ON FORMULATIONS OF DISCONTINUOUS GALERKIN
AND FLUX RECONSTRUCTION METHODS FOR
CONSERVATION LAWS

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High-Order Methods

Discontinuous Galerkin (DG) methods by Reed and Hill 1973, Cockburn and Shu 1990’s, Bassi and Rebay 1997, 2000 …
  • Integral form, stable, powerful machinery
  • Not intuitive

Staggered-Grid methods by Kopriva and Kolias 1996; Spectral Difference (SD) scheme by Liu, Vinokur, and Wang 2004, ...
  • Differential form, simple and intuitive
  • Mildly unstable

  • Differential form, recovers DG, SD, Spectral Volume
  • Simple, economical, and intuitive
  • Stability proofs (Jameson 2010, Vincent el al. 2011,...)
Outline

• Review DG method
• New strong form (approximate delta functions)
• FR methods by integrating the new strong form
• Fourier and energy stability
• Conclusions
Conservation Laws

Conservation law

\[ u_t + f_x = 0 \]

with initial condition

\[ u(x,0) = u_{\text{init}}(x). \]

Calculate the solution \( u(x,t) \)
Legendre Polynomials

Let $P_m$ be the space of polynomials of degree $m$ or less.

On $I = [-1,1]$, for any two continuous functions $v$ and $w$

$$(v, w)_I = (v, w) = \int_{-1}^{1} v(\xi)w(\xi)d\xi$$

Let the Legendre polynomial of degree $i$ be denoted by $L_i$ and defined by

$L_i \perp P_{i-1}$ and $L_i(1) = 1$. 

Projection

On $I = [-1,1]$, the projection of a function $\nu$ onto $P_m$ is

$$\mathcal{P}_m(\nu) = \sum_{i=0}^{m} \frac{(\nu, L_i)}{(L_i, L_i)} L_i.$$
Discretization

For each cell $E_j$, with the local coordinate $\xi$ on $[-1,1]$,

$$u_j(\xi) = \sum_{i=0}^{k} u_{j,i} L_i(\xi)$$

At time $t^n$, (dropping superscript $n$) suppose the data $u_{j,i}$ are known for all $j$ and $i$.

We wish to calculate $f_x$ for $(u_j)_t + (f(u_j))_x = 0$. 
Interface Flux

At each interface $j - 1/2$, using $u^-_{j-1/2}$ and $u^+_{j-1/2}$, define a flux $f^I_{j-1/2}$ (say, Roe's flux) common for the two adjacent cells.
Jumps at interfaces

On $E = E_j$, denote $(u, v)_E = \int_E u(x)v(x)dx$.

Set $[f]_L = f^+_L - f^-_L$ and $[f]_R = f^+_R - f^-_R$.

$f^+_L = f(u_j(x_{j-1/2}))$, $f^-_L = f(u_j(x_{j-1/2}))$

$f^+_R = f(u_j(x_{j+1/2}))$, $f^-_R = f(u_j(x_{j+1/2}))$

$E = E_j$
Review DG Formulation

On $E$, with test function $\phi$ (degree $k$),

$$(u_h, \phi)_t + ((f(u_h))_x, \phi) = 0.$$ 

Integrate by parts,

$$(u_h, \phi)_t + (f\phi)_{\partial E} - (f(u_h), \phi_x) = 0.$$ 

Allow data across cells to interact by

$$(u_h, \phi)_t + (f^I \phi)_{\partial E} - (f(u_h), \phi_x) = 0.$$ 

The above is the weak form. Equivalently,

$$(u_h, \phi)_t + f_R^I \phi_R - f_L^I \phi_L - (f(u_h), \phi_x) = 0.$$
Review DG Formulation

Weak form: on $E$

$$(u_h, \phi)_t + f_R \phi_R - f_L \phi_L - (f(u_h), \phi_x) = 0.$$ 

With $[f]_L = f^I_L - f^+_L$, and $[f]_R = f^I_R - f^-_R$, integrate by parts again, we obtain the strong form

$$(u_h, \phi)_t + ((f(u_h))_x, \phi) + [f]_R \phi_R - [f]_L \phi_L = 0.$$ 

The task is to eliminate $\phi$. 
Approximate Dirac Delta Function

* For a fixed $\alpha$ on $I = [-1,1]$, let the approximate (Dirac) delta function to degree $k$ at $\alpha$ be a linear functional on $P_k$:

$$\delta_\alpha(\phi) = \phi(\alpha).$$

* There exists a polynomial of degree $k$ denoted by $\gamma_{\alpha,k} = \gamma_\alpha$, i.e., $\gamma_\alpha \in P_k$, such that

$$(\gamma_\alpha,\phi) = \phi(\alpha).$$

* Proof. Set $\gamma_\alpha = \sum_{i=0}^k b_i L_i$. Then $(\gamma_\alpha,L_m) = (\sum_{i=0}^k b_i L_i,L_m)$,

or $L_m(\alpha) = b_m(L_m, L_m)$, or $b_m = L_m(\alpha)(2m+1)/2$.

That is,

$$\gamma_\alpha = \delta_\alpha = \delta_{\alpha,k} = \sum_{i=0}^k \frac{2i+1}{2} L_i(\alpha)L_i.$$
Approximate Dirac Delta Function

\[ \delta_{-1, k} = \sum_{i=0}^{k} \frac{2i + 1}{2} (-1)^i L_i \quad \text{and} \quad \delta_{1, k} = \sum_{i=0}^{k} \frac{2i + 1}{2} L_i. \]

\[ \|L_i\| = \sqrt{\frac{2}{2i+1}} \]

\[ \left\| \frac{2i+1}{2} L_i \right\| = \sqrt{\frac{2i+1}{2}} \]
New Strong Form

Standard strong form

\[(u_h, \phi)_t + ((f(u_h))_x, \phi) + [f]_R \phi_R - [f]_L \phi_L = 0.\]

Using the approximate delta functions,

\[(u_h, \phi)_t + ((f(u_h))_x, \phi) + [f]_R (\delta_R, \phi) - [f]_L (\delta_L, \phi) = 0.\]

Using the projection onto \( P_k \),

\[(u_h, \phi)_t + (\mathcal{P}_k [f(u_h)]_x, \phi) + [f]_R (\delta_R, \phi) - [f]_L (\delta_L, \phi) = 0.\]

New strong form

\[(u_h)_t + \mathcal{P}_k ((f(u_h))_x) + [f]_R \delta_R - [f]_L \delta_L = 0.\]
Three Members of a Family of FR Schemes

Scheme DG

\[
(u_h)_t + \mathcal{P}_k((f(u_h))_x) + \\
[f]_R \left( \delta_{R,k-1} + \frac{2k + 1}{2} L_k \right) - [f]_L \left( \delta_{L,k-1} + (-1)^k \frac{2k + 1}{2} L_k \right) = 0.
\]

Scheme \( g_{Ga} \)

\[
(u_h)_t + \mathcal{P}_k((f(u_h))_x) + \\
[f]_R \left( \delta_{R,k-1} + \frac{k + 1}{2} L_k \right) - [f]_L \left( \delta_{L,k-1} + (-1)^k \frac{k + 1}{2} L_k \right) = 0.
\]

Scheme \( g_2 \)

\[
(u_h)_t + \mathcal{P}_k((f(u_h))_x) + \\
[f]_R \left( \delta_{R,k-1} + \frac{k}{2} L_k \right) - [f]_L \left( \delta_{L,k-1} + (-1)^k \frac{k}{2} L_k \right) = 0.
\]
New Strong Forms

Strong form  S1

\[(u_h)_t + \mathcal{P}_k((f(u_h))_x) + [f]_R \delta_R - [f]_L \delta_L = 0.\]

Strong form  S2

\[(u_h)_t + (\mathcal{P}_k(f(u_h)))_x + [f]_R \delta_R - [f]_L \delta_L = 0.\]

- Derivative with no interaction: projection or interpolation; for form S1, interpolate via chain rule: \((f(u))_x = a(u) u_x\)
- Interaction: approximate delta function, exact to degree \(k\).
Energy-Stable FR (ESFR) Schemes

Strong form S1 and S2 for DG method (linear advection),

\[ (u_h)_t + a(u_h)_{\xi} + [f]_R \left( \delta_{R,k-1} + \frac{2k+1}{2} L_k \right) - [f]_L \left( \delta_{L,k-1} + (-1)^k \frac{(2k+1)}{2} L_k \right) = 0. \]

ESFR schemes made simple: \( \alpha_k > 0 \)

\[ (u_h)_t + a(u_h)_{\xi} + [f]_R \left( \delta_{R,k-1} + \alpha_k L_k \right) - [f]_L \left( \delta_{L,k-1} + (-1)^k \alpha_k L_k \right) = 0. \]

Key idea of the proof: Differentiate \( k \) times in \( \xi \)

\[ \left( \frac{d^k u_h}{d \xi^k} \right)_t + [f]_R \left( \alpha_k \frac{d^k L_k}{d \xi^k} \right) - [f]_L \left( (-1)^k \alpha_k \frac{d^k L_k}{d \xi^k} \right) = 0. \]
Reconstructing the Flux by Integrating the Strong Form S1

\[ S1 \quad (u_h)_t + \mathcal{P}_k((f(u_h))_x) + [f]_R \delta_R - [f]_L \delta_L = 0. \]

1. Flux polynomial (no interaction), i.e.,

   discontinuous flux function, deg. \( k + 1 \)

   \[ f_{IPD}(\eta) = f_L^+ + \int_{-1}^{\eta} \mathcal{P}_k((f(u_h))_{\xi}) \, d\xi \]

   \( f_{IPD} \) of degree \( k + 1 \) determined by

   \[ f_{IPD}(-1) = f_L^+, \quad f_{IPD}(1) = f_R^- \]

   and \( \mathcal{P}_{k-1}(f_{IPD}) = \mathcal{P}_{k-1}(f(u_h)) \)
FR: Integrate the Strong Form S1

\[
S1 \quad (u_h)_t + \mathcal{P}_k((f(u_h))_x) + [f]_R \delta_R - [f]_L \delta_L = 0.
\]

2(a). Correction function for the right boundary

\[
g_R(\xi) = \int_{-1}^{\xi} \delta_R(\eta) d\eta
\]

\[
g_R' = \delta_R
\]

\(g_R\) is of degree \(k + 1\):

\[
g_R(-1) = 0,
\]

\[
g_R(1) = 1,
\]

\[
\mathcal{P}_{k-1}(g_R) = 0.
\]
FR: Integrate the Strong Form S1

\[ S1 \quad (u_h)_t + \mathcal{P}_k((f(u_h))_x) + [f]_R \delta_R - [f]_L \delta_L = 0. \]

2(b). Correction function for the left boundary

\[ g_L(\xi) = \int_\xi^1 \delta_L(\eta) d\eta \]

\[ g_L' = -\delta_L \]

\( g_L \) is of degree \( k + 1 \): 

\[ g_L(-1) = 1, \]

\[ g_L(1) = 0, \]

\[ \mathcal{P}_{k-1}(g_L) = 0. \]
Flux Reconstruction Form

On $E$, for nonlinear conservation laws, set

$$F = f_{\text{IPD}} + [f]_L g_L + [f]_R g_R.$$ 

Then $F$ is of degree $k + 1$ determined by

$$F(-1) = f^I_L, \quad F(1) = f^I_R,$$

and

$$\mathcal{P}_{k-1}(F) = \mathcal{P}_{k-1}(f(u_h)).$$

Also,

$$F_\xi = \mathcal{P}_k\left(\left(f(u_h)\right)_\xi\right) + [f]_L \delta_L + [f]_R \delta_R.$$
Reconstructing the Flux

Example: advection equation with $k = 1$. 

(a) Data

(b) DG
A Family of Fourier Stable FR Schemes

Let $g_L$ of deg. $k+1$ be defined by

$$g_L(-1) = 1, \quad g_L(1) = 0,$$

and $k$ additional conditions.

For DG,

$$\mathcal{P}_{k-1}(g_L) = 0.$$

For a family of stable schemes,

$$\mathcal{P}_{k-2}(g_L) = 0.$$
Correction functions \((k = 1)\)

(a) Data

(b) DG (projection)

(c) Scheme \( g_{Ga} \) (SD)

(d) Scheme \( g_2 \)
Second-Order FR Schemes \((k = 1)\)

(a) Data

(b) DG

(c) Spectral Difference (SD)

(d) Scheme \(g_2\)
Correction Functions for Fourier-Stable Schemes

1. \[ g_{DG} = R_{R,k+1} \]

\( g_{DG} \) results in the DG method.

2. \[ g_{Ga} = \frac{k + 1}{2k + 1} R_{R,k+1} + \frac{k}{2k + 1} R_{R,k} \]

\( g_{Ga} \) vanishes at the \( k \) Gauss points

3. \[ g_2 = \frac{k}{2k + 1} R_{R,k+1} + \frac{k + 1}{2k + 1} R_{R,k} \]

\( g_2 \) vanishes at \( k \) of the \( k + 1 \) Lobatto points
Correction functions for $k = 3$

\[ g_{Ga} \]

\[ g_2 \]
Fourier Analysis, $k = 0$
Fourier Analysis, $k = 1$

\[
g_{DG} = \frac{3}\xi^2 - \frac{\xi}{2} - \frac{1}{4}, \quad g_{Ga} = \frac{\xi^2}{2} - \frac{\xi}{2}, \quad \text{and} \quad g_2 = \frac{\xi^2}{4} - \frac{\xi}{2} + \frac{1}{4}.
\]
## Fourier Analysis, $k = 1$

### Orders of accuracy and errors

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Order of accuracy</th>
<th>Coarse mesh error, $w = \pi/8$</th>
<th>Fine mesh error, $w = \pi/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>3</td>
<td>$-3.2 \times 10^{-4} - 3.3 \times 10^{-5}i$</td>
<td>$-2.1 \times 10^{-5} - 1.1 \times 10^{-6}i$</td>
</tr>
<tr>
<td>$g_{Ga}$</td>
<td>2</td>
<td>$-7.1 \times 10^{-4} + 2.4 \times 10^{-3}i$</td>
<td>$-4.6 \times 10^{-5} + 3.1 \times 10^{-4}i$</td>
</tr>
<tr>
<td>$g_2$</td>
<td>2</td>
<td>$-2.5 \times 10^{-3} + 9. \times 10^{-3}i$</td>
<td>$-7.1 \times 10^{-4} + 2.4 \times 10^{-3}i$</td>
</tr>
</tbody>
</table>
Fourier Analysis, $k = 2$
Fourier Analysis, $k = 2$

**Orders of accuracy and errors**

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Order of accuracy</th>
<th>Coarse mesh error, $w = \pi / 8$</th>
<th>Fine mesh error, $w = \pi / 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>5</td>
<td>$-5. \times 10^{-7} - 3.4 \times 10^{-8} i$</td>
<td>$-7.9 \times 10^{-9} - 2.7 \times 10^{-10} i$</td>
</tr>
<tr>
<td>$g_{Ga}$</td>
<td>4</td>
<td>$-1.4 \times 10^{-6} + 8.5 \times 10^{-6} i$</td>
<td>$-2.2 \times 10^{-8} + 2.7 \times 10^{-7} i$</td>
</tr>
<tr>
<td>$g_2$</td>
<td>4</td>
<td>$-3.2 \times 10^{-6} + 1.9 \times 10^{-5} i$</td>
<td>$-5. \times 10^{-8} + 6. \times 10^{-7} i$</td>
</tr>
</tbody>
</table>
Fourier Analysis, $k = 3$
Fourier Analysis, $k = 3$

Orders of accuracy and errors

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Order of accuracy</th>
<th>Coarse mesh error, $w = \pi / 4$</th>
<th>Fine mesh error, $w = \pi / 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DG</td>
<td>7</td>
<td>$-1.0 \times 10^{-7} - 1.0 \times 10^{-8} i$</td>
<td>$-4.0 \times 10^{-10} - 2.0 \times 10^{-11} i$</td>
</tr>
<tr>
<td>$g_{Ga}$</td>
<td>6</td>
<td>$-3.1 \times 10^{-7} + 1.3 \times 10^{-6} i$</td>
<td>$-1.2 \times 10^{-9} + 1.1 \times 10^{-8} i$</td>
</tr>
<tr>
<td>$g_2$</td>
<td>6</td>
<td>$-5.4 \times 10^{-7} + 2.3 \times 10^{-6} i$</td>
<td>$-2.2 \times 10^{-9} + 1.9 \times 10^{-8} i$</td>
</tr>
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</table>
Stability

* For solutions of degree $k$, if $g$ is orthogonal to $P_{k-2}$, then the (family) scheme is Fourier as well as energy-stable.

* The above condition is not necessary: $g_{\text{Lump, Ch-Lo}}$ is not orthogonal to any $P_m$, but the resulting scheme is Fourier-stable.

Open problems

1. The collection of all $g$ resulting in stable schemes remains to be identified.

2. Is Fourier stability equivalent to energy stability?
Energy Stability

- Jameson (2010) proved that a particular SD scheme (recovered via FR) is energy-stable.
- Vincent, Castonguay, and Jameson (2011) proved energy stability for a family of FR schemes.
- Energy-stability proofs for advection and advection diffusion equations in 1D, 2D, and 3D were provided by Vincent, Castonguay, Williams, and Jameson.
- Can the current simplified proof for energy stability be extended to 2D, 3D, and tensor product cases?
Summary

• Review DG method
• New strong forms (approximate delta functions)
• Reconstruct the flux: FR methods
• Simplified energy-stability proof
• Open problems (for grad student, 1 month of study)
• NASA Report TM-2014-218135  June 1014  (pdf)
• There is significant current research activities in FR methods for practical flow problems in CFD.
Thank you for your attention.

Questions/Comments