Effect of Index of Refraction on Radiation Characteristics in a Heated Absorbing, Emitting, and Scattering Layer

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Nomenclature

\[ a = \text{absorption coefficient of layer, m}^{-1} \]
\[ D = \text{thickness of plane layer, m} \]
\[ E_1, E_2, E_3 = \text{exponential integral functions,} \]
\[ n = \text{index of refraction} \]
\[ q = \text{heat flux, W/m}^2 \]
\[ T = \text{absolute temperature, K} \]
\[ T_{\text{in}}, T_{\text{out}} = \text{temperatures of surrounding radiating regions, K} \]
\[ x = \text{coordinate normal to boundary of plane layer, m; } X = x/D \]
\[ \kappa = \text{optical depth, } (a + \sigma_t)x; \]
\[ \kappa_D, \text{optical thickness, } (a + \sigma_t)D \]
\[ \alpha = \text{Stefan-Boltzmann constant, W/m}^2 \text{K}^4 \]
\[ \sigma_t = \text{scattering coefficient of layer, m}^{-1} \]
\[ \rho = \text{reflectivity of interface for internally incident radiation} \]
\[ \sigma^\circ = \text{transmissivity of surface for externally incident radiation} \]
\[ \Phi = \text{dimensionless temperature distribution} \]
\[ \Psi = \text{dimensionless radiative heat flux} \]

Subscripts

\[ 1, 0 = \text{incoming and outgoing radiation} \]
\[ r = \text{radiative quantity} \]
\[ 1, 2 = \text{the hotter and cooler surroundings of the layer} \]

Introduction

The index of refraction can considerably influence the temperature distribution and radiative heat flow in semitransparent materials such as some ceramics. For external radiant heating, the refractive index influences the amount of energy transmitted into the interior of the material. Emission within a material depends on the square of its refractive index, and hence this emission can be many times that for a blackbody radiating into a vacuum. Since radiation exiting through an interface into a vacuum cannot exceed that of a blackbody, it is reduced and placed into angular directions for which there is total internal reflection. This provides a trapping effect for retaining energy within the layer and tends to equalize its temperature distribution.

An analysis of temperature distributions in absorbing-emitting layers, including index of refraction effects, was developed by Gordon (1958) to predict cooling and heat treating of glass plates. The interfaces were optically smooth; the resulting specular reflections were computed from the Fresnel reflection laws. This provides a somewhat different behavior than for diffuse interfaces. A similar application was for heating that occurs in a window of a re-entry vehicle (Fowlie et al., 1969).

Because of the large amount of scattering in many ceramic materials, the interfaces between the ceramic and the surrounding air or vacuum are assumed to be diffuse. The refractive index of the surroundings is unity. As shown in Fig. 1, the layer is subjected to radiation from the surroundings at \( T_{\text{in}} \) and \( T_{\text{out}} \) on the two sides \( x = 0 \) and \( x = D \). It is assumed that the surroundings act as black environments so the incident energies on the two sides are \( q_1(0) = \sigma T_{\text{in}}^4 \) and \( q_2(D) = \sigma T_{\text{out}}^4 \).

Inside the layer there are outgoing and incoming fluxes, \( q_1 \) and \( q_2 \), at each interior surface, as shown on the figure. Since scattering is included, the local optical depth is related to the \( x \) coordinate by \( \kappa = (a + \sigma_t)x \).

The temperature distribution inside the layer is governed by the integral equation given by Siegel and Howell (1981), modified with the index of refraction factor as

\[ \sigma T_{\text{in}}^4 \]

Fig. 1 Layer geometry, coordinate system, and nomenclature of heat fluxes at boundaries

Analysis

A plane layer of ceramic material has thickness \( D \) as shown in Fig. 1. It absorbs, emits, and isotropically scatters radiation. The limiting case is considered here where the layer temperature distribution is dominated by radiation so heat conduction is neglected. The material has a constant index of refraction; it is the effect of the index of refraction that is investigated here. Because of the large amount of scattering in many ceramic materials, the interfaces between the ceramic and the surrounding air or vacuum are assumed to be diffuse. The refractive index of the surroundings is unity. As shown in Fig. 1, the layer is subjected to radiation from the surroundings at \( T_{\text{in}} \) and \( T_{\text{out}} \) on the two sides \( x = 0 \) and \( x = D \). It is assumed that the surroundings act as black environments so the incident energies on the two sides are \( q_1(0) = \sigma T_{\text{in}}^4 \) and \( q_2(D) = \sigma T_{\text{out}}^4 \).

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Fig. 1 Layer geometry, coordinate system, and nomenclature of heat fluxes at boundaries
\[ n^2 a T^4(\kappa) = \frac{1}{2} [q_0(0) E_2(\kappa) + q_0(\kappa) E_1(\kappa \Delta \kappa)] \]
\[ + \frac{n^2}{2} \int_0^{\kappa \Delta \kappa} a T^4(\kappa') E_1(1 - \kappa' \Delta \kappa) d\kappa' \]  
(1)
The radiative flux, which is a constant through the layer for the present conditions of radiative equilibrium, can be found from the temperature distribution as (Siegel and Howell, 1981),
\[ q_r = q_0(0) - 2q_0(\kappa_0) E_1(\kappa_0) - 2n^2 \int_0^{\kappa_0} a T^4(\kappa') E_1(\kappa') d\kappa' \]  
(2)
The following dimensionless groups are now defined:
\[ \Phi(\kappa) = \frac{n^2 a T^4(\kappa) - q_0(0)}{q_0(0) - q_0(\kappa_0)} \]  
(3a)
\[ \Psi = \frac{q_r}{q_0(0) - q_0(\kappa_0)} \]  
(3b)
Equations (1) and (2) then become (note that from the temperature distribution as (Siegel and Howell, 1981), to obtain \( 4' \) and \( \Phi \) for all \( \kappa > 1 \), there are no reflections at the interfaces so that \( q_0(0) = 0 \)
\[ \Phi(X) = \frac{1}{2} E_2(\kappa_0 X) + \frac{1}{2} \kappa_0 \int_0^X \Phi(X') E_1(1 - X' \Delta X') dX' \]  
(4)
\[ \Psi = 1 - 2\kappa_0 \int_0^X \Phi(X') E_2(\kappa_0 X') dX' \]  
(5)
In Eqs. (4) and (5), \( \Phi \) and \( \Psi \) are not functions of \( n \). Hence to obtain \( \Phi \) and \( \Psi \) for all \( n \geq 1 \) it is necessary to solve Eq. (4) only once for each \( \kappa_0 \) and use each result to evaluate Eq. (5).
For the special case when the index of refraction of the layer is 1, there are no reflections at the interfaces so that \( q_0(0) = \sigma T^4_{s1} \), \( q_0(\kappa_0) = \sigma T^4_{s2} \), and the dimensionless groups become
\[ \Phi(\kappa) = \frac{T^4(\kappa) - T^4_{s2}}{T^4_{s1} - T^4_{s2}} \]  
(6a)
\[ \Psi = \frac{q_r}{\sigma T^4_{s1} - \sigma T^4_{s2}} \]  
(6b)
For \( n > 1 \), however, the \( \Phi \) and \( \Psi \) in Eq. (3) contain the outgoing boundary fluxes \( q_0(0) \) and \( q_0(\kappa_0) \) that are unknown, so the solution has not yet provided \( T(\kappa) \) and \( q_r \) for the cases when \( n > 1 \). In order to find these quantities the \( q_0(0) \) and \( q_0(\kappa_0) \) must be obtained in terms of known quantities. This is accomplished by looking at the boundary conditions in detail.
At the diffuse interfaces the internal fluxes are related to the transmission of external flux and the reflection of internal flux by
\[ q_0(0) = \sigma T^4_{s1} \rho^i + q_0(0) \rho^i \]  
(7a)
\[ q_0(\kappa_0) = \sigma T^4_{s2} \rho^i + q_0(\kappa_0) \rho^i \]  
(7b)
At the inside surfaces of the two boundaries there are the following relations between the radiative flux and the outgoing and incoming fluxes:
\[ q_r(0) = q_0(0) \]  
(8a)
\[ q_r(\kappa_0) = -q_0(\kappa_0) + q_0(\kappa_0) \]  
(8b)
Equations (7a) and (8a) are combined to eliminate \( q_0(0) \) and similarly for Eqs. (7b) and (8b) to eliminate \( q_0(\kappa_0) \). This yields
\[ q_0(0) = \frac{1}{1 - \rho} (\rho^i a T^4_{s1} - a \rho^i) \]  
(9a)
\[ q_0(\kappa_0) = \frac{1}{1 - \rho} (\rho^i a T^4_{s2} + a \rho^i) \]  
(9b)
These relations are substituted into Eq. (3b) to eliminate the \( q_0(\kappa_0) \). This yields the radiative flux for any index of refraction in terms of the value of \( \Psi \). Use is also made of the relation at an interface (Richmond, 1963) that \( \rho^i = (1 - \rho)n^2 \). The result for \( q_r \) for any \( n \) as
\[ q_r = \frac{n^2 \Psi}{\sigma (T^4_{s1} - T^4_{s2})} \]  
(10)
Following the same procedure, the temperature distribution is found by starting with Eq. (3a), using Eq. (9) to eliminate the \( q_0(\kappa_0) \) and then using Eq. (10) to eliminate \( q_r \). This yields the \( T(X) \) for any \( n \) as
\[ T^4(X) - T^4_{s2} = \frac{\Phi(X) + \rho^i}{1 - \rho^i} \]  
(11)
To use these relations values of \( \rho^i \) are needed for various refractive indexes. The externally incident radiation is diffuse. Although the interfaces are probably not optically smooth, it is assumed that each bit of roughness acts as a smooth facet so that the reflectivity can be obtained from the usually used interface relations for a nonabsorbing dielectric medium. Then by integrating the reflected energy over all incident directions the relation for \( \rho^i(n) \) is (Richmond, 1963),
\[ \rho^i(n) = \frac{1}{n^2} \left\{ \frac{1}{2} \left( \frac{3n+1}{6n+1} \right) \ln \left( \frac{n-1}{n+1} \right) \right. \]
\[ + \frac{2n^2(2n^2-1)}{(n^2+1)(n^2-1)} \ln(n) \} \]  
(12)
As discussed by Cox (1965), in the fairly transparent spectral regions of ceramic materials, the extinction coefficient in the complex index of refraction is usually not large enough to affect the surface reflectivity significantly, so Eq. (12) for nonattenuating dielectrics gives good results. The extinction

| Table 1 Dimensionless temperature distribution, \( \Phi \) |
|---|---|---|---|---|---|---|
| \( n \) | 0.1 | 0.3 | 1.0 | 3.0 | 10 | 30 | 100 |
| 0.0 | 0.5710 | 0.6419 | 0.7582 | 0.8693 | 0.9495 | 0.9819 | 0.9948 |
| 0.05 | 0.6149 | 0.6229 | 0.7230 | 0.8211 | 0.8966 | 0.9304 | 0.9456 |
| 0.1 | 0.6541 | 0.6072 | 0.5946 | 0.5819 | 0.5511 | 0.5828 | 0.5974 |
| 0.2 | 0.5357 | 0.5786 | 0.6249 | 0.7088 | 0.7627 | 0.7876 | 0.7994 |
| 0.3 | 0.3242 | 0.5517 | 0.5942 | 0.6384 | 0.6750 | 0.6920 | 0.6999 |
| 0.4 | 0.3130 | 0.5257 | 0.5468 | 0.5690 | 0.5874 | 0.5961 | 0.6002 |
| 0.5 | 0.3000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |
| 0.6 | 0.2870 | 0.4743 | 0.4532 | 0.4310 | 0.4125 | 0.4038 | 0.4002 |
| 0.7 | 0.4739 | 0.4483 | 0.4098 | 0.3616 | 0.3249 | 0.3079 | 0.3094 |
| 0.8 | 0.4693 | 0.4214 | 0.3571 | 0.2992 | 0.2372 | 0.2123 | 0.2007 |
| 0.9 | 0.4559 | 0.3928 | 0.3054 | 0.2181 | 0.1488 | 0.1171 | 0.1026 |
| 1.0 | 0.4290 | 0.3581 | 0.2419 | 0.1307 | 0.0505 | 0.0181 | 0.0052 |

| Table 2 Dimensionless heat flux, \( \Psi \) |
|---|---|
| \( \kappa_0 \) | \( \Psi \) |
| 0.1 | 0.9157 |
| 0.3 | 0.7936 |
| 1.0 | 0.5534 |
| 3.0 | 0.3017 |
| 10 | 0.1168 |
| 30 | 0.0419 |
| 100 | 0.0122 |

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Fig. 2 Effect of index of refraction on dimensionless temperature distributions for various optical thicknesses: (a) index of refraction, \( n = 1 \); (b) index of refraction, \( n = 2 \); (c) index of refraction, \( n = 4 \)

Fig. 3 Effect of index of refraction on radiative heat flux through the layer as a function of optical thickness

Results and Discussion

The \( \Phi \) and \( \Psi \) were evaluated from Eqs. (4) and (5). The results are in excellent agreement with the plotted values of Heslet and Warming (1965) and are given for convenience in Tables 1 and 2 for \( 0.1 \leq \kappa_0 \leq 100 \). Equations (10) and (11) were then used to obtain dimensionless radiative heat fluxes and temperatures for \( n > 1 \). The required \( \rho' \) were obtained from Eq. (12).

Figure 2 shows the dimensionless temperature distributions for \( n = 1, 2, \text{ and } 4 \). For small optical thickness the dimensionless distribution approaches 0.5, and as \( \kappa_0 \to \infty \) the profile becomes linear extending from 1.0 to 0. The effect of increasing \( n \) is to decrease the range of the temperature distributions, and for a fixed \( \kappa_0 \) they move closer to 0.5 as \( n \) is made larger. The dimensionless profiles are all rather linear. The fact that the profiles become more uniform with increasing \( n \) is the result of increasing internal reflections within the absorbing and scattering layer and decreasing transmission into the layer. Since each element in the layer is in radiative equilibrium all locally absorbed radiation must be reemitted. Since scattering is assumed isotropic, local scattering is added to the local emission. The large amount of internal reflection tends to equalize the energy throughout the layer and flatten the temperature distributions. The effect of \( \sigma \) is calculated very easily from the simple relation given by Eq. (11), where \( \rho' \) is a function of \( n \).

The effect of \( n \) and \( \kappa_0 \) on the dimensionless radiative heat flux through the layer is in Fig. 3. The heat flux decreases as

optical thickness, \( \kappa_0 = (\alpha + \kappa_0) \lambda D \).
the optical thickness increases, and the effect of $n$ is quite pronounced in altering the heat flux. The curve becomes very flat for $n = 4$. This is because interface reflections and increased internal emission (as a result of the $n^2$ factor) have become quite strong in regulating the heat transfer. The effect of optical thickness is thereby suppressed. At large optical thicknesses this increases the radiative flux as compared with the results for $n = 1$. As the layer becomes transparent $\kappa_{D} = 0$, $\Psi = 1$ and the dimensionless flux from Eq. (10) approaches $n^2(1 - \rho')/(1 + \rho')$.

References

The Prandtl Number Effect on Melting Dominated by Natural Convection

J. S. Lim1 and A. Bejan1,2

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$B$</td>
<td>constant, Eq. (13)</td>
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<tr>
<td>$C_p$</td>
<td>liquid specific heat</td>
</tr>
<tr>
<td>$Fo$</td>
<td>Fourier number = $aH/H^2$</td>
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<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$Gr$</td>
<td>Grashof number = $g\beta\Delta T H^3/\nu^2$</td>
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Objective

In a recent paper, Gobin and Benard (1990) considered the task of correlating the heat transfer data for melting in the presence of natural convection when the Pr value of the liquid phase is considerably smaller than 1. Earlier correlations were reported by Webb and Viskanta (1986), and Beckermann and Viskanta (1989). Additional low-Pr studies were conducted by Webb and Viskanta (1985), Gau and Viskanta (1986), Wolff and Viskanta (1987), and Beckermann (1989). To correlate the low-Pr data is an important and timely task, especially in view of the voluminous work that has been dedicated to situations in which Pr is greater than 1.

For the convection-dominated regime known also as quasi-stationary melting, Gobin and Benard (1990) correlated their low-Pr numerical data with the formula:

$$Nu = 0.29Ra^{0.27}Pr^{0.18}$$

They noted that this correlation does not agree with the $Nu - (RaPr)^{1/4}$ trend that might be expected from the single-phase natural convection scales for low Prandtl numbers (Bejan, 1984). They concluded that:

1. The relevance of the group $(RaPr)$ is not verified by their numerical results for convection-dominated melting, and
2. Further work is required to determine the scaling laws that govern the transition from the initial (conduction) regime to the final (convection) regime of the process of melting by side heating.

These two conclusions defined the work presented in this note. In it we report the correct scales of natural convection melting when the Prandtl number is small. We then construct a scaling-correct heat transfer correlation that spans the entire range of Prandtl numbers.

Scale Analysis

The scales of the natural convection melting process can be determined by extending Jany and Bejan’s (1988) high-Pr theory to the range of low Prandtl numbers represented by liquid metals. The theory is based on the geometric fact that during the transition from conduction to convection-dominated melting the melt region contains two distinct zones. As shown in Fig. 1, the upper zone of height $z$ is ruled by convection (namely, distinct boundary layers, $\delta$), while the lower zone of uniform thickness $s$ and height $(H - z)$ is governed by horizontal conduction. It is assumed that the flow is laminar, and that $Ste \ll 1$.

The conduction thickness is well known,

$$s \sim H(SteFo)^{1/2}$$