Conjunction Assessment Risk Analysis

Late-Notice HIE Investigation

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13 APR 2016
Purpose

• Provide a response to MOWG action item 1410-01:
  – Analyze close approaches which have required mission team action on short notice. Determine why the approaches were identified later in the process than most other events.

• Method:
  – Performed an analysis to determine whether there is any correlation between late notice event identification and space weather, sparse tracking, or high drag objects, which would allow preventive action to be taken
  – Examined specific late notice events identified by missions as problematic to try to identify root cause and attempt to relate them to the correlation analysis
Agenda

• Frequency/correlation of large state changes
  – Individual states of primary and secondary objects
    • Overall frequency of $\varepsilon/\sigma$ and comparison to theoretical frequencies
    • $P_c$ vs primary and secondary $\varepsilon/\sigma$ size
    • Secondary $\varepsilon/\sigma$ vs tracking level and vs energy dissipation rate (EDR)
  – Combined states: component miss distances vs combined covariance
    • Overall frequency of $\varepsilon/\sigma$ and comparison to theoretical frequencies
    • $P_c$ vs $\varepsilon/\sigma$ size; $\varepsilon/\sigma$ vs tracking levels, EDR
    • $\varepsilon/\sigma$ vs $F_{10}$, average $Ap$, and peak $Ap$
  – Preliminary conclusions

• Case studies of four late-notice events

• Overall conclusions and way forward
Dataset to Analyze

• All reported conjunctions for ca. 700 km defended missions
  – Landsat-7, Terra, EO-1, Aqua, Aura, CloudSat, CALIPSO, GCOM-W1, Landsat-8, OCO-2, SMAP

• Data interval from 1 MAY 2015 to 1 FEB 2016
  – New Dynamic Consider Parameter (DCP) functionality installed on ASW on May 1, 2016, so covariance realism purported to be much improved

• All CDM updates examined within every event (not just initial identification)
Broad Investigation of Large State Changes

• Determine actual frequency of large state changes, in both individual and combined states
  – Compare to theoretically expected frequencies

• Determine whether broadly correlated with potential/expected causes
  – Low tracking
  – Harder-to-maintain orbits (larger energy dissipation rate)
  – General levels of solar activity (EUV and Joule atmospheric heating)
• Main parameter to represent size of state change is component position difference divided by associated standard deviation ($\varepsilon/\sigma$)
  – Presumption of OD is that errors are normally distributed and unbiased
  – $\varepsilon$ is difference in component position between subsequent state estimates
  – $\sigma$ is square root of associated variance from first state’s covariance
  – Dividing $\varepsilon$ by $\sigma$ creates standardized normal variable ($\mu=0$ because unbiased)
  – Set of these should thus conform to standard normal distribution

• Same method currently used in CARA daily and HIE reports
Large State Changes: Parameterization (2 of 3)

- However . . . This is only true for the “diagonalized” situation, in which covariance axes and coordinate frame axes align
  - Results meaningful only if ellipse closely aligns with coordinate axes
  - Once ellipse rotated, then component errors are correlated
    - Individual component error distributions no longer independent random variables
- How often are covariance error ellipsoids naturally diagonalized?
  - Not terrible assumption for individual satellites (primary, secondary)
  - More tenuous for combined situation (miss distance vs combined covariance)
- Bottom line: $\varepsilon/\sigma$ statistics at the component level must be used with care
  - When plotted against only positive axis, presume $\varepsilon/\sigma$ to be $\text{abs}(\varepsilon/\sigma)$
• **Comparison alternative: Mahalanobis distance**
  – If individual component errors normally distributed, then sum of squares of individual ratios ($\epsilon^2/\sigma^2$) will constitute a 3-DoF $\chi^2$ distribution
  – Formulary $\epsilon C^{-1}\epsilon^T$ properly considers all correlations and makes the calculation independent of coordinate system
  – Approach less frequently encountered, so less intuition built up around result
  – But will be supplied and examined along with Gaussian variables
  – Can also examine 2-DoF situation for only radial and in-track
    • More information on this later
Issues in Comparison to Theory

• Commonly-known “percentages” for univariate Gaussian distribution consider two-tailed results
  – 95.4% for 2-σ distribution considers results from 2.3% to 97.7%
  – 99.7% for 3-σ distribution considers results from 0.15% to 99.85%

• Potential double-counting of large state changes
  – Subsequent updates analyzed for large state change behavior
  – In a chain of updates, return to normalcy will appear as a second large change
  – Demarcation between one and two events not so easy to define
    (S = small state change; L = large state change)
    • S S L L S S – one or two events?
    • S S S L S L S S – one or two events?
    • S S S S S S L – one or two events (would it have been counted as two if one more update had been available?)
  – For data-mining simplicity, all large changes counted, with the caveat that reported number might be twice as large as “actual” number
Primary and Secondary Treated Individually

STATE-CHANGE FREQUENCY AND COMPARISON TO THEORY
Frequency of Large State Changes: Secondary Objects

**Radial**

![Bar chart showing the percentage of events for different values of $\epsilon/\sigma$. The x-axis represents $\epsilon/\sigma$ with bins for $<1$, $1-2$, $2-3$, $3-5$, and $>5$. The y-axis represents the percentage of events on a logarithmic scale.](chart_radial.png)

**In-Track**

![Bar chart showing the percentage of events for different values of $\epsilon/\sigma$. The x-axis represents $\epsilon/\sigma$ with bins for $<1$, $1-2$, $2-3$, $3-5$, and $>5$. The y-axis represents the percentage of events on a logarithmic scale.](chart_in_track.png)

**Cross-Track**

![Bar chart showing the percentage of events for different values of $\epsilon/\sigma$. The x-axis represents $\epsilon/\sigma$ with bins for $<1$, $1-2$, $2-3$, $3-5$, and $>5$. The y-axis represents the percentage of events on a logarithmic scale.](chart_cross_track.png)

- **Legend**:
  - All Updates
  - Subset with Event PcMax > 1E-05
  - Subset of PcMax with > 1 OoM $\Delta P_c$
  - Subset of PcMax with > 2 OoM $\Delta P_c
Secondary Large State Changes: 3-σ Comparison to Theory

• Radial frequency a little more than twice theoretical prediction
  – 1.3% of updates > 3-σ change
  – If all events were double-counted, then would expect 0.6%--have a little more than double this

• In-track frequency greater than theory
  – ~3.3%; greater than theory by perhaps factor of five
  – Component most heavily affected by expected modeling errors

• Cross-track component performs surprisingly badly
  – ~7.0% of updates; substantially greater than theory
  – Can be largely attributed to very small sigma values
    • Will be addressed in more depth later

• How often do these matter?
  – If occur nearly always for low-Pc events, then perhaps not much
Secondary Large State Changes: Frequency in Situations that Matter

• Change > $3\sigma$ and max Pc for event $>$ 1E-05
  – Radial = 0.3%, in-track = 0.4%, cross-track = 0.9%
• Change > $3\sigma$, max Pc for event $>$ 1E-05, and Pc change after “large” change $>$ 1 order of magnitude
  – Radial = 0.07%, in-track = 0.13%, cross-track = 0.15%
• Change > $3\sigma$, max Pc for event $>$ 1E-05, and Pc change after “large” change $>$ 2 orders of magnitude
  – Radial = 0.05%, in-track = 0.12%, cross-track = 0.09%
• Percentages perhaps as much as double “real” values
• Sampling is not large enough to allow robust comparison to theory at the $3\sigma$ level

Overall, prevalence is greater than theory would predict. However, presence in events of significance notably reduced.
Frequency of Large State Changes: Primary Objects

**Radial**

- Percentage of Events
- $\epsilon/\sigma$
- Categories: $<1$, 1-2, 2-3, 3-5, >5

**In-Track**

- Percentage of Events
- $\epsilon/\sigma$
- Categories: $<1$, 1-2, 2-3, 3-5, >5

**Cross-Track**

- Percentage of Events
- $\epsilon/\sigma$
- Categories: $<1$, 1-2, 2-3, 3-5, >5

Legend:
- All Updates
- Subset with Event PcMax $> 1E-05$
- Subset of PcMax with $> 1$ OoM $\Delta$Pc
- Subset of PcMax with $> 2$ OoM $\Delta$Pc
• Data summary
  – Table below reports situation for which abs(ε/σ) > 3
  – Performance tabulated for all events, events with Pc > 1E-05, and additionally with both one and two orders-of-magnitude (OoM) change in Pc value after large update

• Percentages for primaries surprisingly large
  – Very similar for radial component; much larger differences with other two
    • Perhaps a little comfort in this, as radial generally most important component for CA

<table>
<thead>
<tr>
<th></th>
<th>Overall</th>
<th>Event Pc &gt; 1E-05</th>
<th>&gt; 1E-05 &amp; ΔPc &gt; 1 OoM</th>
<th>&gt; 1E-05 &amp; ΔPc &gt; 2 OoM</th>
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<td>Primary</td>
<td>Secondary</td>
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<td>1.57</td>
<td>1.33</td>
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<td>Cross-Track</td>
<td>13.53</td>
<td>7.10</td>
<td>2.64</td>
<td>0.88</td>
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</table>
Comparison of $\varepsilon/\sigma$ to Theory: Primary and Secondary Objects

![Graphs showing comparison of $\varepsilon/\sigma$ to theory for different objects.](image-url)
Comparison of $\varepsilon/\sigma$ to Theory: Interpretation

• Radial behaves reasonably well—better than theory until more extreme part of tails reached
  – Cannot see tail behavior very well in provided plots
• In-track has non-theoretical distribution beyond about $\varepsilon/\sigma > 1$
  – As remarked previously, worse for secondaries than for primaries
• Cross-track highly leptokurtic—peaked with very long tails
  – Does not match a Gaussian distribution at all
• In using chi-squared distribution, 2-DoF framework gives more sanguine situation
  – Eliminates effect of large cross-track differences
  – Nonetheless, non-theory outliers dominate performance in the tails
• None of these results sets match the theory particularly well
• Immediate conclusion difficult
  – OD residuals suspected to be leptokurtic
  – Present trend could be extension of this
Error and Sigma CDFs by Component: Primary (left) and Secondary (right)

- Theory-vs-praxis spread largest for cross-track
- Cross-over for cross-track and in-track for primary results
  - Sigmas coming out of OD probably not large enough; probably could use some kind of adjustment

Cross-track performance due to unreasonably small sigmas
Primary and Secondary Treated Individually
CORRELATION INVESTIGATIONS
Correlation Indices

• Correlation tests indicate the degree to which two variables are correlated
  – Range of values is -1 to 1
  – 1 indicates perfect correlation;
  – -1 indicates perfect inverse correlation;
  – 0 indicates no correlation

• Three correlation indices in general use
  – Pearson correlation coefficient: tests for linear correlation
    • What most people learn in statistics class
    • Evaluates how well two datasets exhibit a linear relationship
  – Kendall’s tau and Spearman’s rho: test for rank correlation
    • Much less frequently taught but more powerful
    • Evaluate how well two datasets move in the same (or opposite) directions

• All three used in present analysis to examine data correlations
Pearson Correlation Coefficient

- Evaluates the degree of a linear relationship between two variables
- Usually evaluated by the formula (s is sample standard deviation), with range of interesting and often not helpful outcomes

\[ r = \frac{1}{n-1} \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{s_x} \right) \left( \frac{y_i - \bar{y}}{s_y} \right) \]

- Some interpretive guidance via relationship to \( r^2 \) value from linear regression: square of Pearson = regression \( r^2 \)
  - Pearson value of 0.5 would equate to \( r^2 \) of 0.25—not very impressive
- Really would like something that reveals even non-linear correlation
Kendall’s Tau

• Rank correlation test
  – With two vectors of data X and Y, compares \((X_i,Y_i)\) to every other \((X_j,Y_j)\)
  – Pair is concordant if, when \(X_i > X_j\), \(Y_i > Y_j\); discordant if the opposite
  – Parameter is \((\# \text{ concordant pairs} - \# \text{ discordant pairs}) / \text{(total pairs)}\)
    • So same range of values (-1 to 1) with same meaning

• Much more robust test
  – Will find both linear and nonlinear correlation
  – Computationally expensive \([\sim O(n^2)]\), but computers are doing the work

• Tied situations create problems
  – In present analysis, arises when comparing continuous to discrete distribution
    • \(\varepsilon/\sigma\) to tracking levels (because tracking levels are counting numbers, so can have multiple \(\varepsilon/\sigma\) values aligned with same tracking level)
  – Even more computationally expensive modifications to adjust for ties
  – Spot-checked these and saw no difference in computed result
Spearman’s Rho

• Test of monotonicity, computed by summing squares of differences in rank
  – Mapped into same -1 to 1 range of values, with same interpretation
• Computational formula

\[ \rho = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n(n^2 - 1)} \]

• Computationally easier but more vulnerable to outlier data
• Usually larger than Kendall’s tau
• Included here for consistency/contrast

Main factor to consult is Kendall’s Tau
Positive Change in Pc vs Secondary $\varepsilon/\sigma$

**Pc vs Radial $\varepsilon/\sigma$**
- Pearson $= 0.43469$
- Kendall $= 0.50466$
- Spearman $= 0.69706$

**Pc vs In-Track $\varepsilon/\sigma$**
- Pearson $= 0.52282$
- Kendall $= 0.53653$
- Spearman $= 0.7269$

**Pc vs Cross-Track $\varepsilon/\sigma$**
- Pearson $= 0.1677$
- Kendall $= 0.37367$
- Spearman $= 0.54043$

**Pc vs 2 DoF $\chi^2$**
- Pearson $= 0.51092$
- Kendall $= 0.59777$
- Spearman $= 0.78916$
Positive Change in Pc vs Primary $\epsilon/\sigma$

**Pc vs Radial $\epsilon/\sigma$**
- Pearson = 0.20681
- Kendall = 0.15573
- Spearman = 0.22593

**Pc vs In-Track $\epsilon/\sigma$**
- Pearson = 0.22612
- Kendall = 0.34377
- Spearman = 0.48671

**Pc vs Cross-Track $\epsilon/\sigma$**
- Pearson = 0.16239
- Kendall = 0.14005
- Spearman = 0.20664

**Pc vs 2 DoF $\chi^2$**
- Pearson = 0.22865
- Kendall = 0.34363
- Spearman = 0.4837
Positive Pc change vs $\epsilon/\sigma$:

Summary

- Summary table (Kendall’s Tau):

<table>
<thead>
<tr>
<th></th>
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<th>In-Track</th>
<th>Cx-Track</th>
<th>Chi-Sq</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary</td>
<td>0.15</td>
<td>0.34</td>
<td>0.14</td>
<td>0.34</td>
</tr>
<tr>
<td>Secondary</td>
<td>0.50</td>
<td>0.54</td>
<td>0.37</td>
<td>0.60</td>
</tr>
</tbody>
</table>

- Pc correlation with large state changes in primary not very strong
- Pc correlation with large state changes in secondary of some significance, but still not overwhelming

Large state changes in the secondary do correlate to large changes in Pc, but not all that strongly
ε/σ vs # of Tracks in Secondary OD: All Data
$\varepsilon/\sigma$ vs # of Tracks in Secondary OD: Radial or In-Track $\text{abs}(\varepsilon/\sigma) > 3$

**Radial $\varepsilon/\sigma$ vs Tracks in OD**

- Pearson = 0.10557
- Kendall = 0.13202
- Spearman = 0.19891

**In-Track $\varepsilon/\sigma$ vs Tracks in OD**

- Pearson = 0.10111
- Kendall = 0.055855
- Spearman = 0.081305

**Cross-Track $\varepsilon/\sigma$ vs Tracks in OD**

- Pearson = 0.081292
- Kendall = 0.038176
- Spearman = 0.054476

**2 DoF $\chi^2$ vs Tracks in OD**

- Pearson = 0.035694
- Kendall = 0.0148076
- Spearman = 0.023365
ε/σ vs # of Tracks in Secondary OD: Radial or In-Track $\text{abs}(\varepsilon/\sigma) > 5$

**Radial $\varepsilon/\sigma$ vs Tracks in OD**
- Pearson=0.15745
- Kendall=-0.1966
- Spearman=0.28947

**In-Track $\varepsilon/\sigma$ vs Tracks in OD**
- Pearson=0.25113
- Kendall=0.17411
- Spearman=0.2548

**Cross-Track $\varepsilon/\sigma$ vs Tracks in OD**
- Pearson=0.063594
- Kendall=0.029761
- Spearman=0.039885

**2 DoF $\chi^2$ vs Tracks in OD**
- Pearson=-0.029983
- Kendall=0.0099744
- Spearman=0.016747
• Correlations low, even for high-change events
• More tellingly, should produce inverse correlation but instead see positive correlation
• Difficult to conclude that unexpectedly large errors are associated with light tracking

Sparse tracking for secondary does not correlate with large state errors
Energy Dissipation Rate

• In 2001 study, Omitron investigated single-value representation of effect of atmospheric drag

• Proposed concept of the energy dissipation rate (EDR)
  – Dot product of inertial drag acceleration vector and inertial velocity vector

\[
EDR(t) = -\vec{A}_D \cdot \vec{V}
\]

  – Units of watts per kilogram
  – Represents amount of energy being removed from the orbit
  – Occasionally amount of energy being added to orbit, such as by solar wind

• EDR can serve as single-value encapsulation of ballistic coefficient and atmospheric density
ε/σ vs EDR of Secondary: All Data
ε/σ vs EDR of Secondary:
Radial or In-Track \( \text{abs}(\frac{\varepsilon}{\sigma}) > 3 \)

**Radial \( \frac{\varepsilon}{\sigma} \) vs EDR**
- Pearson: 0.24765
- Kendall: 0.18999
- Spearman: 0.28372

**In-Track \( \frac{\varepsilon}{\sigma} \) vs EDR**
- Pearson: 0.0084687
- Kendall: 0.020528
- Spearman: 0.030568

**Cross-Track \( \frac{\varepsilon}{\sigma} \) vs EDR**
- Pearson: 0.052167
- Kendall: 0.027585
- Spearman: 0.042118

**2 DoF \( \chi^2 \) vs EDR**
- Pearson: 0.22928
- Kendall: 0.097304
- Spearman: 0.14481

M.D. Hejduk | Late-Notice HIEs | 13 APR 2016 | 33
$\varepsilon/\sigma$ vs EDR of Secondary:
Radial or In-Track $\text{abs}(\varepsilon/\sigma) > 5$

**Radial $\varepsilon/\sigma$ vs EDR**
- Pearson: 0.31425
- Kendall: 0.23025
- Spearman: 0.34881

**In-Track $\varepsilon/\sigma$ vs EDR**
- Pearson: -0.072463
- Kendall: -0.071696
- Spearman: -0.10309

**Cross-Track $\varepsilon/\sigma$ vs EDR**
- Pearson: 0.0066154
- Kendall: 0.073035
- Spearman: 0.1086

**2 DoF $\chi^2$ vs EDR**
- Pearson: 0.24163
- Kendall: 0.083323
- Spearman: 0.12658
• Correlation rather weak
• Somewhat stronger correlation when dataset limited to the higher $\varepsilon/\sigma$ values, but even then not significant
Combined Situation

STATE-CHANGE FREQUENCY AND COMPARISON TO THEORY
Frequency of Large State Changes: Miss vs Combined Sigma

Radial

In-Track

Cross-Track

Percentage of Events

$\epsilon / \sigma$

<1 1-2 2-3 3-5 >5

Percentage of Events

$\epsilon / \sigma$

<1 1-2 2-3 3-5 >5

Percentage of Events

$\epsilon / \sigma$

<1 1-2 2-3 3-5 >5

Legend:

- All Updates
- Subset with Event PcMax > 1E-05
- Subset of PcMax with > 1 OoM $\Delta$Pc
- Subset of PcMax with > 2 OoM $\Delta$Pc
### Frequency of Large State Changes: Tabular Summary

<table>
<thead>
<tr>
<th>Overall</th>
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<th>$&gt; 1E-05 &amp; \Delta P_c &gt; 1 \text{ OoM}$</th>
<th>$&gt; 1E-05 &amp; \Delta P_c &gt; 2 \text{ OoM}$</th>
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<td>13.53</td>
<td>7.10</td>
<td>0.90</td>
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</table>

- Values much closer to theoretical expectation, especially for radial and cross-track
  - In-track is expected to be the most vulnerable to modeling errors, so not surprising that non-compliance largest in this component
Comparison of $\varepsilon/\sigma$ to Theory: Miss Component vs Combined Sigma
Better behaved than individual satellite results, especially for cross-track component.
Combined Situation

CORRELATION INVESTIGATIONS
Positive Change in Pc vs $\varepsilon/\sigma$

Pc vs Radial $\varepsilon/\sigma$

- Pearson: 0.48745
- Kendall: 0.51059
- Spearman: 0.68859

Pc vs In-Track $\varepsilon/\sigma$

- Pearson: 0.22736
- Kendall: 0.36328
- Spearman: 0.52244

Pc vs Cross-Track $\varepsilon/\sigma$

- Pearson: 0.21204
- Kendall: 0.55408
- Spearman: 0.73889

Pc vs 2 DoF $\chi^2$

- Pearson: 0.39457
- Kendall: 0.47018
- Spearman: 0.64603
Positive Change in Pc vs $\epsilon/\sigma$: Tabular Summary

<table>
<thead>
<tr>
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<th>Radial</th>
<th>In-Track</th>
<th>Cx-Track</th>
<th>Chi-Sq</th>
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<tr>
<td>Combined</td>
<td>0.51</td>
<td>0.36</td>
<td>0.55</td>
<td>0.47</td>
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</table>

- Notable levels of correlation endure here, although less impressive than for secondaries alone
- Large state changes are with some frequency associated with positive change in Pc
  - The larger the state change, the larger the Pc change

Large unexpected changes in miss distance are somewhat often associated with larger changes in resultant Pc value
Correlations with OD Tracking and EDR: Tabular Summary

• Correlations are in most cases not significant and in an inverse relationship to what is expected
  – Expect inverse correlation with tracking level and get regular correlation
  – Expect regular correlation with EDR and get inverse correlation

• Difficult to see meaningful correlations with these indices

<table>
<thead>
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<th>In-Track</th>
<th>Cx-Track</th>
<th>Chi-Sq</th>
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<tr>
<td>Tracks in OD: Kendall</td>
<td></td>
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<tr>
<td>All Data</td>
<td>0.17</td>
<td>0.11</td>
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<tr>
<td>ε/σ &gt; 3</td>
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<td>0.01</td>
<td>0.29</td>
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<td>ε/σ &gt; 5</td>
<td>-0.19</td>
<td>-0.02</td>
<td>-0.39</td>
<td>-0.06</td>
</tr>
</tbody>
</table>

No meaningful correlation between secondary EDR value and large unexpected changes in relative miss
Correlations with Solar Activity

- Elevated levels of solar activity can produce an unstable atmosphere whose density is difficult to model
  - More strongly true with geomagnetic storms (Dst, Ap)
  - Can also be observed with EUV (F10, M10, S10, Y10, &c.)

- Different possibilities for essence of the problem
  - Higher solar activity *simpliciter*
  - Mismatch between predicted and realized solar activity

- Will investigate the former with correlation studies
  - Median F10 and Ap over prediction interval
  - Peak Ap over prediction interval

- Will investigate the latter with case studies
Combined $\epsilon/\sigma$ vs Median $F_{10}$: All Data

- **Radial $\epsilon/\sigma$ vs $F_{10}$**
  - Pearson=-0.0028185
  - Kendall=0.0081204
  - Spearman=0.012195

- **In-Track $\epsilon/\sigma$ vs $F_{10}$**
  - Pearson=-0.011297
  - Kendall=-0.0096707
  - Spearman=-0.01425

- **Cross-Track $\epsilon/\sigma$ vs $F_{10}$**
  - Pearson=-0.012868
  - Kendall=-0.0055951
  - Spearman=-0.0081742

- **3 DoF $\chi^2$ vs $F_{10}$**
  - Pearson=-0.0034562
  - Kendall=0.016049
  - Spearman=0.023996
Combined $\varepsilon/\sigma$ vs Median $F_{10}$: Any Component $\text{abs}(\varepsilon/\sigma) > 3$
Combined $\varepsilon/\sigma$ vs Median $F_{10}$: Any Component $\text{abs}(\varepsilon/\sigma) > 5$
Combined $\epsilon/\sigma$ vs Solar Indices: Tabular Summary

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- Correlations are essentially nonexistent in all areas

Simple elevated levels of solar activity do not correlate with large changes in relative miss
CASE STUDIES
Late-Notice HIE Case Studies

• Examined four late-notice events that fell within data investigation period of current study
  – 1 MAY 2015 to 1 FEB 2016

• Events examined
  – Terra vs 38192, TCA 24 JUN 201
  – Aura vs 89477; TCA 29 AUG 2015
  – Terra vs 37131; TCA 19 DEC 2015
  – GPM vs 28685; TCA 5 SEP 2015

• Will look at
  – $\varepsilon/\sigma$ vs time (same as $\Delta$ position to uncertainty plots from daily/HIE report, like at right)
  – $P_c$ vs time (same as from daily/HIE report)
  – Dst and Ap; prediction vs actual
    • Segmented by what is available in support of each update
JSpOC Space Weather Information Files: Delivery Information

• Calculated externally at two different sites, through an arrangement with SET
• Forwarded three times a day to JSpOC
• Nominally at 0600, 1500, and 2100 Zulu
• Delivery goals met most of the time, but not always
  – Occasionally files late or missing, as shown in graph at right
  – 4.7% of file spacings deviate from nominal cadence of 9 hrs, 9 hrs, and 6 hrs
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JSpOC Space Weather Information Files: Data Currency

• Three types of data in file
  – “Issued” – definitive values for the solar/geomagnetic index, subjected to full availability of feeder data and consistency tests
  – “Nowcast” – initial observations of values, hand-scaled and not subject to consistency tests
    • Measurements stay in “nowcast” status for typically 24 hours
  – “Predicted” – values are predicted
    • EUV predicted values from 54- and sometimes 108-day autoregression analyses of past data
    • Geomagnetic indices are predicted from observed solar activity earlier in the solar rotation (and thus expected to become georelevant at a given future time)

• Data type timing
  – Issued/Nowcast data used in propagating states from epoch to current time
    • Scaled/debiased with HASDM results
  – Predicted data used in propagating states from current time to TCA
  – Accuracy of predicted data can influence propagated result substantially
Space Weather Evolution Charts

- Upper left shows Dst; lower left shows Ap
- Black line is “issued” (definitive) data
- Colored lines are predicted data
  - Each line begins when a given OD update executed
  - Each line shows predicted values of the geomagnetic index of choice
    - When Dst lines move to small positive value, prediction stops (zeroes in file)
    - When Ap lines move to small negative value, prediction stops (ones in file)
- Dst threshold for solar storm compensation engagement also shown
- Upper right shows $\varepsilon/\sigma$ for each component
  - Miss distance vs combined covariance
- Lower right shows $P_c$ vs time
Case Study #1:
Terra vs 38192, TCA 24 JUN 2015
• About half a day before spike in Ap/Dst begins
  – Some predicted increased Dst activity, but not of severity actually realized
  – Predictions at very end of storm over-predict Dst
  – Final prediction and shrinking covariance produces Pc dropoff

• SWTS indicates conjunction vulnerable to large Pc changes due to density mismodeling

• Bottom line: missed solar storm and subsequent prediction failures produced late changes
Space Weather Trade-Space Result: Aura vs 89477; 56 Hours to TCA

- Run from update right as spike in Ap/Dst is beginning
  - No predicted spike in relevant ASW space weather file
- Indicates that conjunction vulnerable to large Pc changes due to atmospheric mismodeling

- Bottom line: space weather predictions missed significant solar storm
  - Most likely cause of late-breaking change in Pc
Case Study #3: Terra vs 37131; TCA 19 DEC 2015

Predicted and Definitive Dst History

Predicted and Definitive Ap History

$\epsilon / \sigma$ vs Time to TCA

$P_c$ vs Time to TCA
Space Weather Trade-Space Result: Terra vs 37131; 28 Hours to TCA

- Run from update before 2 OoM change in Pc observed
  - Strange actual behavior in Dst
  - Modest unmodeled increase in Ap
- SWTS indicates that conjunction vulnerable to Pc changes due to atmospheric mismodeling

- Bottom line: odd space weather behavior, and deviation from predication, probably responsible for modest increase in Pc
Case Study #4: GPM vs 28685; TCA 5 SEP 2015

Predicted and Definitive Dst History

Predicted and Definitive Ap History

$\epsilon/\sigma$ vs Time to TCA

Pc vs Time to TCA

Days to TCA

M.D. Hejduk | Late-Notice HIEs | 13 APR 2016 | 62
Case Study #4: Situation

- No particular space weather anomalies here
- Rather, simply a late-notice event
- When did this event first appear in the screenings?
  - Because of ongoing screening-volume study, results from “large screening” experiment available during this period
    - 50 x 250 x 250
  - Can see how object fared against operational screening volume
Case Study #4: Screening Results (TCA on 09/05)
Case Study #4: Screening Results Discussion

• Late-notice situation here mostly bad luck
  – R and C components within screening volume ~6 days before TCA
  – In-track component above threshold for several days, barely above threshold two days before TCA, and violates it about 17 hours before TCA
  – CDM appears to have been generated as part of run subsequent to screening

• Increased speed of current processes will improve this situation somewhat in future
  – 3 screenings/day will find such edge cases faster
  – Faster processing will get CDM out faster after conjunction discovery
SUMMARY AND WAY FORWARD
Late-Breaking HIEs: Overall Summary

• Occur more often than theory would indicate
• Do not correlate at global level with any obvious causal condition
  – Light tracking, hard-to-maintain orbits, or generally elevated solar activity
• Case studies indicate two culprits
  – Failure of JSpOC space weather predicted indices to predict solar storms
  – Edge cases for general screenings
• Is there any good news?
  – No, not really
• CARA member of NASA LWS space weather expert panel
  – Dr. Matt Hejduk as CA expert panel representative
  – Dr. Yihua Zheng as GSFC space physics representative, also representing mission interests

• Purpose of panel to recommend NASA research investments to improve prediction and modeling
  – Will issue formal report of recommendations by December, as well as accompanying journal article
  – Will attempt to focus at least part of recommendation to address JSpOC situation

• Hope to leverage report to push state of the art at JSpOC
  – However, from their perspective, a large investment was just made in atmospheric density prediction modeling; need to focus on other items
Solar Storm Predictions: What are we Doing? (2 of 2)

- Will investigate whether file update frequency can be accelerated
  - Brief JSpOC on these results to show the problems that latencies create
    - See if there are mechanisms to improve efficiencies
  - Use SWTS function to determine whether such intervention is needed
    - Events that are not vulnerable to atmospheric density mismodeling would not require out-of-cycle updates
    - Would not have helped cases investigated here, as entire solar storms were missed
  
- However, probably a fairly long time before there is much improvement with such scenarios
Screening Volume Sizing: What are we Doing?

- First, must recognize that edge cases will exist with any volume size
- Study of proper screening volume sizing presently in work
  - Preliminary results set briefed to JSpOC in March
  - Set of refinements presently being performed; expect to issue a definitive update in next couple of months
- Certain philosophical issues need to be adjudicated
  - e.g., increasing screening volume size to produce a large number of additional events, all with a Pc of 0, does nothing to provide better situational awareness
- In the end, two changes will probably take place
  - Screening volume sizes will increase
  - Some sort of additional winnowing test on captured candidates will be performed in order to identify events worthy of increased attention/tasking
- Stay tuned!
BACKUP SLIDES
Normal Deviates and Chi-squared Variables

• Let \( q \) and \( r \) be vectors of values that conform to a Gaussian distribution
  – These collection of values are called normal deviates

• A normal deviate set can be transformed to a standard normal deviate by subtracting the mean and dividing by the standard deviation
  – This produces the so-called Z-variables

\[
Z_q = \frac{q - \mu_q}{\sigma_q}, \quad Z_r = \frac{q - \mu_r}{\sigma_r}
\]

• The sum of the squares of a series of standard normal deviates produces a chi-squared distribution, with the number of degrees of freedom equal to the number of series combined

\[
Z_q^2 + Z_r^2 = \chi^2_{2dof}
\]
Normal Deviates in State Estimation

• In a state estimate, the errors in each component (u, v, and w here) are expected to follow a Gaussian distribution
  – If all systematic errors have been solved for, only random error should remain

• These errors can be standardized to the Z-formulation
  – Mean presumed to be zero (OD should produce unbiased results), so no need for explicit subtraction of mean

\[ Z_u = \frac{u}{\sigma_u}, \quad Z_v = \frac{v}{\sigma_v}, \quad Z_w = \frac{w}{\sigma_w} \]

• Sum of squares of these standardized errors should follow a chi-squared distribution with three degrees of freedom

\[ Z_u^2 + Z_v^2 + Z_w^2 = \chi^2_{3, dof} \]
State Estimation Example Calculation

• Let us presume we have a precision ephemeris, state estimate, and covariance about the state estimate
  – For the present, further presume covariance aligns perfectly withuvw frame (no off-diagonal terms)

• Error vector $\varepsilon$ is position difference between state estimate and precision ephemeris, and covariance consists only of variances along the diagonal
  – Inverse of covariance matrix is straightforward

\[
\varepsilon = \begin{bmatrix} \varepsilon_u \\ \varepsilon_v \\ \varepsilon_w \end{bmatrix}, \quad C = \begin{bmatrix} \sigma_u^2 & 0 & 0 \\ 0 & \sigma_v^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1/\sigma_u^2 & 0 & 0 \\ 0 & 1/\sigma_v^2 & 0 \\ 0 & 0 & 1/\sigma_w^2 \end{bmatrix}
\]

• Resultant simple formula for chi-squared variables

\[
\varepsilon C^{-1} \varepsilon^T = \frac{\varepsilon_u^2}{\sigma_u^2} + \frac{\varepsilon_v^2}{\sigma_v^2} + \frac{\varepsilon_w^2}{\sigma_w^2} = \chi^2_{3 \text{dof}}
\]

• Extension to case with off-diagonal terms straightforward