice models available at [http://www.ssec.wisc.edu/~baum/Cirrus/MODIS_V2models.html](http://www.ssec.wisc.edu/~baum/Cirrus/MODIS_V2models.html) uses a weighted combination of different ice crystal habits (see [1,2]). For small sizes ($r<30$ μm) MODIS uses 100% droxtals [15] while for sizes between 30 μm and 500 μm, there are 15% of bullet rosettes, 50% solid columns and 35% plates. In Fig. 5 we used crystal sizes from both categories with $r<60$ μm. As Fig. 5 illustrates, the single scattering albedo of the MODIS ice model for weakly absorbing wavelengths also follows the spectral invariant relationship (3) with slope $p$ predicted by Eq. (11) with $r_0=10$ μm and 15 μm < $r$ < 55 μm.

5. Concluding remarks

It was shown numerically and confirmed theoretically that the spectrum of single scattering albedo $ω_0(r)$ of water droplets of effective radius $r$ at weakly absorbing wavelengths $\lambda$ can be well approximated by another spectrum $ω_0(r_0)$ of effective radius $r_0$ using Eq. (15) assuming that the slope $p(r,r_0)$ between the ratio $ω_0(r)/ω_0(r_0)$ and $ω_0(r_0)$ in the linear relationship (3) is known. The slope $p(r,r_0)$ can be approximated by a linear function of $r_0/r$ (see Eq. (11)). In addition to spherical water droplets, it was also shown that the linear relationship (3) (and thus Eq. (15)) is a good approximation for irregular ice crystals of different shapes [14]. For each effective radius $r$, by weakly absorbing we mean all wavelengths with $k_\lambda r < 0.1$ where $k_\lambda=4\pi m_\lambda/\lambda$ is the bulk absorption coefficient ($m_\lambda$ is the imaginary part of the refractive index and $\lambda$ is wavelength).

The linear relationship (3) and its counterpart (15) allow us to relate radiances as a function of single scattering albedo $ω_0(r_0)$ to radiances at other single scattering albedos $ω_0(r)$ in the spectrally invariant relationship between radiances discussed in [10]. This study is yet in progress. The linear relationship (3) also helps to interpret a spectrally invariant relationship in zenith radiances observed near cloud edges [9] and confirmed with intensive radiative transfer calculations [5].

Finally, we note that a spectral invariant relationship for single scattering albedo of green leaves was theoretically predicted by Lewis and Disney [8]. It was found that in the weakly absorbing spectral interval between 710 nm and 790 nm, spectra of green leaves from different species (hazelnut, aspen, jack pine, etc.) were related to one fixed spectrum, called a reference spectrum, via Eq. (13); this property was used in classification of forest types from hyperspectral data [7,11].

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References