Effect of Static Strains on Diffusion

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A theory is developed that gives the diffusion coefficient in strained systems as an exponential function of the strain. This theory starts with the statistical theory of the atomic jump frequency as developed by Vineyard. The parameter determining the effect of strain on diffusion is related to the changes in the interatomic forces with strain. Comparison of the theory with published experimental results for the effect of pressure on diffusion shows that the experiments agree with the form of the theoretical equation in all cases within experimental error.

I. INTRODUCTION

SINCE the diffusion rate in a crystal depends on the atomic interaction energy, and since this energy depends on the interatomic distances, it is to be expected that the diffusion coefficient of a migrating species will be altered by a strain superimposed on the crystal. Experimental evidence shows that the change in the diffusion coefficients resulting from strains can be considerable. Uniaxial elastic strain can increase the self-diffusion coefficient by as much as a factor of two\(^1\) and large hydrostatic pressures may decrease the self-diffusion coefficient by as much as an order of magnitude.\(^2\)-\(^5\)

The theory of the effect of pressure on diffusion has been examined on the basis of the dynamic theory of diffusion.\(^6\),\(^7\) In this theory, the pressure effect is represented by a parameter that is a function of the normal mode vibrations of the atoms in the crystal, and the diffusion coefficient is an exponential function of the pressure.


The dynamic theory of diffusion was developed as an alternative to the absolute rate theory of diffusion, since it was believed that the absolute rate theory depended on the postulate that the jumping atom spends a long time at the top of the potential barrier. However, it can be shown that the theory of the jump frequency can be developed without reference to such a postulate\(^6\) by considering the motion of a representative point in phase space. The jump frequency then depends on the rate at which phase points move over the potential maximum in configuration space, and not on the length of time the phase points spend at the maximum. In view of this situation, it is of interest to investigate the effect of strain on diffusion in terms of the statistical rate theory.

The statistical rate theory of diffusion in strained crystals as developed in this paper shows that the diffusion coefficient is an exponential function of strain, and that the strain effect can be represented by a parameter that is a function of the interatomic forces. The rate theory, therefore, has an advantage over the dynamic theory in two respects: First, the effect of strain on diffusion in different materials can be correlated with the interatomic potential energy, and second, the interatomic forces provide a basis on which to calculate the magnitude of the strain effect for different diffusion mechanisms. Accordingly, the possibility pre-
sents itself of deciding among alternative diffusion mechanisms from a comparison of the results of experiments on the effect of diffusion in strained systems with theoretical calculations. Such a program would be considerably more difficult in the framework of the dynamic theory.

The general equation for the diffusion coefficient for the flow of a single species in an isotropic solid may be written

\[ D = a \lambda^2 n \Gamma, \]  

where \( D \) is the diffusion coefficient, \( \lambda \) is the lattice parameter, \( n \) is the concentration of carrier defects, \( \Gamma \) is the jump frequency, and \( \alpha \) is a constant that is determined by the crystal structure. In the following sections expressions are derived for the effect of homogeneous static strains on the jump frequency and vacancy concentration. The resulting equations are put into a form in which comparisons can be made with existing experimental data.

II. DEPENDENCE OF JUMP FREQUENCY ON STRAIN

According to the statistical theory of rate processes, the jump frequency is determined by the ratio of two configurational integrals, one referring to the activated state and the other referring to the normal state. In analyzing the effect of strain on the jump frequency, the formulation of the rate process theory in solids given by Vineyard is used, in which the jump frequency is given in terms of these integrals by

\[ \Gamma = \frac{kT}{2\pi m} \int e^{-\varphi(q_0)/kT}d\sigma / \int e^{-\varphi(q)/kT}dA, \]  

where \( k \) is Boltzmann's constant, \( T \) is the temperature, and \( \varphi \) is the potential energy of the system as a function of all the coordinates of all the atoms in the crystal. The integral in the numerator of Eq. (2) is evaluated over a hypersurface \( \sigma \) in the configuration space such that the surface passes through the point corresponding to the diffusing atom at its activated position with all other atoms at their equilibrium positions. The hypersurface is also required to be perpendicular to contours of constant potential energy in the configuration space. The hypersurface defined in this manner divides the configuration space into two symmetric parts. The integral in the denominator is evaluated over the configuration volume \( A \) of one of these symmetric parts.

Equation (2) was derived for the case of an unstrained crystal. However, it is applicable to strained crystals if the potential energy \( \varphi \) is taken to be a function of the six strain components \( \epsilon_{\alpha\beta} \) as well as the atomic coordinates \( q_\alpha \). A similar procedure has been used by Born in an analysis of the statistical mechanics of crystal lattices. Thus, the potential energy in Eq. (2) is given by

\[ \varphi = \varphi(q_\alpha \epsilon_{\alpha\beta}), \]  

where \( q_\alpha \) represents the set of all atomic coordinates and \( \epsilon_{\alpha\beta} \) represents the set of six independent strain components.

The potential \( \varphi \) can be expanded as a Taylor series in the strains about the point of zero strain with the result that

\[ \varphi(q_\alpha \epsilon_{\alpha\beta}) = \varphi(q_\alpha 0) + \sum_{\alpha,\beta} C_{\alpha\beta} \epsilon_{\alpha\beta} \]  

\[ + \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} \epsilon_{\alpha\beta} \epsilon_{\alpha\gamma} + \cdots, \]  

where the coefficients \( C_{\alpha\beta} \) and \( C_{\alpha\beta\gamma} \) are defined by

\[ C_{\alpha\beta} = \frac{\partial \varphi}{\partial \epsilon_{\alpha\beta}} q_\alpha 0, \]  

\[ C_{\alpha\beta\gamma} = \frac{1}{2} \frac{\partial^2 \varphi}{\partial \epsilon_{\alpha\beta} \partial \epsilon_{\alpha\gamma}} q_\alpha 0. \]  

The subscripts indicate that the derivatives are evaluated when the strains are zero and the coordinates have the value \( q_\alpha \).

Substituting Eq. (4) into Eq. (2) gives the jump frequency in terms of the strain:

\[ \Gamma(\epsilon_{\alpha\beta}) \]  

\[ = \left( \frac{kT}{2\pi m} \right)^4 \int e^{-\varphi(q_\alpha 0)/kT} \exp \left( -\frac{1}{kT} \sum_{\alpha,\beta} C_{\alpha\beta} \epsilon_{\alpha\beta} \right) \exp \left( -\frac{1}{kT} \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} \epsilon_{\alpha\beta} \epsilon_{\alpha\gamma} \right) \cdots, \]  

where terms of order higher than the first have been ignored. It will be shown later that the first-order contribution of the strain to the jump frequency depends on the difference of the average value of \( C_{\alpha\beta} \) evaluated near the normal configuration and near the activated configuration, and on similar differences in the averages of \( C_{\alpha\beta\gamma} \), etc. It is extremely difficult to give an \textit{a priori} estimate of the relative magnitudes of these differences. At any rate, for small enough strains the first-order terms predominate and the higher order terms can be neglected. It will be seen later that the form of experimental results is adequately described by considering only the first-order terms in the strains. For zero strain, Eq. (7) gives the jump frequency as

\[ \Gamma_0 = \left( \frac{kT}{2\pi m} \right)^4 \int e^{-\varphi(q_\alpha 0)/kT} d\sigma / \int e^{-\varphi(q_\alpha 0)/kT} dA. \]  

\( ^{\text{9 Max Born, Proc. Cambridge Phil. Soc. 36, 160 (1940).}} \)
Now take the ratio of Eqs. (7) and (8). The result is
\[
\frac{\Gamma(\epsilon_{AB})}{\Gamma_0} = \left\langle \exp\left( -\frac{1}{kT_{\sigma,\sigma}} \sum C_{\sigma\beta}\epsilon_{\alpha\beta} \right) \right\rangle_{\sigma} / \left\langle \exp\left( -\frac{1}{kT_{\sigma,\sigma}} \sum C_{\sigma\beta}\epsilon_{\alpha\beta} \right) \right\rangle_{\sigma} \tag{9}
\]

The angular brackets indicate that a statistical average has been taken of the quantity within the brackets, and the subscripts \( \sigma \) and \( \alpha \) indicate that the averages are taken over the regions of configuration space \( \sigma \) and \( \alpha \), respectively. The explicit expressions for these averages are
\[
\left\langle \exp\left( -\frac{1}{kT_{\sigma,\sigma}} \sum C_{\sigma\beta}\epsilon_{\alpha\beta} \right) \right\rangle_{\sigma} = \int e^{-\psi(q_{i,0})/kT} \exp\left( -\frac{1}{kT_{\sigma,\sigma}} \sum C_{\sigma\beta}\epsilon_{\alpha\beta} \right) d\sigma / \int e^{-\psi(q_{i,0})/kT} d\sigma, \tag{10}
\]
\[
\left\langle \exp\left( -\frac{1}{kT_{\sigma,\alpha}} \sum C_{\sigma\beta}\epsilon_{\alpha\beta} \right) \right\rangle_{\alpha} = \int e^{-\psi(q_{i,0})/kT} \exp\left( -\frac{1}{kT_{\sigma,\sigma}} \sum C_{\sigma\beta}\epsilon_{\alpha\beta} \right) d\alpha / \int \int e^{-\psi(q_{i,0})/kT} d\alpha d\sigma. \tag{11a}
\]

For small strains and high temperatures, the conditions under which the experimental effects of strain on diffusion are usually determined, the exponents in Eq. (9) can be expanded into a series, and only the first two terms need be retained. Thus, Eq. (9) can be written as
\[
\frac{\Gamma(\epsilon_{AB})}{\Gamma_0} = \left( 1 - \frac{1}{kT_{\sigma,\sigma}} \sum (C_{\sigma\beta})_{\sigma} \epsilon_{\alpha\beta} \right) / \left( 1 - \frac{1}{kT_{\sigma,\alpha}} \sum (C_{\sigma\beta})_{\alpha} \epsilon_{\alpha\beta} \right), \tag{11b}
\]

where \( (C_{\sigma\beta})_{\sigma} \) and \( (C_{\sigma\beta})_{\alpha} \) are given by
\[
(C_{\sigma\beta})_{\sigma} = \int_{\sigma} \frac{\partial \phi}{\partial \epsilon_{\alpha\beta}^{(q_i,0)}} e^{-\psi(q_{i,0})/kT} d\sigma / \int_{\sigma} e^{-\psi(q_{i,0})/kT} d\sigma, \tag{12}
\]
\[
(C_{\sigma\beta})_{\alpha} = \int_{\alpha} \frac{\partial \phi}{\partial \epsilon_{\alpha\beta}^{(q_i,0)}} e^{-\psi(q_{i,0})/kT} d\alpha / \int_{\alpha} e^{-\psi(q_{i,0})/kT} d\alpha. \tag{13}
\]

Taking logarithms of Eq. (11) and utilizing the fact that \( \ln(1-x) = -x \) for small \( x \), gives
\[
\ln \frac{\Gamma(\epsilon_{AB})}{\Gamma_0} = -\frac{1}{kT_{\sigma,\sigma}} \sum (C_{\sigma\beta})_{\sigma} \epsilon_{\alpha\beta} \left( (C_{\sigma\beta})_{\alpha} - (C_{\sigma\beta})_{\sigma} \right), \tag{14}
\]
\[
\Gamma(\epsilon_{AB}) = \Gamma_0 \exp \left( \frac{1}{kT_{\sigma,\sigma}} \sum m_{\alpha\beta} \epsilon_{\alpha\beta} \right). \tag{16}
\]

Since \( \Gamma_0 \) can always be written as
\[
\Gamma = \nu^* e^{-\Delta E^*/kT}, \tag{17}
\]
where \( \Delta E^* \) is the energy of activation for the atomic jump and \( \nu^* \) is an effective frequency, it is evident from Eq. (16) that the strain affects the jump frequency by an effective change in the energy of activation.

Equation (16) shows that the jump frequency has a simple exponential dependence on the strains and that this dependence is controlled by the derivatives of the potential energy with respect to the strains evaluated at the saddle point of the activated state.

Equation (16) gives the general relation between the jump frequency and the strain that will be used in this paper.

To illustrate the application of Eq. (16), three special cases will be considered:

1. Uniform compression or expansion, in which
\[
\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon, \tag{18}
\]

2. Simple shear, in which
\[
\epsilon_{xy} = \epsilon_{yx} = \epsilon, \tag{19}
\]

3. Simple elastic tension or compression in the \( x \) direction, in which
\[
\epsilon_{xx} = \epsilon_{\sigma x}, \quad \epsilon_{yy} = \epsilon_{\sigma y}, \quad \epsilon_{zz} = -\mu \epsilon_{\sigma x}, \tag{20}
\]

where \( \mu \) is Poisson's ratio. For these three cases, Eq. (16) gives the following results: For uniform compression or expansion,
\[
\Gamma(\epsilon) = \Gamma_0 e^{(m/n kT)} \epsilon, \tag{21}
\]
where
\[
m = (C)_{\alpha} - (C)_{\sigma}, \tag{22}
\]
and
\[
(C)_{\sigma} = \int_{\sigma} \frac{\partial \phi}{\partial \epsilon_{\alpha\beta}^{(q_i,0)}} e^{-\psi(q_{i,0})/kT} d\sigma / \int\int e^{-\psi(q_{i,0})/kT} d\sigma d\sigma, \tag{23}
\]
\[
(C)_{\alpha} = \int_{\alpha} \frac{\partial \phi}{\partial \epsilon_{\alpha\beta}^{(q_i,0)}} e^{-\psi(q_{i,0})/kT} d\alpha / \int\int e^{-\psi(q_{i,0})/kT} d\alpha d\sigma. \tag{24}
\]
For simple shear, 

\[ \Gamma(\epsilon_s) = \Gamma_0 e^{(2m_s/kT)\epsilon_s}, \] 

where 

\[ m_s = \langle C_s \rangle_s - \langle C_s \rangle_s, \] 

\( \langle C_s \rangle_s \) and \( \langle C_s \rangle_s \) are given by statistical averages similar to Eqs. (23) and (24), i.e., 

\[ \langle C_s \rangle_s = \left\langle \left( \frac{\partial \varphi}{\partial \epsilon_s} \right)_{q_s,0} \right\rangle_s, \] 

\[ \langle C_s \rangle_s = \left\langle \left( \frac{\partial \varphi}{\partial \epsilon_s} \right)_{q_s,0} \right\rangle_s. \] 

For simple elastic tension or compression in the z direction, 

\[ \Gamma(\epsilon_L) = \Gamma_0 e^{(m_L/kT)(1-2\nu)}, \] 

where 

\[ m_L = \langle C_L \rangle_s - \langle C_L \rangle_s, \] 

or 

\[ m_L = \left\langle \left( \frac{\partial \varphi}{\partial \epsilon_L} \right)_{q_s,0} \right\rangle_s - \left\langle \left( \frac{\partial \varphi}{\partial \epsilon_L} \right)_{q_s,0} \right\rangle_s. \] 

### III. DEPENDENCE OF VACANCY CONCENTRATION ON STRAIN

The \( n \) appearing in Eq. (1) has a different significance for different mechanisms of diffusion. Broadly speaking, \( n \) is the probability that a diffusing particle has a site available to jump into. For dilute interstitial diffusion this probability is nearly unity whether or not the system is strained. For diffusion by a vacancy mechanism, however, \( n \) is the atomic fraction of vacancies in the crystal, given by 

\[ n = n_v/N_T, \] 

where \( n_v \) is the vacancy concentration, and \( N_T \) is the total number of lattice sites per cubic centimeter. It is therefore necessary to investigate the variation of \( n_v \) with strain.

The atomic fraction of vacancies in a crystal at equilibrium is given by (see Appendix) 

\[ n_v = \int \cdots \int e^{-\varphi(q_j, \epsilon_s)/kT} \prod_i d\epsilon_s dq_j / \int \cdots \int e^{-\varphi(q_j, \epsilon_s)/kT} \prod_i d\epsilon_s dq_j, \] 

where \( \varphi \) is the energy of the crystal containing a vacancy and \( \psi_0 \) is the energy of the perfect crystal. Performing the integrations over the momenta \( p_j \) converts Eq. (32) into 

\[ n_v = \int \cdots \int e^{-\varphi(q_j, \epsilon_s)/kT} \prod_i dq_j \prod_k (\nu_j)_k \int \cdots \int e^{-\varphi(q_j, \epsilon_s)/kT} \prod_i dq_j \prod_k (\nu_j)_k, \] 

where \( \varphi(q) \) and \( \varphi_0(q) \) are the potential energies in a crystal containing a vacancy and in a perfect crystal, respectively, each taken as a function of all the coordinates; \( (\nu_j)_k \) is the frequency of the \( k \)th vibrational mode in the crystal containing a vacancy; and \( (\nu_j)_k \) is the frequency of the \( k \)th normal mode in a perfect crystal. In a strained crystal, the \( \varphi \) and the \( \nu \) must be written as functions of strain, so that (33) becomes 

\[ n_v(\epsilon_s) \] 

\[ = \int \cdots \int \exp[-\varphi(q_j, \epsilon_s)/kT] \prod_i dq_j \prod_k (\nu_j)_k \int \cdots \int \exp[-\varphi(q_j, \epsilon_s)/kT] \prod_i dq_j \prod_k (\nu_j)_k. \] 

(34)

An estimate of the effect of strain on the frequency ratios can be made from Grüneisen's relation

\[ d \ln \nu/d \ln V = -\gamma, \] 

where \( V \) is the volume and \( \gamma \) is a positive constant. Integrating Eq. (35) for each vibrational mode as the crystal goes from the strained to the unstrained state 

\[ (\nu_j)_k \] 

\[ = \frac{1}{1 + \delta V/V} \gamma, \] 

where \( \delta V \) is the volume change arising from the strain. Grüneisen's relation, therefore, leads to an equality of frequency ratios in the strained and unstrained systems:

\[ \Pi_k (\nu_j)_k / \Pi_k (\nu_j)_k = \frac{\Pi_k (\nu_j)_k / \Pi_k (\nu_j)_k}{\Pi_k (\nu_j)_k / \Pi_k (\nu_j)_k}. \] 

(38)

Therefore, the ratio of Eqs. (34) and (33) is 

\[ \frac{n_v(\epsilon_s)}{n_v} = \frac{\int \cdots \int \exp[-\varphi(q_j, \epsilon_s)/kT] \prod_i dq_j / \int \cdots \int e^{-\varphi(q_j, \epsilon_s)/kT} \prod_i dq_j}{\int \cdots \int e^{-\varphi(q_j, \epsilon_s)/kT} \prod_i dq_j / \int \cdots \int \exp[-\varphi(q_j, \epsilon_s)/kT]} \prod_i dq_j. \] 

(39)

The potential-energy functions \( \varphi_e(\mathbf{q}, \varepsilon_{a\beta}) \) and \( \varphi_o(\mathbf{q}, \varepsilon_{a\beta}) \) can be expanded as Taylor series in the strain just as in the development beginning with Eq. (4) and leading to Eq. (16). The result is

\[
\frac{n_e(\varepsilon_{a\beta})}{n_o} = \exp\left(\frac{1}{kT} \sum_{a,\beta} \omega_{a\beta} \varepsilon_{a\beta}\right), \tag{40}
\]

where

\[
w_{a\beta} = \left\langle \frac{\partial \varphi_o}{\partial \varepsilon_{a\beta}} \right\rangle_{q_{j0}} - \left\langle \frac{\partial \varphi_e}{\partial \varepsilon_{a\beta}} \right\rangle_{q_{j0}}. \tag{41}
\]

For the special case of uniform compression or expansion, Eqs. (40) and (41) become

\[
n_e(\varepsilon) = n_o \exp(2\omega/VT), \tag{42}
\]

\[
w = \left\langle \frac{\partial \varphi_o}{\partial \varepsilon} \right\rangle_{q_{j0}} - \left\langle \frac{\partial \varphi_e}{\partial \varepsilon} \right\rangle_{q_{j0}}, \tag{43}
\]

where \( \varepsilon \) is the volume strain.

IV. PROOF THAT THE STATISTICAL AVERAGES \( \left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle_{q_{j0}} \) DO NOT VANISH

The preceding theory depends on the statistical averages of the derivatives of the potential energy of the crystal with respect to strain. It has been assumed that these averages are not zero, and that a first-order expansion in the strains is therefore adequate for small strains. This assumption can be justified by expanding the crystal energy in normal coordinates. The statistical averages of interest all have the form

\[
\left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle = \int \cdots \int \left( \frac{\partial \varphi}{\partial \varepsilon} \right)_o \exp(-\omega/VT) \prod_i dq_i / \int \cdots \int \exp(-\omega/VT) \prod_i dq_i, \tag{44}
\]

\[
\left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle = -3 \int \cdots \int \left( \sum_i \gamma_i \omega_i^2 q_i^2 \exp\left(-\frac{1}{2kT} \sum_i \omega_i^2 q_i^2\right) \right) \prod_i dq_i / \int \cdots \int \exp\left(-\frac{1}{2kT} \sum_i \omega_i^2 q_i^2\right) \prod_i dq_i. \tag{51}
\]

Now perform a coordinate transformation according to the following definition:

\[
u_i = \omega_i q_i. \tag{52}
\]

Then, after a few simple algebraic manipulations, Eq. (51) becomes

\[
\left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle = -3 \sum_i \gamma_i \int_{-\infty}^{\infty} u_i^2 \exp\left(-\frac{u_i^2}{2kT}\right) du_i / \int_{-\infty}^{\infty} \exp\left(-u_i^2/2kT\right) du_i \tag{53}
\]

where \( \varphi \) is the total potential energy of the crystal as a function of all the coordinates \( q_i \), and the subscript zero means that the derivative is evaluated at zero strain. If the \( q_i \) are taken to be the normal coordinates, \( \varphi \) can be written to the second order as

\[
\varphi = \varphi(0) + \frac{1}{2} \sum_i \omega_i^2 q_i^2, \tag{45}
\]

where \( \varphi(0) \) is the potential energy when all the atoms are at their mean positions, and the \( \omega_i \) are the normal mode frequencies. Differentiating Eq. (45) with respect to strain gives

\[
\frac{\partial \varphi}{\partial \varepsilon} = \frac{\partial \varphi(0)}{\partial \varepsilon} + \sum_i \omega_i \frac{d\omega_i}{d\varepsilon} q_i^2, \tag{46}
\]

and, since at zero strain the first term on the right is zero,

\[
\left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle = \sum_i \frac{d\omega_i}{d\varepsilon} q_i^2. \tag{47}
\]

For the purposes of this discussion, \( \varepsilon \) will be taken to be the strain corresponding to uniform compression or expansion, so that for small strains the volume is given by

\[
V = V_o(1 + 3\varepsilon), \tag{48}
\]

\( V_o \) being the volume at zero strain. Introducing the Grüneisen parameter \( \gamma_j \) by the relation

\[
d \ln \omega_i / d \ln V = -\gamma_j, \tag{49}
\]

where the \( \gamma_j \) are a set of positive constants, and using Eq. (48), Eq. (47) becomes

\[
\left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle = -3 \sum_i \gamma_i \omega_i^2 q_i^2. \tag{50}
\]

Substituting Eqs. (45) and (50) into Eq. (44) gives

\[
\left\langle \frac{\partial \varphi}{\partial \varepsilon} \right\rangle = -3kT \sum_i \gamma_i. \tag{54}
\]

Equation (54) shows that the averages of the first derivatives are never zero and that these averages are proportional to the temperature.

It is extremely difficult to make any \textit{a priori} decisions concerning the signs of \( m_{a\beta} \) and \( w_{a\beta} \) defined by Eqs. (15) and (41). Such a decision requires a detailed in-
nvestigation of the variation of localized normal mode vibrations with strain in the vicinity of a defect. However, on the basis of general physical considerations, it is to be expected that both $m_{\alpha\beta}$ and $w_{\alpha\beta}$ are positive.

V. EFFECT OF PRESSURE ON DIFFUSION CONSTANT

Using Eqs. (1), (21), and (42), and the fact that the lattice parameter in the strained system is $(1+\varepsilon)$ times the lattice parameter in the unstrained system, the relation between the diffusion coefficients in the strained and unstrained systems for uniform compression or expansion is

$$D(\varepsilon) = D_u (1+\varepsilon)^2 \exp \left( \frac{M}{kT} \frac{3\varepsilon}{1-\varepsilon} \right), \quad (55)$$

where $D_u$ is the diffusion coefficient in the unstrained system, and $M$ is given by

$$M = m + w \quad \text{(vacancy mechanism),} \quad (56)$$

$$M = m \quad \text{(interstitial, ring or exchange mechanism).} \quad (57)$$

In terms of the volume strain, $\varepsilon = \frac{1}{3} (\Delta V/V_0)$ for small strains, where $\Delta V$ is the initial volume, so that Eq. (55) takes the form

$$D(\Delta V/V_0) = D_u \left(1 + \frac{\Delta V}{V_0} \right)^4 \exp \left[ \frac{M}{kT} \frac{\Delta V}{V_0} \right]. \quad (58)$$

Therefore, it is evident that a plot of $\ln[D(\Delta V/V_0) \times (1+\Delta V/V_0)^{-1}]$ against $\Delta V/V_0$ should be linear with a slope $a$ given by

$$a = M/kT, \quad (59)$$

and an intercept given by $\ln D_u$.

Several investigators have obtained data on the variation of the diffusion coefficients with pressure that is suitable for testing Eq. (58). Reference 2 presents data for the self-diffusion coefficient as a function of pressure for sodium, phosphorous, mercury, gallium, and lead up to 12000, 4000, and 8000 atmospheres, respectively. The self-diffusion coefficient of liquid gallium up to pressures of 10000 atmospheres is given in reference 3. The self-diffusion coefficient for single-crystal zinc was computed from the data of Richards and Boyer assuming that the form of $\Delta V/V_0$ as a function of pressure is the same as that for mercury. The values of $\Delta V/V_0$ for white phosphorus were computed from data in reference 16 assuming that the variation of the fractional volume change with pressure has the same form as that observed for black and red phosphorus.

In all cases, the available compressibility data were extrapolated to the diffusion temperature.

The linearity of the plots presented in Figs. 1 to 3 shows that the form of Eq. (58) is valid for those systems investigated within the probable inaccuracies of the experiments and the calculations. The slopes of the plots are given in Table I, where $a = M/kT$ and $\alpha T$ are shown for the various materials.

The fact that $\alpha T$ is so much smaller for the liquid metals than for any of the solids including sodium is indicative of the difference in the mechanism of diffusion in liquids and solids. In a liquid, the atoms are not constrained to remain at lattice positions, so that diffusion occurs by a cooperative process involving the migrating atom and its nearest neighbors. Thus, the change in the interatomic forces can be kept to a minimum throughout the diffusion process, and consequently $\alpha T$ would be very low.

From Eqs. (54) and (59) it is seen that $a$ should be temperature independent. For the self-diffusion of lead for which pressure data are available at two temperatures, the value of $a$ is reasonably constant.
Fig. 1. Variation of \( \log[D_v(1+\Delta V/V_0)^{-2/3}] \) plotted against volume change \( (\Delta V/V_0) \) for self-diffusion of various elements. (a) Sodium at 363°K [see enclosed graph]. (b) White phosphorous at 314°K. (c) Liquid mercury at 303°K. (d) Liquid gallium at 303°K. (e) Lead at 526.2°K — ○; Lead at 574.2°K — □.
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Fig. 2. Variation of \( \log [D_0 (1 + \Delta V/V_0)^{-2/3}] \) plotted against fractional change in lattice parameter \( \Delta \lambda/\lambda_0 \) for self-diffusion in zinc. (a) Zinc at 580°K, perpendicular to \( \epsilon \) axis. (b) Zinc at 580°K, parallel to \( \epsilon \) axis.

Fig. 3. Variation of \( \log (1/K) \) plotted against volume change \( \Delta V/V_0 \) for mobility of silver at 573°K. (a) Silver chloride. (b) Silver bromide.

Activation Volume

The activation volume is ordinarily calculated from the relation

\[
\Delta V = \left[ \frac{\partial (\Delta G)}{\partial P} \right]_T = -kT \left[ \frac{\partial \ln (D/\alpha \lambda^* \gamma)}{\partial P} \right]_T, \tag{60}
\]
TABLE I. Comparison of values of \( aT \) for various solids.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Temperature, ( T ) (°K)</th>
<th>( a )</th>
<th>( aT )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sodium</td>
<td>279.9</td>
<td>10.130</td>
<td></td>
</tr>
<tr>
<td>Zinc (1)</td>
<td>86.6</td>
<td>50.230</td>
<td></td>
</tr>
<tr>
<td>Zinc (II)</td>
<td>34.7</td>
<td>20.130</td>
<td></td>
</tr>
<tr>
<td>Mercury (liquid)</td>
<td>6.5</td>
<td>1970</td>
<td></td>
</tr>
<tr>
<td>Gallium (liquid)</td>
<td>6.5</td>
<td>1970</td>
<td></td>
</tr>
<tr>
<td>Silver in silver chloride</td>
<td>57.9</td>
<td>53.240</td>
<td></td>
</tr>
<tr>
<td>Silver in silver bromide</td>
<td>128.0</td>
<td>73.340</td>
<td></td>
</tr>
<tr>
<td>Lead</td>
<td>121.5</td>
<td>69.740</td>
<td></td>
</tr>
</tbody>
</table>

where \( P \) is the pressure and \( \Delta G \) refers to the Gibbs free-energy changes for vacancy formation and for the formation of the activated state configuration. This free-energy change is calculated from the measured diffusion coefficient as a function of pressure. It follows from Eq. (58) that (60) may be written

\[
\Delta V^I = -kT \left[ \frac{(M/kT) \partial (\Delta V/V_0)}{\partial P} \right]_T
\]

(61)

Since

\[
\left[ \frac{\partial (\Delta V/V_0)}{\partial P} \right]_T = -\beta
\]

where \( \beta \) is the compressibility, the activation volume defined by (61) can be calculated from the simple formula

\[
\Delta V^I = a\beta kT.
\]

Table II presents values of the activation volume calculated from Eq. (62) at atmospheric pressure for those systems for which data are available.

CONCLUSIONS

A statistical mechanical theory was developed that relates the diffusion coefficient to strain in terms of the atomic properties of the system. The theory makes the following statements:

1. For diffusion as a function of hydrostatic pressure, the diffusion coefficient is an exponential function of the volume strain.
2. The rate of change of the diffusion coefficient with strain is related to the interatomic forces. The relation is explicit enough that the variation of the diffusion coefficient with pressure can be interpreted in terms of the interatomic potential-energy functions of the material.
3. For diffusion under hydrostatic pressure, the activation volume can be calculated from the compressibility and the rate of change of the diffusion coefficient with volume strain.

In every case for which data are available, these conclusions are in agreement with experiment.

The general framework of the theory provides a basis for understanding the effect of strain on diffusion in terms of the atomic properties of the system and should provide a valuable tool for comparing diffusion rates for different states of strain, as well as for investigating the mechanism of diffusion.

APPENDIX

The Vacancy Concentration Formula

Consider a canonical ensemble containing \( X \) member systems, each system being a crystal containing \( N \) atoms and 1 vacancy. Let \( E_j \) be the \( j \)th energy level of a system containing 1 vacancy and let \( \Omega_j \) be the corresponding degeneracy. Then the number of systems containing 1 vacancy is

\[
N_1 = \frac{\sum \Omega_j \exp(-E_j/kT)}{Z},
\]

(A1)

where \( Z \) is the total partition function for the ensemble.

The number of vacancies in the ensemble is

\[
N_v = \sum_i lN_i,
\]

(A2)

and the number of atoms in the ensemble is

\[
N_A = N \sum_i N_i.
\]

(A3)

The atomic fraction of vacancies is given by \( n_v = N_v / (N_A + N_v) \). Since \( N_v \ll N_A \), \( n_v \) is given by the ratio of Eqs. (A2) to (A3) to an excellent approximation, and therefore

\[
n_v = \frac{\sum_i l\Omega_j \exp(-E_j/kT)}{\sum_i l\Omega_j}.
\]

(A4)

where \( \Omega_j \) is defined by

\[
\Omega_j = \sum_i \Omega_j \exp(-E_j/kT).
\]

(A5)

\( \Omega_j \) is the partition function of a system containing \( l \) vacancies.
Carrying out the division in (A4) and retaining only the leading term give

\[ n_\ast = (1/N)Q_1/Q_0, \]  

(A6)

which is an excellent approximation, since the energy of formation of a vacancy is of the order of 1 ev, and therefore the higher terms in the series are very small. \( Q_0 \) is the partition function of a perfect crystal and \( Q_1 \) is the partition function of a crystal containing a vacancy.

In the semiclassical approximation,

\[ \frac{Q_1}{Q_0} = (N+1) \int \ldots \int e^{-\psi_0/P_j} \prod_i dP_i dQ_i / \int \ldots \int e^{-\psi_0/P_j} \prod_i dP_i dQ_i, \]  

(A7)

where \( \psi_0 = \psi_0(p,q) \) is the energy of a perfect crystal. The integrations are carried out over all values of the momenta and coordinates \( P_j \) and \( Q_i \). The factor \((N+1)\) arises from the fact that \( N \) indistinguishable atoms can be placed in \((N+1)\) numbered lattice sites in \((N+1)\) ways so that \( Q_1 \) is proportional to \((N+1)\).

Combining Eqs. (A6) and (A7) gives

\[ n_\ast = \int \ldots \int e^{-\psi_0(P_j, Q_i)/kT} \prod_i dP_i dQ_i / \int \ldots \int e^{-\psi_0(P_j, Q_i)/kT} \prod_i dP_i dQ_i, \]  

(A8)

where unity has been neglected relative to \( N \).

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