Structural Dynamic Analysis of Rocket Engine Turbomachinery

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AIAA Greater Huntsville Section Short Course Series
August 6, 2015
Motivation for Structural Dynamic Analysis of Turbomachinery

How Turbomachinery is used in Rocket Engines

Overview and Introduction to Structural Dynamics
  1. SDOF Systems
  2. MDOF Systems

Application to Turbomachinery
  1. Fourier Characterization of Excitation
  2. Temporal Fourier series
  3. Modeling and Modal Analysis of Displacements and Stresses
  4. Campbell Diagrams
  5. Damping
  6. Forced Response Analysis in the Frequency Domain

Some Additional Necessary Details
  1. Cyclic Symmetry
  2. Spatial Characterization of Excitation, Orthogonality with Mode Shape
  3. Mistuning

Conclusion
How turbomachinery is used in Rocket Engines

- Liquid Fuel (LH2, Kerosene) and Oxidizer (LO2) are stored in fuel tanks at a few atmospheres.
- Turbines, driven by hot gas created by mini-combustors, tied with shaft to pump, which sucks in propellants and increases their pressures to several hundred atm.
- High pressure propellants sent to Combustion Chamber, which ignites mixture with injectors.
- Hot gas directed to converging/diverging nozzle to give flow very high velocity for thrust.
Crack found during ground-test program can stop engine development

- If crack propagates, it could liberate a piece, which at very high rotational speeds could be catastrophic (i.e., engine will explode)
Structural Dynamics Basics
A) Equation of Motion for Undamped System

1) Model Spring-Mass System to represent real structure

For cantilever beam, \( \delta = \frac{PL^3}{3EI} \)

Derive \( k_{beam} \) (in class):
Here, \( P=k\delta \) is analogous to typical \( F=kx \), so \( k=p/\delta \)

\[ k = \frac{P}{\frac{pl^3}{3EI}} \]

\[ k_{beam} = \frac{3EI}{L^3} \]

2) Generate Equation of Motion (EoM) using Newton’s 2nd Law

\[ \Sigma F_x = m\ddot{u} \]

\[ -ku + F_{external} = m\ddot{u} \]

\[ m\ddot{u} + ku = F_{external} \]
Solution to Equation of Motion (2\textsuperscript{nd} Order ODE)

\[ u(t)_{\text{total}} = u_{\text{particular, steady state, nonhomogeneous}} + u_{\text{complimentary, transient, homogeneous}} \]

Where

• \( u_{\text{complimentary, transient, homogeneous}} \) is the component of the solution when the RHS of the EoM is zero.
  - In physical terms, this is the response due to the internal dynamic characteristics of the structure, and comes about when there are non-zero Initial conditions (I.C.’s).

• \( u_{\text{particular, steady state, nonhomogeneous}} \) is the component of the solution when the RHS of the EoM is non-zero.
  - In physical terms, this is the response due to external forcing functions.
1) simplest, worth remembering:

- Assume solution \( u = u(t) \) is of form \( u(t) = A \cos(\omega t) \)

\[
\begin{align*}
\dot{u}(t) &= -A\omega \sin(\omega t) \\
\ddot{u}(t) &= -A\omega^2 \cos(\omega t)
\end{align*}
\]

- Now plug these equalities into eq of motion:

\[
\begin{align*}
m(-A\omega^2 \cos \omega t) + k(A \cos \omega t) &= 0 \\
A \cos \omega t(k - \omega^2 m) &= 0
\end{align*}
\]

For \( A \cos \omega t = 0 \), \( A \) has to = 0 \( \), i.e., no response (“trivial solution”)

Therefore,

\[
\begin{align*}
k - \omega^2 m &= 0 \\
\omega^2 &= \frac{k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}} \text{ Rad/sec}
\end{align*}
\]

Prize question: is the natural frequency of a system the same on the moon as on earth?

Yes

Define \( \lambda \equiv \text{Eigenvalue} = \omega^2 \equiv \text{Natural Frequency}^2 \)

So, solution for \( u = u(t) \) is \( u(t) = A \cos(\sqrt{\frac{k}{m}} t) \) where \( A \) depends on the initial conditions (IC)
Natural Frequency Units, Period Relationship

Question: What is relationship of the natural frequency \( \omega \) (sometimes called “circular natural frequency) to the natural frequency \( f \) in hz?

\[
f \text{ Hz} = \frac{\omega \text{ rad/sec}}{2\pi} \text{ Hz}
\]

Question: What is relationship of the period \( T \) to the natural frequency \( f \) and the circular natural frequency \( \omega \)?

\[
T = \frac{1}{f \text{ Hz}} = \frac{2\pi}{\omega \text{ sec}}
\]

E.G.: \( u = 2\sin(3t) \)

\[
\omega = 3 \text{ Rad/sec}
\]

\[
f = \frac{3 \text{ Rad/sec}}{2\pi} \text{ Hz} = 0.477 \text{ Hz} \Rightarrow u = 2\sin(2\pi(0.477)t)
\]

\[
T = \frac{1}{f} \text{ sec} = \frac{2\pi}{\omega} \text{ sec} = \frac{2\pi}{3 \text{ Rad/sec}} s = 2.094 s
\]
Damped Free Vibration of SDOF Systems

Damper has parameter $c \text{ lb/sec}$

$$\sum F_{\text{external}} = m\ddot{u}$$

$$-c\dot{u} - ku = m\ddot{u}$$

$$m\ddot{u} + c\dot{u} + ku = 0$$
Damping Categories

• 3 cases of solutions:

  1) Critical Damping

\[ c = 2\sqrt{km} \equiv c_{\text{critical}} \]

  2) Overdamped

\[ c > 2\sqrt{km} \]
\[ \Rightarrow c > c_{\text{critical}} \]

If we give a critically damped or overdamped SDOF system an initial displacement \( u_o \), we get

\[ \text{No Oscillation} \]
3) Underdamped

\[ c < 2\sqrt{km} \]

\[ C < C_{\text{critical}} \]

Now, to talk about underdamped systems, go back to the damped eq. of motion:

\[ m\ddot{u} + c\dot{u} + ku = 0 \]

divide by m

\[ \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0 \]

Now define viscous damping ratio \( \zeta \) as a percentage of critical damping

\[ \zeta = \frac{c}{c_{\text{critical}}} \]

Using \( \omega^2 = \frac{k}{m} \), we get

\[ \ddot{u} + \frac{\zeta c_{cr}}{m}\dot{u} + \omega^2 u = 0 \]

Use \( c_{cr} = 2\sqrt{km} \) to get

\[ \ddot{u} + \frac{2\zeta \sqrt{km}}{m}\dot{u} + \omega^2 u = 0 \]

\[ \ddot{u} + 2\zeta \omega \dot{u} + \omega^2 u = 0 \]

VERY IMPORTANT FORM, ALL YOU NEED ARE DYNAMIC CHARACTERISTICS!
Response to Harmonic Excitation

\[ \Omega = \text{Excitation Frequency} \]
\[ f = \text{Excitation Force} \text{ (NOT enforced}\]
\[ \omega = \text{System Natural Frequency} = \sqrt{\frac{k}{m}} \]

\[ \text{eq. of motion:} \quad m\ddot{u} + c\dot{u} + ku = F_0 \cos \Omega t \]

Solution: \[ u(t) = u_t(t) + u_{ss}(t) \]

where

\[ u_t(t) = \text{transient solution} = \text{homogeneous (rhs=0) sol'n} \]

\[ u(t) = e^{-\zeta \omega_d t} \left( u_o \cos \omega_d t + \frac{\dot{u}_o + \zeta \omega_o}{\omega_d} \sin \omega_d t \right) \]

which decays to zero after a few cycles,

If the excitation frequency stays constant or slowly varies.
- Response $U$ as a function of excitation frequency is

$$\bar{U}(\Omega) = \bar{H}(\Omega)\bar{U}_{st}$$

where we define the "Complex Frequency Response"

$$\bar{H}(\Omega) = \frac{\text{Dynamic Response } \bar{U}}{\text{Static Response } \bar{U}_{st}}$$

and where we determine the static response $U_{st}$ to force $F_o$ using

$$kU_{st} = F_o \quad \rightarrow \quad U_{st} = \frac{F_o}{k}$$

After a long derivation, we get

$$|\bar{H}(\Omega)| = \sqrt{\frac{1}{(1 - r^2)^2 + (2\zeta r)^2}}$$

and

phase lag $\phi = \tan^{-1}\left(\frac{-2\zeta r}{1 - r^2}\right)$

where we define the Frequency Ratio $r = \frac{\Omega}{\omega}$

Resonance is defined when $\Omega = \omega$, ie, $r=1$.

At $r=1$, $|\bar{H}(\Omega)| = \frac{1}{2\zeta}$ = Quality Factor $Q$
Example:

F=2; c=0.6; m=1; k=9

\( \omega = \sqrt{\frac{k}{m}} = 3 \)

\( \zeta = \frac{c}{2\sqrt{km}} = 0.1 \)

\( U_{\text{static}} = \frac{F_0}{k} = 0.222 \)

At resonance, \( |U| = Q \quad U_{\text{static}} = \)

\( |U| = \frac{1}{2\zeta} (0.222) = 1.111 \)

For \( \Omega = 2.8 \), \( r = \frac{\Omega}{\omega} = \frac{2.8}{3} = 0.9333 \), so

\( |\vec{H}(\Omega)| = \sqrt{\frac{1}{(1-r^2)^2 + (2\zeta r)^2}} = \sqrt{\frac{1}{(1-0.93333^2)^2 + (2*0.1*0.93333)^2}} = 4.408 \)

\( \phi = \tan^{-1}\left[ \frac{-2\zeta r}{1-r^2} \right] = \tan^{-1}\left[ \frac{-2(0.1)*0.9333}{1-0.9333^2} \right] = \tan^{-1}\left( \frac{-0.01866}{0.12889} \right) = -0.9665 \)
Given Damped system shown below:

![Damped system diagram](image)

- **P = 5 cos(3t) lb**
- **W = 386.1 lb**
- **k = 4 lb/in**
- **C = 0.3 lb-sec/in**
- **c_{critical} = 2\sqrt{km}**

**a)** What is the frequency of the steady-state dynamic response \( u(t) \)?

3 Rad/sec

**b)** If the load is applied statically, what is the displacement?

\[
 u_{static} = \frac{F}{k} = \frac{5}{4} = 1.25''
\]

**c)** What is the approximate maximum response of the mass for any frequency of excitation?

\[
c_{cr} = 2\sqrt{km} = 2\sqrt{4*1} = 4
\]

\[
\zeta = \frac{c}{c_{cr}} = 0.3/4 = 0.075 \Rightarrow Q \approx \frac{1}{2\zeta} = \frac{1}{2*0.075} = 6.667
\]

\[
u_{dynamic} = Qu_{static} = 6.667 * 1.25'' = 8.333''
\]

**d)** At what approximate frequency does this occur?

\[
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2 \text{ rad/s}
\]

**e)** Are there any excitation frequencies where the response would be close to the static response, and if so, what would be one of these frequencies?

Yes, frequencies < 0.5*\(\omega\), therefore < 1 rad/s

2) The Complex Frequency Response \( H \) is the ratio of the dynamic response to the static response as a function of frequency ratio \( r=\frac{\Omega}{\omega} \).
• Structures discretized using finite elements (rigorous mathematical representation of little piece of a solid)

• Machines, other mechanical systems can be modeled fairly accurately with lumped parameter MDOF models (i.e., lumped rigid masses, massless springs & dampers).

• Equations of Motion (EoM)
  - Newton method: easiest to use for translational systems, but very difficult for rotational motion
Modal Analysis – Obtain Natural Frequencies and Modes

Solutions for Undamped, Free Vibration of MDOF Systems with N dof's.

\[
[M]\{\ddot{u}\} + [K]\{u\} = \{0\}
\]

Assume solution of form

\[
\{u\}_m = \{\phi\}_m e^{i(\omega_m t + \alpha_m)}
\]

\(m=1,M\) where \(M \leq N\) (can choose to "use" less than N solutions)

where \(m\) spatial solutions \(\equiv\) Eigenvectors \(\equiv\) Mode Shapes

Discrete MDOF modes

\[
\{\phi\}_m
\]

Alternate nomenclature: \(\phi_{ij}\), where \(i\) is dof, \(j\) is mode number
Example: Natural Frequencies and Modes of Axial Vibration of Cantilever Bar

E=2.9e7 lb/in²

ρ = 0.1 \frac{lb·sec²}{in³}

A) Discretize into 2 finite elements, draw coordinate system:

Lump mass at nodes: m_{element} = ρ A L

So

m_1 = ρAL/2
m_2 = ρAL
m_3 = ρAL/2

Element stiffness:
k_1 = k_2 = AE/L

Write Eq’s of Motion:

\[ -k_1(u_1 - u_2) = m_1\ddot{u}_1 \]
\[ m_1\ddot{u}_1 - k_1u_2 + k_1u_1 = 0 \]
\[ m_1\ddot{u}_1 + k_1u_1 - k_1u_2 = 0 \]
Equation of Motion in Matrix Form:

\[
\frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Apply Boundary Conditions (BC’s): Since \( u_3 = 0 \), can cross out corresponding row & column.

Write system matrix \([D]=([K]-\omega^2[M])\), let \( \lambda = \omega^2 \)

\[
[D] = \left( \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) a
\]

4) Divide through by AE/L;

Let

\[
\mu = \frac{\rho AL}{AE} \lambda = \frac{\rho L^2}{2E} \lambda
\]

then \([D] = \begin{bmatrix} 1 - \mu & -1 \\ -1 & 2 - 2\mu \end{bmatrix} \)
5) Set $|D|=0$

$$
\begin{vmatrix}
1 - \mu & -1 \\
-1 & 2 - 2\mu
\end{vmatrix} = 0
$$

$(1 - \mu)(2 - 2\mu) - 1 = 0$

$2\mu^2 - 4\mu + 1 = 0$

6) Solve for roots $\mu_m$; solve for $\omega_m$

$$
\mu_{1,2} = \frac{4 \pm \sqrt{16 - (4 \times 2 	imes 1)}}{2} = 1 \pm \frac{\sqrt{2}}{2}
$$

$\mu_1 = 0.2928 \Rightarrow \lambda_1 = \frac{2E}{\rho L^2} \mu = \frac{2(2.9e7 \frac{lb}{in^2})}{0.1 \frac{lbsec^2}{in^4} (480in)^2} = 0.2928 = 737.1 \Rightarrow \omega_1 = 27.15 \frac{Rad}{sec}$

$\mu_2 = 1.707 \Rightarrow \lambda_2 = \frac{2(2.9e7 \frac{lb}{in^2})}{0.1 \frac{lbsec^2}{in^4} (480in)^2} = 1.707 = 4297.5 \Rightarrow \omega_2 = 65.5 \frac{Rad}{sec}$
7) Plug \( \mu_i \) in to \( [D]\{\phi\} = 0 \Rightarrow \{\phi\} \), use "max" normalization

\[ \mu_i : (1-0.2928)\phi_{11} - \phi_{21} = 0 \]
\[ .7071 \phi_{11} - \phi_{21} = 0 \Rightarrow \phi_{21} = .7071 \phi_{11} \]

Let \( \phi_{11} = 1 \Rightarrow \phi_{21} = .7071 \Rightarrow \{\phi\}_1 = \left\{ \frac{1}{.7071} \right\} \]

2nd equation in \( [D]\{\phi\} = 0 \) gives same thing (not linearly independent)

\[ \mu_2 : (1-1.7071)\phi_{12} - \phi_{22} = 0 \]
\[ -.7071 \phi_{12} = \phi_{22} \]

Let \( \phi_{12} = 1 \Rightarrow \phi_{22} = -.7071 \Rightarrow \{\phi\}_2 = \left\{ \frac{1}{-.7071} \right\} \]

\[ \therefore [\Phi] = \begin{bmatrix} 1 & 1 \\ .7071 & -.7071 \end{bmatrix} \]

Max normalized mode shapes are useful for visualization

8) If desired, obtain Mass Normalized Mode Shapes

\( s_1 = 0.10206 \) and \( s_2 = 0.10206 \), so

\[ [\Phi]_{mass} = \begin{bmatrix} .10206 & .10206 \\ .07216 & -.07216 \end{bmatrix} \]
See great animations of MDOF systems by Dr. Dan Russell, Graduate Program in Acoustics, Penn State. Or https://www.youtube.com/watch?v=kvG7OrjBirl

Eigenvalue $\lambda = \text{natural frequency } \omega^2$

$\lambda_1 = \omega_1^2 \Rightarrow \{\phi\}_1$

$\lambda_2 = \omega_2^2 \Rightarrow \{\phi\}_2$

$\lambda_3 = \omega_3^2 \Rightarrow \{\phi\}_3$

Displacements for all locations of mode shape are either in-phase or $180^\circ$ out-of-phase with each other, but have phase relationship of $\alpha_i$ with excitation.

Modal Matrix $\Phi = \begin{bmatrix} \{\phi\}_1 & \{\phi\}_2 & \{\phi\}_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} & \cdots & \phi_{13} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{n3} \end{bmatrix}$
Application of Structural Dynamics to Turbomachinery
First obtain speed range of operation from performance group.

- For Rocket Engines, there are generally several “nominal” operating speeds dependent upon phase of mission (e.g., reduce thrust during “Max Q”).
- However, since flow is the controlling parameter, actual rotational speeds are uncertain (especially during design phase).
- For new LPS engine being built at MSFC, assuming possible variation +/-5% about each of two operating speeds.

In addition, speed generally isn’t constant, but instead “dithers”.†

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†Implementation of Speed Variation in the Structural Dynamic Assessment of Turbomachinery Flow Path Components
Andrew M. Brown, R. Benjamin Davis and Michael K. DeHaye
Forces are not, in general, perfect sine waves (although sometimes they’re close)

We can deal with these in two ways:
  - Represent forces as sum of Sines (Spectral, Frequency Domain Approach), sum response to each Sine
  - Calculate response to actual temporal (time history) loading using “impulse response function”

Spectral Approach: given a periodic but non-harmonic excitation

\[ p(\tau) = p(\tau + T) \]
Jean Fourier realized we can write loading $p(t)$ as sum of average, cosines, & sines:

$$p(t) = a_o + \sum_{n=1}^{\infty} [a_n \cos(n\Omega_1 t) + b_n \sin(n\Omega_1 t)]$$

where

$$a_0 = \frac{1}{T_1} \int_{\tau}^{\tau+T} p(t) dt = \text{avg value of } p(t)$$

$$a_n = \frac{2}{T_1} \int_{\tau}^{\tau+T} p(t) \cos(n\Omega_1 t) dt$$

$$b_n = \frac{2}{T_1} \int_{\tau}^{\tau+T} p(t) \sin(n\Omega_1 t) dt$$

Textbook Example: Using Fourier Series, represent square wave excitation as:

$$p(t) = \frac{4p_0}{\pi} \sum_{n=1,3,5,\ldots} \left( \frac{1}{n} \right) \sin(n\Omega_1 t)$$
• Since all real machinery has some level of unbalance and shaft whirl, sinusoidal ("harmonic") loads at 1 and 2 times rotational speed ("1 and 2N") will be generated, along with up to their 3rd multiples (also called "harmonics") 3-6N throughout turbopump.

*Space Shuttle Main Engine Powerhead Cross-Section*
Harmonic excitation at engine order = Number of flow distortions and up to their 3\textsuperscript{rd} multiples arising from adjacent upstream and downstream blade and vane counts.

- Use CFD to generate Loading
Engine Example of Application of Fourier Series

- Excitation wave based upon a pump with 3 primary distortions (e.g. diffusers), within slightly asymmetric overall field.
- Let primary excitation at 3N have an amplitude of 1, and asymmetric primary distortion have an amplitude of 0.1.
  - each of these will have a harmonic, since they aren’t perfect sinusoidal distortions, such that the harmonic of the asymmetric is 0.05, and the amplitude of the harmonic of the primary distortion is 0.25.
  - So \( p(t) = b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + b_6 \sin(6t) \)

Where
\[ b_1 = 0.1, \ b_2 = 0.05, \ b_3 = 1.0, \ b_6 = 0.25 \]

- Have to assess dynamics for each frequency component of excitation.
Now Structure: Create FEM of component, Modal Analysis

Example:
Turbine Blades

Mode 12 at 36850 hz

Mode 13 at 38519 hz

Modal Animations very useful for identifying problem modes, optimal damper locations
Create “Campbell Diagram”

- Simplest Version of Campbell Diagram is just a glorified Resonance Chart.
Modal Analysis has Multiple Uses

- Redesign Configuration to move excitations ranges away from natural frequencies
- Redesign component to move resonances out of operating range.
- Put in enough damping to significantly reduce response
- Use as first step in “Forced Response Analysis” (applying forces and calculating structural response).
• Modal analysis of original design indicated resonance with primary mode by primary forcing function.
  - Since excitation simultaneously from upstream and downstream blades, critical to change design to avoid resonance.
  - Extensive optimization effort performed to either move natural frequency out of range and/or change count of turbine blades to move excitation.

Stator Airfoil Thickness Changes

- **Initial (R02)**
- **R03b2**
- **R03b2_t2**
- **R03b2_t4**
- **R03b2_t5**
- **R03b2_t6**

Since excitation simultaneously from upstream and downstream blades, critical to change design to avoid resonance. Extensive optimization effort performed to either move natural frequency out of range and/or change count of turbine blades to move excitation.

Modal analysis of original design indicated resonance with primary mode by primary forcing function.
Range of +/- 5% on natural frequencies to account for modeling uncertainty.
Damping

• Damping is critical parameter for forced response prediction, so “whirligig” test program used to obtain data.

• Whirligig is mechanically-driven rotor with bladed-disk excited by pressurized orifice plate simulate blade excitation.

• Key assumption is that this reflects true configuration.

• SDOF Curve fit technique applied to selected top-responding blades to derive damping from response.
Data shows wide-variation in damping, but reasonable population (15-20 acceptable samples) for statistical characterization.

Lognormal distribution fits obtained for each mode.
Can Also Use Modal Analysis in Failure Investigations

- Examination of **Modal Stress** Plots provides link to location of observed cracking.

\[
\phi^m = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \quad \rightarrow \quad \phi^m_{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{pmatrix}
\]

SSME HPFTP 1st Stage Turbine Blade

Modal displacement

Modal stress

Crack Origination Site at Inside of tip core

Mode 25 @ 55,523 Hz

Maximum Normalized Modal Stress

Minimum Normalized Modal Stress
Frequency and Transient Response Analysis uses Concept of Modal Superposition using Generalized (or Principal Coordinates).

\[
\begin{bmatrix} M \end{bmatrix}\{\ddot{u}\} + \begin{bmatrix} C \end{bmatrix}\{\dot{u}\} + \begin{bmatrix} K \end{bmatrix}\{u\} = \{P(t)\}
\]

Mode Superposition Method – transforms to set of uncoupled, SDOF equations that we can solve using SDOF methods.

First obtain \([\Phi]_{\text{mass}}\). Then introduce coordinate transformation:

\[
\{u\} = N \begin{bmatrix} \Phi \end{bmatrix}^{M} \{\eta\}
\]

\[
\begin{bmatrix} M \end{bmatrix}\begin{bmatrix} \Phi \end{bmatrix}\{\ddot{\eta}\} + \begin{bmatrix} C \end{bmatrix}\begin{bmatrix} \Phi \end{bmatrix}\{\dot{\eta}\} + \begin{bmatrix} K \end{bmatrix}\begin{bmatrix} \Phi \end{bmatrix}\{\eta\} = \{P(t)\}
\]

\[
\begin{bmatrix} \Lambda \end{bmatrix}\{\ddot{\eta}\} + \begin{bmatrix} \Theta \end{bmatrix}\{\dot{\eta}\} + \begin{bmatrix} \kappa \end{bmatrix}\{\eta\} = \begin{bmatrix} \Phi \end{bmatrix}^{T}\{P(t)\}.
\]
for the SDOF equation of motion,

\[ m \ddot{u} + c \dot{u} + ku = F \rightarrow \ddot{u} + 2\zeta \omega \dot{u} + \omega^2 u = F \]

\[
|U(\Omega)| = \left( \frac{F_0}{k} \right) \sqrt{\frac{1}{\left(1 - \left(\frac{\Omega}{\omega}\right)^2\right)^2 + \left(2\zeta \left(\frac{\Omega}{\omega}\right)\right)^2}}
\]

So we get the same equations in \( \eta \):

\[
\ddot{\eta}_m + 2\zeta \omega_m \dot{\eta}_m + \lambda \eta_m = \left\{ \phi \right\}_m^T \{ P(t) \}
\]

\[
|\eta_m(t)| = \frac{\left\{ \phi \right\}_m^T \{ F \}}{\lambda_m} \sqrt{\frac{1}{\left(1 - \left(\frac{\Omega}{\omega_m}\right)^2\right)^2 + \left(2\zeta \frac{\Omega}{\omega_m}\right)^2}}
\]

- For “Frequency Response” Analysis, apply Fourier coefficients coming from CFD such that excitation frequencies match Campbell crossovers.
Forced Response Analysis in Failure Investigations

- SSME HPFTP 1\textsuperscript{st} Stage Impeller.

Mode shape

Crack location 1\textsuperscript{st} splitter

Frequency Response Analysis
Many structures possess some kind of symmetry that can be used to simplify their analysis.

A cyclically symmetric structure possesses rotational symmetry, i.e., the original configuration is obtained after the structure is rotated about the axis of symmetry by a given angle.

Instead of modeling entire structure, only model one sector.

For turbomachinery structures, structural analysis is generally only possible by taking advantage of huge reduction in model size by using cyclic symmetry.

Section courtesy of Dr. Eric Christensen, DCI Inc.
Characteristics of Cyclic Symmetric Modes

- Most Nodal Diameter modes exist in pairs, same shape but rotated by $\pi/ND$
- For purely cyclic symmetric sections, highest Nodal Diameter possible is $N/2$ for even # sectors, $(N-1)/2$ for odd # sectors.

First family of modes:
- Has unique eigenvalues
- Has unique eigenvectors
- All segments have same mode shape

The next family of modes:
- Pairs of degenerate (identical) eigenvalues
- Non-unique eigenvectors

The last family of modes:
- Only exist if $N$ is even
- Has unique eigenvalues & eigenvectors.
Cyclically symmetric sections have $N = 6$, so max ND = $N/2 = 3$ for those sections.

However, much of impeller is disk-like, with axisymmetry, having infinite # ND’s.

Codes generate “Harmonic Families” of modes which only contain Nodal Diameter modes according to following stair-step pattern:

- $H_0$ contains ND = 0, 7, 14, ...
- $H_1$ contains ND = 1, 6, 8, 13, 15, ...
- $H_2$ contains ND = 2, 5, 9, 12, 16, ...
- $H_3$ contains ND = 3, 4, 10, 11, 17,
• **Generalized (or Modal) Force** defined as

\[ \{f\}_m = \{\Phi\}_m^T \{F\}. \]

• This is just the dot product of each mode with the excitation force vector and means that the response is directly proportional to the similarity of the spatial shape of each mode with the spatial shape of the force.

• For pure harmonic waves, the “Orthogonality Principle” states

\[
\int_{-\pi}^\pi \sin(n\theta)\sin(m\theta) = \begin{cases} 
\pi & \text{when } n=m \\
0 & \text{otherwise}
\end{cases}
\]

• Think of the \{\Phi\} as a continuous function, and the force the same way.
  - Then the Dot Product is the same as an integration of the product of the two functions.
  - So this says that the only non-zero result of an excitation wave shaped like a Sine and a mode shaped like a Sine is for the components of those waves that have the same wave number!
Have to determine Nodal Diameter of Modes to identify Resonance

• For disks and disk dominated modes, 5ND Traveling Wave will excite a 5ND mode

5ND travelling wave Mode of Bladed-Disc

5ND standing wave mode of Impeller (modal test using holography)

• On the other hand, 3ND excitation (perhaps from pump diffusers) will not excite a 5ND structural mode.

\[ p(\theta) = P_0 \sin 3\theta \]
• “Triple Crossover Points” (speed, $\omega$, and ND) needed for resonance of pure shroud (disk) modes.

• ND mode at exact spatial number of distortions in excitation.
Sampling by discrete number of points on structure of pressure oscillation results in spatial Nodal Diameter excitation at the difference of the two counts.

E.g., a 74 wave number pressure field (coming from 2x37 vanes), exciting 69 blades results in a Nodal Diameter mode of 69-74=-5, where sign indicates direction of traveling 5ND wave (plot courtesy Anton Gagne).

“Blade/Vane” Interaction causes different ND excitation.
Tyler–Sofrin Blade–Vane Interaction Charts

- Chart identifies Nodal Diameter families that can be excited

<table>
<thead>
<tr>
<th>Upstream Nozzle Multiples</th>
<th>37</th>
<th>74</th>
<th>111</th>
<th>148</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downstream Stator Multiples</td>
<td>57</td>
<td>114</td>
<td>171</td>
<td>228</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Blade multiples</th>
<th>69</th>
<th>32</th>
<th>-5</th>
<th>N/A</th>
<th>N/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade multiples</td>
<td>69</td>
<td>12</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Upstream Nozzle Multiples</th>
<th>138</th>
<th>N/A</th>
<th>N/A</th>
<th>27</th>
<th>-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downstream Stator Multiples</td>
<td>138</td>
<td>N/A</td>
<td>24</td>
<td>-33</td>
<td>N/A</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Upstream Nozzle Multiples</th>
<th>207</th>
<th>N/A</th>
<th>N/A</th>
<th>N/A</th>
<th>N/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downstream Stator Multiples</td>
<td>207</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>-21</td>
</tr>
</tbody>
</table>

- All modes in Campbell have to be from these families
  - E.g., Nodal Diameter 5, blade mode 3 (torsion)

- **Temporal Frequency** of Excitation is at the engine order of the distortion.

- Much of chart is marked “N/A – not applicable” because.....
  - Highest number of ND waves in a cyclic symmetric structure is N/2 or (N-1)/2
Example – LPSP Turbine Blisk Aliasing Tables, Non–Problematic Modal Evaluation

• 123 blades allow \((123-1)/2=61\) Nodal Diameters

<table>
<thead>
<tr>
<th></th>
<th>1st Stage Nozzles</th>
<th>2nd Stage Stators</th>
<th>Exit Guide Vanes</th>
</tr>
</thead>
<tbody>
<tr>
<td>blades</td>
<td>87</td>
<td>93</td>
<td>97</td>
</tr>
<tr>
<td>123</td>
<td>36 51 138</td>
<td>30 63 156</td>
<td>26 71 168</td>
</tr>
<tr>
<td>246</td>
<td>159 72 15</td>
<td>153 60 33</td>
<td>149 52 45</td>
</tr>
<tr>
<td>369</td>
<td>282 195 108</td>
<td>276 183 90</td>
<td>272 175 78</td>
</tr>
<tr>
<td>0</td>
<td>87 174 261</td>
<td>93 186 279</td>
<td>97 194 291</td>
</tr>
</tbody>
</table>

• Many crossings judged low risk, or acceptable risk for non-flight program
  - 15ND, 33ND, and 45ND crossing modes eliminated due to probable low 3X forcing function, high frequency
  - 52ND modes extremely complicated, high frequency
  - Many modes eliminated due to non-adjacency of forcing function

• Also have to consider mechanical excitations order 1-6N
  • In this case, no 1-6ND modes have crossings with appropriate forcing function.
Turbine Blisk Campbell for Problematic Modes in 100% PL Range

LPSP 1st Stage Turbine Blisk Campbell Diagram 100% PL Range

- Pump Speed (kRPM)
- Frequency (kHz)

- 27500 RPM +/- 5%
- ND 60, pair 16 (186 EO), 92836 Hz
- ND 60, pair 16 (186 EO), 86036 Hz
- N51, pair 14 (174 EO), 85972 Hz
- ND 60, pair 12 (186 EO), 78349 Hz
- ND 60, pair 11 (174 EO), 75845 Hz
- ND 60, pair 13 (186 EO), 79524 Hz

186 = 2 x Stators
174 = 2 x Nozzles
### Turbine Blisk Problematic Modes Possible Resolutions

<table>
<thead>
<tr>
<th>ND</th>
<th>mode pair number</th>
<th>Natural Frequencies</th>
<th>Natural Frequencies</th>
<th>mode shape description</th>
<th>excitation source and order</th>
<th>potential solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>70% PL</td>
<td>100% PL</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>5</td>
<td><strong>28955.47</strong></td>
<td>28964.54</td>
<td>1&lt;sup&gt;st&lt;/sup&gt; Torsion</td>
<td>12/31/13- 2&lt;sup&gt;nd&lt;/sup&gt; blade -are nozzles problem?</td>
<td>87=1 x 1st st upstream nozzles</td>
</tr>
<tr>
<td>51</td>
<td>9</td>
<td><strong>58040.23</strong></td>
<td>58048.62</td>
<td>1&lt;sup&gt;st&lt;/sup&gt; blade chordwise 2&lt;sup&gt;nd&lt;/sup&gt; bending</td>
<td>174= 2 x 1st st upstream nozzles</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>11</td>
<td>75845.08</td>
<td><strong>75878.04</strong></td>
<td>1&lt;sup&gt;st&lt;/sup&gt; blade spanwise TE 2 wave</td>
<td>174= 2 x 1st st upstream nozzles</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>12</td>
<td>78335.00</td>
<td><strong>78349.16</strong></td>
<td>2nd blade bending</td>
<td>186 = 2 x upstream 2nd st stator</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>13</td>
<td>79511.67</td>
<td><strong>79524.37</strong></td>
<td>2nd blade ?</td>
<td>186 = 2 x upstream 2nd st stator</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>14</td>
<td>85944.41</td>
<td><strong>85972.20</strong></td>
<td>1&lt;sup&gt;st&lt;/sup&gt; blade TE spanwise 1 wave</td>
<td>174= 2 x 1st st upstream nozzles</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>14</td>
<td>86007.76</td>
<td><strong>86035.52</strong></td>
<td>1&lt;sup&gt;st&lt;/sup&gt; blade TE 1.5 wave</td>
<td>186 = 2 x downstream 2nd st stator</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>16</td>
<td>92827.38</td>
<td><strong>92835.74</strong></td>
<td>1&lt;sup&gt;st&lt;/sup&gt; blade TE 0.5 wave</td>
<td>186 = 2 x downstream 2nd st stator</td>
<td></td>
</tr>
</tbody>
</table>
Spatial Fourier Analysis helpful to identify ND number of both excitation and modes

- Let’s say we have **measured** pressure field

\[ p(t, \theta) \]

that has the **unknown** temporal and spatial form:

\[ s = \text{Sin}[3t + 2\theta] + \text{Cos}[4t + 3\theta] \]

- First perform Temporal Fourier Analysis at each value of \( \theta \), using Complex Form (more efficient than harmonic form)

\[
p(t) = \sum_{n=-\infty}^{\infty} \left[ c_n e^{in\Omega_1 t} \right] \quad \text{where}
\]

\[
c_n = \frac{1}{T_1} \int_{0}^{T} p(t) e^{-in\Omega_1 t} \, dt
\]
Now look at obtaining spatial components for each of the temporal Fourier components ("bins").

\[ p(\theta) = \sum_{n=-\infty}^{\infty} \left[ c_n e^{in\Omega_t \theta} \right] \]

where

\[ c_n = \frac{1}{2\pi} \int_{0}^{2\pi} p(\theta) e^{in\Omega_t \theta} d\theta \]
2-D Fourier Transform Shows Spatial Complexity of J2X turbine flow field and response, used in evaluation of forced response methodologies.
Due to manufacturing tolerances, the turbine blades on a bladed disk will never be identical.

Result is that the bladed-disk modes (e.g., ND17, 1\textsuperscript{st} Torsion) will have slightly varying natural frequencies and mode shapes; typical variation is +/- 1.5%.

Variation itself called \textit{Mistuning}, which generates two effects in addition to bifurcation of individual modes into n*individual modes:

- \textit{Localization} – mode shapes warp such that maximum deflection is at a single location, rather than at every high point in a “tuned” nodal diameter mode.

- \textit{Amplification} – most important effect – the maximum resonant “mistuned” response is frequently up to twice as much as the “tuned” maximum resonant response.

Probabilistic analysis techniques required since every bladed-disk will be different.

Innumerable papers and Ph.D. theses have been devoted to this topic over the last 40 years.

Tractable techniques for predicting level of amplification for a design have only existed since 2004.
Localization due to Mistuning

Fig. 3b  One ND tuned and mistuned mode shapes

Fig. 3c  Two ND tuned and mistuned mode shapes

Ref: Rao, J.S., Mistuning of Bladed Disk Assemblies to Mitigate Resonance, Altair Engineering, 2006
• Maxiumum amplified blade is not blade with most mistuning, appears to be most pronounced near locations of “eigenvalue veering” on Nodal Diameter plot.

• Can predict amplification for a given design using “MISER” by assuming a std deviation of frequency mistuning and generating a nodal diameter diagram for modes of interest.

Conclusion

• Structural Dynamic Analysis of Turbomachinery is critical aspect of design, development, test, and failure analysis of Rocket Engines.

• Process of Analysis starts consists of modeling, modal analysis and characterization, comparison with excitation field, and forced response analysis if necessary.

• Thorough understanding of Fourier Analysis, Vibration Theory, Finite Element Analysis critical.

• Knowledge of Turbomachinery Design and Fluid Dynamics very useful.