Conjunction Assessment Risk Analysis

OD Covariance in Conjunction Assessment: Introduction and Issues

M.D. Hejduk / M. Duncan | 1 JUN 2015
Why You’re Receiving this Briefing

• Since inception, CARA has required owner/operator predicted ephemerides, at least for maneuverable satellites
• Around 2012, CARA began requesting predicted covariance as well
  – If/as possible for existing missions
  – Included in ICD/OA for new missions
• CARA recently received two actions
  – MOWG Action Item 1309-05
  – ESMO Maneuver Process Review RFA-02
• Primary objectives of actions
  – Why is CARA asking for predicted covariance from o/o
  – Any implementation recommendations
• This briefing is in response to those actions
• Right table indicates which ESC missions are currently providing predicted covariance to CARA

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Agenda

- Covariance basics
- Use of covariance in probability of collision (Pc) calculation
- Covariance generation and propagation methods
- Covariance tuning
- Covariance theory compatibility
- CARA O/O covariance needs
- Conclusions
• **Purpose of OD**
  – Generate estimate of the object’s state at a given time (called the *epoch time*)
  – Generate additional parameters and constructs to allow object’s future states to be predicted (accomplished through orbit *propagation*)
  – Generate a statement of the estimation error, both at epoch and for any predicted state (usually accomplished by means of a *covariance matrix*)

• **Error types**
  – OD approaches (either batch or filter) presume that they solve for all significant systematic errors
  – Remaining solution error is thus presumed to be random (Gaussian) error
  – Sometimes this error can be intentionally inflated to try to improve the fidelity of the error modeling
  – Nonetheless, presumed to be Gaussian in form and unbiased
OD Parameters Generated by ASW Solutions

- **Solved for: State parameters**
  - Six parameters needed to determine 3-d state fully
  - Cartesian: three position and three velocity parameters in orthogonal system
  - Element: six orbital elements that describe the geometry of the orbit

- **Solved for: Non-conservative force parameters**
  - Ballistic coefficient \((C_D A/m)\); describes vulnerability of spacecraft state to atmospheric drag
  - Solar radiation pressure (SRP) coefficient \((C_R A/m)\); describes vulnerability of spacecraft state to visible light momentum from sun

- **Considered: ballistic coefficient and SRP consider parameter**
  - Not solved for but “considered” as part of the solution
  - Derived from information outside of the OD itself
  - Discussed later
OD Uncertainty Modeling

- Characterizes the overall uncertainty of the OD epoch and/or propagated state
  - Uncertainty of each estimated parameter and their interactions

- This is a characterization of a multivariate statistical distribution

- In general, need the four cumulants to characterize the distribution
  - Mean, variance, skewness, and kurtosis; and their mutual interactions
  - Requires higher-order tensors to do this for a multivariate distribution

- Assumptions about error distribution can simplify situation substantially
  - Presuming the solution is unbiased places the mean error values at zero
  - Presuming the error distribution is Gaussian eliminates the need for the third and fourth cumulants
  - Error distribution can thus be expressed by means of variances of each solved-for component and their cross-correlations
  - Thus, error can be fully represented by means of a covariance matrix
Covariance Matrix Construction: Symbolic Example

- Three estimated parameters (a, b, and c)
- Variances of each along diagonal
- Off-diagonal terms the product of two standard deviations and the correlation coefficient ($\rho$); matrix is symmetric

$$
\begin{array}{cccc}
  a & b & c & \ldots \\
  a & \sigma_a^2 & \rho_{ab}\sigma_a\sigma_b & \rho_{ac}\sigma_a\sigma_c & \ldots \\
  b & \rho_{ab}\sigma_a\sigma_b & \sigma_b^2 & \rho_{bc}\sigma_a\sigma_c & \ldots \\
  c & \rho_{ac}\sigma_a\sigma_c & \rho_{bc}\sigma_a\sigma_c & \sigma_c^2 & \ldots \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
$$
Example Covariance from CDM

- 8 x 8 matrix typical of most ASW updates
  - Some orbit regimes not suited to solution for both drag and SRP; these covariances 7 x 7

- Mix of different units often creates poorly conditioned matrices
  - Condition number of matrix at right is 9.8E+11—terrible!

- Often better numerically (and more intuitive) to separate matrix into sections

- First 3 x 3 portion (amber) is position covariance—often considered separately

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Position Covariance Ellipse

- **Position covariance defines an “error ellipsoid”**
  - Placed at predicted satellite position
  - Square root of variance in each direction defines each semi-major axis (UVW system used here)
  - Off-diagonal terms rotate the ellipse from the nominal position shown

- **Ellipse of a certain “sigma” value contains a given percentage of the expected data points**
  - 1-σ: 19.9%
  - 2-σ: 73.9%
  - 3-σ: 97.1%
  - Note how much lower these are than the univariate normal percentage points
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Covariance in Calculation of Probability of Collision (Pc)

- Primary and secondary covariances combined and projected into conjunction plane (plane perpendicular to relative velocity vector at TCA)
- Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii
- Z-axis perpendicular to x-axis in conjunction plane
- Pc is portion of combined error ellipsoid hard-body radius circle

\[ P_c = \frac{1}{\sqrt{(2\pi)^2|C^*|}} \int \int_A \exp\left(-\frac{1}{2} \vec{r}^T C^{-1} \vec{r}\right) dXdZ \]

Covariance essential to Pc calculation, which is the most important factor in collision risk assessment
Pc vs Miss Distance Calculations

- Pc is the best single-parameter encapsulation of the risk
- Without Pc, have only the miss distance
- Correlation between miss distance and Pc very poor
  - Four Pc bands shown below; correlation with miss distance poor in all cases
- **Important to have Pc, and covariance necessary for its calculation**
Pc Sensitivity to Scaling of Primary Covariance

- If covariance of primary inadequately sized, Pc affected
- Graph below shows Pc differences between nominal value and recalculation with primary covariance rescaled (SF 0.5 – 2)
- ~2-5% of cases show differences greater than an order of magnitude—can affect operational conclusions
- **Important to get primary covariance right**
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Batch Epoch Covariance Generation (1 of 2)

- **Batch least-squares update (ASW method)** uses the following minimization equation
  \[ dx = (A^TWA)^{-1}A^Twb \]
  - \( dx \) is the vector of corrections to the state estimate
  - \( A \) is the time-enabled partial derivative matrix, used to map the residuals into state-space
  - \( W \) is the “weighting” matrix that provides relative weights of observation quality (usually \( 1/\sigma \), where \( \sigma \) is the standard deviation generated by the sensor calibration process)
  - \( b \) is the vector of residuals (observations – predictions from existing state estimate)

- **Covariance is the collected term** \( (A^TWA)^{-1} \)
  - \( A \) the product of two partial derivative matrices:
    \[ A = \frac{\partial (obs)}{\partial x_0} = \frac{\partial (obs)}{\partial x} \cdot \frac{\partial x}{\partial x_0} \]
    - First term: partial derivatives of observations with respect to state at obs time
    - Second term: partial derivatives of state at obs time with respect to epoch state
Batch Epoch Covariance Generation (2 of 2)

• Formulated this way, this covariance matrix is called an *a priori* covariance
  – A does not contain actual residuals, only transformational partial derivatives
  – So \((A^TWAW)^{-1}\) is a function only of the amount of tracking, times of tracks, and sensor calibration relative weights among those tracks
    • Not a function of the actual residuals from the correction

• Limitations of *a priori* covariance
  – Does not account well for unmodeled errors, such as transient atmospheric density prediction errors
    • Because not examining actual fit residuals
  – W-matrix only as good as sensor calibration process
    • Principal weakness of present process, but expected to be improved eventually with JSpOC Mission System (JMS) upgrades
Covariance Propagation Methods

• Full Monte Carlo
  – Perturb state at epoch (using covariance), propagate each point forward to $t_n$ with full non-linear dynamics, and summarize distribution at $t_n$

• Sigma point propagation
  – Define small number of states to represent covariance statistically, propagate set forward by time-steps, reformulate sigma point set at each time-step, and use sigma point set at $t_n$ to formulate covariance at $t_n$

• Linear mapping
  – Create a state-transition matrix by linearization of the dynamics and use it to propagate the covariance to $t_n$ by pre- and post-multiplication

• All three of above methods legitimate
  – List moves from highest to lowest fidelity and computational intensity

More Info
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Covariance Tuning

• For CA, position covariance needs to be a realistic representation of the state uncertainty volume at the propagation point of interest

• Two aspects to this requirement
  – Does the position error volume conform to a trivariate Gaussian distribution?
  – If so, is it of the proper dimensions and orientation?

• Regarding the first item, extensive study has confirmed that this is not an issue for high-PC events (Pc>1E-04)
  – Ghrist and Plakalovic (2012)
  – 248 cases examined in different orbit regimes, with prop times of 2 to 7 days
  – 2-d Pc calculation compared to Monte Carlo (with 4E+07 trials)
  – Only one case of more than 10% deviation between 2-d and MC calculation
    • And 10% deviation not considered operationally significant
  – Explanation: high Pc requires covariance overlap near the centers of the covariances—a part that is not affected by non-Gaussian alterations

• Second item is area of legitimate concern
Covariance Tuning:
Covariance Realism Evaluation Method

- Presume reference orbit (or precision observation) available for a satellite
- Position differences between predicted ephemeris and precision position (from reference orbit or observation) are $dU$, $dV$, and $dW$
  - Can be collected into vector $\varepsilon$
- Mahalanobis distance $(\varepsilon * C^{-1} * \varepsilon^T)$ represents the ratio of the difference to the covariance’s prediction
  - For a trivariate distribution, expected value is 3
- A group of such calculations should conform to a chi-squared distribution with three degrees of freedom
- This method (distribution testing of groups of such calculations) used to determine if covariance properly sized
Covariance Tuning: Covariance Irrealism Remediation

- Examine individual component performance of covariance modeling to determine principal sources of the irrealism
  - Deviation probably stems from non-conservative force modeling (drag and/or solar radiation pressure)

- If using process noise, tune/modify process noise matrix to attempt to compensate
  - Originally directed at geopotential mismodeling; but with common use of higher-order theories, no longer the principal source of errors

- If using batch methods, include consider parameters
  - Additive value applied to either the drag or solar radiation pressure variances (or both) in order to make them larger
    - Poor modeling of these phenomena requires larger uncertainty estimate
  - Through cross-correlation terms, these variances will affect the other covariance parameters through the linear state transition

- Continue tuning process until proper distribution of calculated Mahalanobis distances achieved
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Covariance Theory Compatibility

- Batch covariance is governed by the amount and quality of tracking data used in the OD
- Propagated covariance is a product of the particular propagation technique used and tuning applied
  - Tuning itself a function of the adequacy of the OD force modeling
- Thus, **important that covariance be generated from same OD basis that produced the state estimate**
- This is not possible for O/O ephemerides that lack a covariance
  - Forced to use O/O state estimate and ASW covariance (or, worse, a synthesized covariance when no ASW covariance exists)
  - Such a covariance a questionable representation of O/O ephemeris error
    - $\sigma_{\text{O/O}}^2 = \sigma_{\text{ASW}}^2 + \sigma_{\text{Diff}}^2$
    - The difference variance is unknown, so using an ASW covariance with an O/O ephemeris underestates the uncertainty but by an unknown amount
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CARA O/O Covariance Needs

• O/O ephemerides need to contain accompanying covariances
  – Especially true for ephemerides that contain planned maneuvers
    • No parallel ASW solution from which to “borrow” covariance for primary
  – One covariance entry for each ephemeris point is standard
    • Could possibly accommodate less frequent spacing, but would not conform to
      CCSDS standard and probably more difficult than the default approach
  – Full covariance (8 x 8) preferred; 3 x 3 (position covariance) usable with
    certain assumptions

• Delivery of covariance can form basis for including maneuver
  execution error in maneuver trade-space analyses
  – Open area for collaborative analysis with O/Os

• Daily delivery of ephemerides desirable
  – Propagation error can effect large changes in the Pc
  – This error minimized for both states and covariances through daily updates
    • Propagation time to TCA reduced
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Conclusions

• Properly tuned O/O covariance very important to CA
  – Incorporation into daily deliveries of O/O ephemerides highly desirable

• Covariance theory compatibility very important
  – Applaud recent efforts (e.g., ESMO) to develop covariance generation capabilities

• Variety of methods for covariance production, propagation, and tuning
  – CARA ready to assist with advice for production and tuning implementation

• Can incorporate O/O covariances into CARA operational software and processes as soon as such products are ready
BACK-UP SLIDES
Calculating Probability of Collision (Pc): Situation at Time of Closest Approach (TCA)
Calculating Pc: 2-D Approximation (1 of 3)
Relative Position Covariance

• Assumptions
  – Covariances of primary and secondary objects are uncorrelated

• Result
  – All of the relative position error can be centered at one of the two satellite positions
    • Position of the secondary is typically used
  – Relative position error can be expressed as the additive combination of the two position covariances (proof given in Chan 2008)
    • $C_a + C_b = C_c$
    • Both covariances must be transformed into a common coordinate frame before combination
Calculating Pc: 2-D Approximation (2 of 3)

Projection to Conjunction Plane

• Combined covariance centered at position of secondary at TCA
• Primary path shown as curved “soda straw”
• If conjunction duration is very short
  – Motion can be considered to be rectilinear—soda straw is straight
  – Conjunction will take place in 2-d plane normal to the relative velocity vector and containing the secondary position
  – Problem can thus be reduced in dimensionality from 3 to 2
• Need to project covariance and primary path into “conjunction plane”
Calculating Pc: 2-D Approximation (3 of 3)
Conjunction Plane Construction

- Combined covariance projected into plane normal to the relative velocity vector and placed at origin
- Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii
- Z-axis perpendicular to x-axis in conjunction plane
Probability of Collision Computation

- $P_c$ is the portion of the density that falls within the HBR circle ($r$ is $[x \ z]$ and $C^*$ is the projected covariance)

\[
P_c = \frac{1}{\sqrt{(2\pi)^2 |C^*|}} \iint_A \exp \left( -\frac{1}{2} \tilde{r}^T C^{-1} \tilde{r} \right) dXdZ
\]

Conclusion: covariance essential to $P_c$ calculation, which is the most important factor in collision risk assessment
Covariance Propagation
Method 1: Full Monte Carlo (1 of 2)

• Creates \( n \) state (position and velocity) perturbations at epoch
  – Covariance at epoch describes uncertainty of state at epoch
  – Can use this to create set of \( n \) possible realizations of the epoch state,
    conforming to the distribution parameters specified by the covariance

• Propagates each of these forward to the time of interest
  – Use the full non-linear dynamics of the propagator
  – Thus produce \( n \) states at TCA (for CA application)

• Summarizes set of \( n \) states statistically
  – Usually empirically, through non-parametric techniques (e.g., percentiles,
    empirical distribution functions)
Covariance Propagation
Method 1: Full Monte Carlo (2 of 2)

- **Advantages**
  - Most accurate method of characterizing uncertainty, as there are no inherent simplifying assumptions or activities (such as linearization)

- **Disadvantages**
  - Very large number of samples required to characterize tails of distribution
  - Far more computationally intensive than other methods
Covariance Propagation
Method 2: Sigma Point Propagation (1 of 2)

• Usually applied to unscented Kalman filter (UKF) OD processes
• Generates a (relatively small) set of sample states (called *sigma points*) about the nominal state, which represent the uncertainty
  – Sample covariance of sigma points should approximate covariance from state estimate
  – Theory says 2L+1 sigma points needed, where L is state degrees of freedom
    • Can increase this somewhat if prior information available; will improve accuracy of uncertainty volume reconstruction

• Propagates *sigma points* to next time step
• Constructs covariance (and state) at this future state from sigma points
  – Weighting functions often assembled to assist in reconstruction
Covariance Propagation
Method 2: Sigma Point Propagation (2 of 2)

• Advantages
  – Greatly reduces number of non-linear propagations
    • However, has to perform sigma-point construction at each time-step

• Disadvantages
  – Makes (and imposes) a priori determination of future uncertainty volume distribution
  – Still requires multiple non-linear propagations
• Non-linear dynamics of orbit propagation can be linearized
  – These linear approximations valid for “short” periods about epoch state

• State transition matrix ($\Phi$) the encapsulation of this linearization
  – Can be used for state propagations (but often is not)
  – Can also be used for propagation of covariance $[\Phi(t,t_o)\ast C(t_o)\ast \Phi^T(t,t_o)]$

• Covariance propagation can also be augmented via the addition of “process noise”
  – Process noise matrix (Q) formulated, which specifies acceleration uncertainty in each coordinate principal direction
    • Intent is to compensate for unmodeled and inadequately-modeled perturbations
    • Can potentially remediate some of the limitations introduced by the linearization
  – Process noise matrix propagated through use of a process noise transition matrix, in a manner similar to state transition: $[\Gamma(t,t_o)\ast Q(t_o)\ast \Gamma^T(t,t_o)]$
Covariance Propagation
Method 3: Linear Mapping (2 of 2)

• **Advantages**
  – Much faster and less computationally intensive than other methods
  – Process noise provides mechanism for covariance tuning/realism adjustments

• **Disadvantages**
  – Least accurate, especially for long propagations
  – Imposes *a priori* statistical structure upon uncertainty volume
  – Use of process noise requires careful tuning process