Why You’re Receiving this Briefing

• Since inception, CARA has required owner/operator predicted ephemerides, at least for maneuverable satellites
• Around 2012, CARA began requesting predicted covariance as well
  – If/as possible for existing missions
  – Included in ICD/OA for new missions
• CARA recently received two actions
  – MOWG Action Item 1309-05
  – ESMO Maneuver Process Review RFA-02
• Primary objectives of actions
  – Why is CARA asking for predicted covariance from o/o
  – Any implementation recommendations
• This briefing is in response to those actions
• Right table indicates which ESC missions are currently providing predicted covariance to CARA

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Agenda

• Covariance basics
• Use of covariance in probability of collision (Pc) calculation
• Covariance generation and propagation methods
• Covariance tuning
• Covariance theory compatibility
• CARA O/O covariance needs
• Conclusions
• **Purpose of OD**
  – Generate estimate of the object’s state at a given time (called the *epoch time*).
  – Generate additional parameters and constructs to allow object’s future states to be predicted (accomplished through orbit *propagation*).
  – Generate a statement of the estimation error, both at epoch and for any predicted state (usually accomplished by means of a *covariance matrix*).

• **Error types**
  – OD approaches (either batch or filter) presume that they solve for all significant systematic errors.
  – Remaining solution error is thus presumed to be random (Gaussian) error.
  – Sometimes this error can be intentionally inflated to try to improve the fidelity of the error modeling.
  – Nonetheless, presumed to be Gaussian in form and unbiased.
OD Parameters Generated by ASW Solutions

- **Solved for: State parameters**
  - Six parameters needed to determine 3-d state fully
  - Cartesian: three position and three velocity parameters in orthogonal system
  - Element: six orbital elements that describe the geometry of the orbit

- **Solved for: Non-conservative force parameters**
  - Ballistic coefficient (C_D A/m); describes vulnerability of spacecraft state to atmospheric drag
  - Solar radiation pressure (SRP) coefficient (C_R A/m); describes vulnerability of spacecraft state to visible light momentum from sun

- **Considered: ballistic coefficient and SRP consider parameter**
  - Not solved for but “considered” as part of the solution
  - Derived from information outside of the OD itself
  - Discussed later
OD Uncertainty Modeling

• Characterizes the overall uncertainty of the OD epoch and/or propagated state
  – Uncertainty of each estimated parameter and their interactions

• This is a characterization of a multivariate statistical distribution

• In general, need the four cumulants to characterize the distribution
  – Mean, variance, skewness, and kurtosis; and their mutual interactions
  – Requires higher-order tensors to do this for a multivariate distribution

• Assumptions about error distribution can simplify situation substantially
  – Presuming the solution is unbiased places the mean error values at zero
  – Presuming the error distribution is Gaussian eliminates the need for the third and fourth cumulants
  – Error distribution can thus be expressed by means of variances of each solved-for component and their cross-correlations
  – Thus, error can be fully represented by means of a covariance matrix
Covariance Matrix Construction: Symbolic Example

- Three estimated parameters (a, b, and c)
- Variances of each along diagonal
- Off-diagonal terms the product of two standard deviations and the correlation coefficient (ρ); matrix is symmetric

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Example Covariance from CDM

- **8 x 8 matrix typical of most ASW updates**
  - Some orbit regimes not suited to solution for both drag and SRP; these covariances 7 x 7

- **Mix of different units often creates poorly conditioned matrices**
  - Condition number of matrix at right is 9.8E+11—terrible!

- **Often better numerically (and more intuitive) to separate matrix into sections**

- **First 3 x 3 portion (amber) is position covariance—often considered separately**

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Position Covariance Ellipse

- Position covariance defines an “error ellipsoid”
  - Placed at predicted satellite position
  - Square root of variance in each direction defines each semi-major axis (UVW system used here)
  - Off-diagonal terms rotate the ellipse from the nominal position shown

- Ellipse of a certain “sigma” value contains a given percentage of the expected data points
  - 1-σ: 19.9%
  - 2-σ: 73.9%
  - 3-σ: 97.1%
  - Note how much lower these are than the univariate normal percentage points
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Covariance in Calculation of Probability of Collision (Pc)

- Primary and secondary covariances combined and projected into *conjunction plane* (plane perpendicular to relative velocity vector at TCA)
- Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii
- Z-axis perpendicular to x-axis in conjunction plane
- Pc is portion of combined error ellipsoid hard-body radius circle

\[
P_c = \frac{1}{\sqrt{(2\pi)^2|C^*|}} \int \int \exp \left( -\frac{1}{2} \tilde{r}^T C^{*-1} \tilde{r} \right) dX dZ
\]

Covariance essential to Pc calculation, which is the most important factor in collision risk assessment
• Pc is the best single-parameter encapsulation of the risk
• Without Pc, have only the miss distance
• Correlation between miss distance and Pc very poor
  – Four Pc bands shown below; correlation with miss distance poor in all cases

• Important to have Pc, and covariance necessary for its calculation
Pc Sensitivity to Scaling of Primary Covariance

• If covariance of primary inadequately sized, Pc affected
• Graph below shows Pc differences between nominal value and recalculation with primary covariance rescaled (SF 0.5 – 2)
• ~2-5% of cases show differences greater than an order of magnitude—can affect operational conclusions

• Important to get primary covariance right
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Batch Epoch Covariance Generation (1 of 2)

• **Batch least-squares update (ASW method)** uses the following minimization equation

\[ \Delta x = (A^TWA)^{-1}A^Twb \]

- \( \Delta x \) is the vector of corrections to the state estimate
- \( A \) is the time-enabled partial derivative matrix, used to map the residuals into state-space
- \( W \) is the “weighting” matrix that provides relative weights of observation quality (usually \( 1/\sigma \), where \( \sigma \) is the standard deviation generated by the sensor calibration process)
- \( b \) is the vector of residuals (observations – predictions from existing state estimate)

• **Covariance is the collected term** \((A^TWA)^{-1}\)

- \( A \) the product of two partial derivative matrices:
  \[ A = \frac{\partial (obs)}{\partial x_0} = \frac{\partial (obs)}{\partial x} \frac{\partial x}{\partial x_0} \]
  - First term: partial derivatives of observations with respect to state at obs time
  - Second term: partial derivatives of state at obs time with respect to epoch state
• Formulated this way, this covariance matrix is called an *a priori* covariance
  – A does not contain actual residuals, only transformational partial derivatives
  – So \((ATWA)^{-1}\) is a function only of the amount of tracking, times of tracks, and sensor calibration relative weights among those tracks
    • Not a function of the actual residuals from the correction
• Limitations of *a priori* covariance
  – Does not account well for unmodeled errors, such as transient atmospheric density prediction errors
    • Because not examining actual fit residuals
  – W-matrix only as good as sensor calibration process
    • Principal weakness of present process, but expected to be improved eventually with JSpOC Mission System (JMS) upgrades
Covariance Propagation Methods

• **Full Monte Carlo**
  – Perturb state at epoch (using covariance), propagate each point forward to $t_n$ with full non-linear dynamics, and summarize distribution at $t_n$

• **Sigma point propagation**
  – Define small number of states to represent covariance statistically, propagate set forward by time-steps, reformulate sigma point set at each time-step, and use sigma point set at $t_n$ to formulate covariance at $t_n$

• **Linear mapping**
  – Create a state-transition matrix by linearization of the dynamics and use it to propagate the covariance to $t_n$ by pre- and post-multiplication

• **All three of above methods legitimate**
  – List moves from highest to lowest fidelity and computational intensity

More Info
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Covariance Tuning

• For CA, position covariance needs to be a realistic representation of the state uncertainty volume at the propagation point of interest

• Two aspects to this requirement
  – Does the position error volume conform to a trivariate Gaussian distribution?
  – If so, is it of the proper dimensions and orientation?

• Regarding the first item, extensive study has confirmed that this is not an issue for high-PC events (Pc>1E-04)
  – Ghrist and Plakalovic (2012)
  – 248 cases examined in different orbit regimes, with prop times of 2 to 7 days
  – 2-d Pc calculation compared to Monte Carlo (with 4E+07 trials)
  – Only one case of more than 10% deviation between 2-d and MC calculation
    • And 10% deviation not considered operationally significant
  – Explanation: high Pc requires covariance overlap near the centers of the covariances—a part that is not affected by non-Gaussian alterations

• Second item is area of legitimate concern
Covariance Tuning: Covariance Realism Evaluation Method

• Presume reference orbit (or precision observation) available for a satellite
• Position differences between predicted ephemeris and precision position (from reference orbit or observation) are dU, dV, and dW
  – Can be collected into vector ε
• Mahalanobis distance (ε * C⁻¹ * εᵀ) represents the ratio of the difference to the covariance’s prediction
  – For a trivariate distribution, expected value is 3
• A group of such calculations should conform to a chi-squared distribution with three degrees of freedom
• This method (distribution testing of groups of such calculations) used to determine if covariance properly sized
Covariance Tuning: Covariance Irrealism Remediation

- Examine individual component performance of covariance modeling to determine principal sources of the irrealism
  - Deviation probably stems from non-conservative force modeling (drag and/or solar radiation pressure)

- If using process noise, tune/modify process noise matrix to attempt to compensate
  - Originally directed at geopotential mismodeling; but with common use of higher-order theories, no longer the principal source of errors

- If using batch methods, include consider parameters
  - Additive value applied to either the drag or solar radiation pressure variances (or both) in order to make them larger
    - Poor modeling of these phenomena requires larger uncertainty estimate
  - Through cross-correlation terms, these variances will affect the other covariance parameters through the linear state transition

- Continue tuning process until proper distribution of calculated Mahalanobis distances achieved
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Covariance Theory Compatibility

• Batch covariance is governed by the amount and quality of tracking data used in the OD
• Propagated covariance is a product of the particular propagation technique used and tuning applied
  – Tuning itself a function of the adequacy of the OD force modeling
• Thus, important that covariance be generated from same OD basis that produced the state estimate
• This is not possible for O/O ephemerides that lack a covariance
  – Forced to use O/O state estimate and ASW covariance (or, worse, a synthesized covariance when no ASW covariance exists)
  – Such a covariance a questionable representation of O/O ephemeris error
    • $\sigma_{O/O}^2 = \sigma_{ASW}^2 + \sigma_{Diff}^2$
    • The difference variance is unknown, so using an ASW covariance with an O/O ephemeris understates the uncertainty but by an unknown amount
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CARA O/O Covariance Needs

• O/O ephemerides need to contain accompanying covariances
  – Especially true for ephemerides that contain planned maneuvers
    • No parallel ASW solution from which to “borrow” covariance for primary
  – One covariance entry for each ephemeris point is standard
    • Could possibly accommodate less frequent spacing, but would not conform to CCSDS standard and probably more difficult than the default approach
  – Full covariance (8 x 8) preferred; 3 x 3 (position covariance) usable with certain assumptions

• Delivery of covariance can form basis for including maneuver execution error in maneuver trade-space analyses
  – Open area for collaborative analysis with O/Os

• Daily delivery of ephemerides desirable
  – Propagation error can effect large changes in the Pc
  – This error minimized for both states and covariances through daily updates
    • Propagation time to TCA reduced
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Conclusions

- Properly tuned O/O covariance very important to CA
  - Incorporation into daily deliveries of O/O ephemerides highly desirable
- Covariance theory compatibility very important
  - Applaud recent efforts (e.g., ESMO) to develop covariance generation capabilities
- Variety of methods for covariance production, propagation, and tuning
  - CARA ready to assist with advice for production and tuning implementation
- Can incorporate O/O covariances into CARA operational software and processes as soon as such products are ready
BACK-UP SLIDES
Calculating Probability of Collision (Pc): Situation at Time of Closest Approach (TCA)
Calculating \( P_c \): 2-D Approximation (1 of 3)
Relative Position Covariance

- **Assumptions**
  - Covariances of primary and secondary objects are uncorrelated

- **Result**
  - All of the relative position error can be centered at one of the two satellite positions
    - Position of the secondary is typically used
  - Relative position error can be expressed as the additive combination of the two position covariances (proof given in Chan 2008)
    - \( C_a + C_b = C_c \)
    - Both covariances must be transformed into a common coordinate frame before combination
Calculating Pc: 2-D Approximation (2 of 3)
Projection to Conjunction Plane

- Combined covariance centered at position of secondary at TCA
- Primary path shown as curved “soda straw”
- If conjunction duration is very short
  - Motion can be considered to be rectilinear—soda straw is straight
  - Conjunction will take place in 2-d plane normal to the relative velocity vector and containing the secondary position
  - Problem can thus be reduced in dimensionality from 3 to 2
- Need to project covariance and primary path into “conjunction plane”
Conjunction Plane Construction

- Combined covariance projected into plane normal to the relative velocity vector and placed at origin
- Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii
- Z-axis perpendicular to x-axis in conjunction plane
Probability of Collision Computation

- $P_c$ is the portion of the density that falls within the HBR circle ($r$ is $[x \ z]$ and $C^*$ is the projected covariance)

$$P_c = \frac{1}{\sqrt{(2\pi)^2 |C^*|}} \int \int_A \exp \left( -\frac{1}{2} \bar{\mathbf{r}}^T C^{*-1} \bar{\mathbf{r}} \right) dXdZ$$

Conclusion: covariance essential to $P_c$ calculation, which is the most important factor in collision risk assessment
Covariance Propagation
Method 1: Full Monte Carlo (1 of 2)

• Creates $n$ state (position and velocity) perturbations at epoch
  – Covariance at epoch describes uncertainty of state at epoch
  – Can use this to create set of $n$ possible realizations of the epoch state, conforming to the distribution parameters specified by the covariance

• Propagates each of these forward to the time of interest
  – Use the full non-linear dynamics of the propagator
  – Thus produce $n$ states at TCA (for CA application)

• Summarizes set of $n$ states statistically
  – Usually empirically, through non-parametric techniques (e.g., percentiles, empirical distribution functions)
Covariance Propagation
Method 1: Full Monte Carlo (2 of 2)

• Advantages
  – Most accurate method of characterizing uncertainty, as there are no inherent simplifying assumptions or activities (such as linearization)

• Disadvantages
  – Very large number of samples required to characterize tails of distribution
  – Far more computationally intensive than other methods
Covariance Propagation

Method 2: Sigma Point Propagation (1 of 2)

- Usually applied to unscented Kalman filter (UKF) OD processes
- Generates a (relatively small) set of sample states (called \textit{sigma points}) about the nominal state, which represent the uncertainty
  - Sample covariance of sigma points should approximate covariance from state estimate
  - Theory says 2L+1 sigma points needed, where L is state degrees of freedom
    - Can increase this somewhat if prior information available; will improve accuracy of uncertainty volume reconstruction
- Propagates sigma points to next time step
- Constructs covariance (and state) at this future state from sigma points
  - Weighting functions often assembled to assist in reconstruction
Covariance Propagation
Method 2: Sigma Point Propagation (2 of 2)

• Advantages
  – Greatly reduces number of non-linear propagations
    • However, has to perform sigma-point construction at each time-step

• Disadvantages
  – Makes (and imposes) \textit{a priori} determination of future uncertainty volume distribution
  – Still requires multiple non-linear propagations
• Non-linear dynamics of orbit propagation can be linearized
  – These linear approximations valid for “short” periods about epoch state

• **State transition matrix** (Φ) the encapsulation of this linearization
  – Can be used for state propagations (but often is not)
  – Can also be used for propagation of covariance \[ Φ(t,t_o)*C(t_o)*Φ^T(t,t_o) \]

• **Covariance propagation can also be augmented via the addition of “process noise”**
  – Process noise matrix (Q) formulated, which specifies acceleration uncertainty in each coordinate principal direction
    • Intent is to compensate for unmodeled and inadequately-modeled perturbations
    • Can potentially remediative some of the limitations introduced by the linearization
  – Process noise matrix propagated through use of a process noise transition matrix, in a manner similar to state transition: \[ Γ(t,t_o)*Q(t_o)*Γ^T(t,t_o) \]
Covariance Propagation
Method 3: Linear Mapping (2 of 2)

• Advantages
  – Much faster and less computationally intensive than other methods
  – Process noise provides mechanism for covariance tuning/realism adjustments

• Disadvantages
  – Least accurate, especially for long propagations
  – Imposes \textit{a priori} statistical structure upon uncertainty volume
  – Use of process noise requires careful tuning process