Conjunction Assessment Risk Analysis

Evaluating Probability of Collision (Pc) Uncertainty

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Agenda

• Conjunction Assessment Basics
• Probability of Collision (Pc) calculation outline
• Pc uncertainty overview
• Pc uncertainty component: covariance uncertainty
  – Covariance realism assessment
  – Covariance realism PDF generation
• Pc uncertainty component: hard-body radius uncertainty
  – Primary objects using projected-area sampling
  – Secondary objects using radar cross-section values
• Pc uncertainty component: natural variation in Pc calculation
• Example output
• Conclusions and future work
How are Satellite Collision Risks Determined/Mitigated?

• Certain spacecraft are determined to be “defended assets”
  – Will be evaluated for collision risk with other objects
• For seven days into the future, the expected positions of the defended asset and the rest of the objects in the space catalogue are determined
• “Keep-out volume” box drawn around the defended asset at each time-step
  – Typically 5km x 5km x 25km in size, with the longer dimension along the orbit path
• Any satellite that penetrates the keep-out volume during the 7-day analysis is considered a possible “conjunctor”
• Particulars of the close approach analyzed to determine actual conjunction risk
“Fly By” Ephemeris Comparison

- Ephemerides generated for primary and secondaries that are possible threats
- Screening volume box (or ellipsoid) constructed about primary
- Box “flown” along the primary’s ephemeris
- Any penetrations of box constitute possible conjunctions
- For these conjunctions, Conjunction Data Message (CDM) generated
  - State estimates and covariances at TCA
  - Relative encounter information
  - OD information
- CDM data used to calculate probability of collision (Pc)
Calculating Probability of Collision (Pc): 3D Situation at Time of Closest Approach (TCA)

Figure taken from Chan (2008)
Calculating Pc: 2-D Approximation (1 of 3)  
Combining Error Volumes

• Assumptions
  – Error volumes (position random variables about the mean) are uncorrelated

• Result
  – All of the relative position error can be centered at one of the two satellite positions
    • Secondary satellite is typically used
  – Relative position error can be expressed as the additive combination of the two satellite position covariances (proof given in Chan 2008)
    • $C_a + C_b = C_c$
  – Must be transformed into a common coordinate system, combined, and then transformed back
Calculating Pc: 2-D Approximation (2 of 3)
Projection to Conjunction Plane

- Combined covariance centered at position of secondary at TCA
- Primary path shown as “soda straw”
- If conjunction duration is very short
  - Motion can be considered to be rectilinear—soda straw is straight
  - Conjunction will take place in 2-d plane normal to the relative velocity vector and containing the secondary position
  - Problem can thus be reduced in dimensionality from 3 to 2
- Need to project covariance and primary path into “conjunction plane”
Calculating $P_c$: 2-D Approximation (3 of 3)

Conjunction Plane Construction

- Combined covariance projected into plane normal to the relative velocity vector and placed at origin
- Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii (“hard-body radius” or HBR’’)
- Z-axis perpendicular to x-axis in conjunction plane

Figure taken from Chan (2008)
2-D Probability of Collision Computation

• Rotate axes until they align with principal axes of projected covariance ellipse
• $P_c$ is then the portion of the density that falls within the HBR circle
  – $r$ is $[x \ z]$ and $C^*$ is the projected covariance

$$P_c = \frac{1}{\sqrt{(2\pi)^2|C^*|}} \iint_A \exp\left(-\frac{1}{2} \tilde{r}^T C^{*-1} \tilde{r}\right) dXdZ$$

![Diagram of 2-D Probability of Collision](image)
2-D vs. 3-D Conjunction Geometry

2-D Geometry

3-D Geometry
Low Relative Velocity or Long Conjunction Duration Situation

- 2-D approximation not valid
- Can attempt 3-D integral
  - Messy, but Coppola (2012) outlines methodology with Lebedev quadrature
- Can use Monte Carlo
  - From TCA
    - Propagate both satellites’ states and covariances to nominal TCA
    - Take position (and maybe velocity) perturbations from each covariance to define new states for primary and secondary
    - Find new TCA and record miss distance
    - Tabulate all miss distances; percent that are smaller than HBR is Pc
  - From epoch
    - Similar procedure to above, but perturbations performed at epoch
    - Perturbed states propagated forward to new TCA with full non-linear dynamics
Conjunction Event Canonical Progression

- Conjunction typically first discovered 7 days before TCA
  - Covariances large, so typically $P_c$ below maximum
- As event tracked and updated, changes to state estimate are relatively small, but covariance shrinks
  - Because closer to TCA, less uncertainty in projecting positions to TCA
- Theoretical maximum $P_c$ encountered when 1-sigma covariance size to miss distance ratio is $1/\sqrt{2}$
  - After this, $P_c$ usually decreases rapidly
- Behavior shown in graph at right
  - X-axis is covariance / miss distance
  - Y-axis is $\log_{10}(P_c / \text{max}(P_c))$
  - Order of magnitude change in $P_c$ considered significant, thus log-space more appropriate
Probability of Collision Calculation

- **Pc is only a nominal solution for the conjunction**
  - Derived from estimates of the mean
    - If underlying distributions not symmetric, then this is not an expression of central tendency
  - Does not include uncertainties on the inputs
    - “Uncertainty of uncertainty volumes” or uncertainty in HBR
- **Thus, while representing the risk, nominal Pc is just a point estimate**
- **Want to know how much variation or uncertainty in the Pc calculated for any given conjunction**
  - Determine uncertainty PDFs for the Pc calculation inputs
  - Through Monte Carlo trials, vary above inputs to the Pc calculation
  - Include a resampling technique to determine natural variation in the calculation
  - Generate a probability density of resultant Pc values
  - Characterize this distribution empirically
Uncertainty in the Probability

• Generate a $P_c$ distribution, using Monte Carlo (MC) trials of the underlying uncertainties
  – Determine uncertainty for each of the $P_c$ parameters

Underlying Uncertainties

- **HBR Uncertainty**: Draw from projected area distribution (primary) and RCS PDF (secondary)
- **Covariance Uncertainty**: Draw from scale factor distributions for both objects
- **Natural Sampling Variability**: Draw from 2D scaled Gaussian covariance

Generate $P_c$ distribution
COVARIANCE REALISM AND SCALE FACTORS
Covariance Realism

• Ways a typical covariance can be unrealistic
  – Much larger or smaller than the “real” error volume
  – Differently oriented from the “real” error volume
  – Representing a different distribution from the “real” error distribution

• This last item not addressed in present study
  – Current form of covariance promotes Gaussian assumption
  – *A priori* arguments for presuming component error distributions close to Gaussian
  – *A posteriori* evidence for component errors following a symmetric distribution
  – Study indicates large-Pc events not affected by “bending” covariances*

• Large covariances not inherently problematic
  – Rather, quite appropriate if errors themselves are large

• Covariance realism assessment approach is combined evaluation of size and orientation, presuming error volume is Gaussian ellipsoid

JSpOC State and Covariance Accuracy Utility

• **Truth ephemeris produced for every satellite**
  – Similar to methodology used for generating precision satellite laser ranging orbits
  – “Stitched together” pieces of ephemeris from a “judiciously chosen” portion of the fit-spans of subsequent batch ODs
    • Methodology to minimize overlap of portions drawing from same observation base
  – Covariance for reference orbit also preserved (epoch covariances from generating ODs)

• **Each produced precision vector for each object compared to its reference orbit at propagation states of interest**
  – Position comparisons at 6, 12, 18, 24, 36, 48, 72, 120, and 168 hrs
  – Propagated position covariance also calculated and retained at each comparison point

• **Raw materials for covariance realism investigations thus available:**
  – State errors
  – Propagated covariance at point of comparison and reference orbit covariance
Reference Orbit Formation Approaches: Previous and Present

Old Approach

- a. 0% → 25% → 50% → abutment
- b.

New Approach

- a. 0% → 100% → abutment
- b.

Fit Span
Covariance Realism:
Normal Deviates and Chi-squared Variables

• Let q and r be vectors of values that conform to a Gaussian distribution
  – Commonly called normal deviates

• A normal deviate set can be transformed to a standard normal deviate by subtracting the mean and dividing by the standard deviation
  – This produces the so-called Z-variables

\[ Z_q = \frac{q - \mu_q}{\sigma_q}, \quad Z_r = \frac{q - \mu_r}{\sigma_r} \]

• The sum of the squares of a series of standard normal deviates produces a chi-squared distribution, with the number of degrees of freedom equal to the number of series combined

\[ Z_q^2 + Z_r^2 = \chi^2_{2 \, \text{dof}} \]
• In a state estimate, the errors in each component (u, v, and w here) are expected to follow a Gaussian distribution
  – If all systematic errors have been solved for, only random error should remain
• These errors can be standardized to the Z-formulation
  – Mean presumed to be zero (OD should produce unbiased results), so no need for explicit subtraction of mean

\[ Z_u = \frac{u}{\sigma_u}, \quad Z_v = \frac{v}{\sigma_v}, \quad Z_w = \frac{w}{\sigma_w} \]

• Sum of squares of these standardized errors should follow a chi-squared distribution with three degrees of freedom

\[ Z_u^2 + Z_v^2 + Z_w^2 = \chi^2_{3 \text{dof}} \]
Covariance Realism: State Estimation Example Calculation

- Let us presume we have a precision ephemeris, state estimate, and covariance about the state estimate
  - For the present, further presume covariance aligns perfectly with uvw frame (no off-diagonal terms)
- Error vector $\epsilon$ is position difference between state estimate and precision ephemeris, and covariance consists only of variances along the diagonal
  - Inverse of covariance matrix is straightforward

\[
\begin{bmatrix}
\epsilon_u \\
\epsilon_v \\
\epsilon_w
\end{bmatrix}, \quad
\begin{bmatrix}
\sigma_u^2 & 0 & 0 \\
0 & \sigma_v^2 & 0 \\
0 & 0 & \sigma_w^2
\end{bmatrix}
\]

\[
C^{-1} = \begin{bmatrix}
1/\sigma_u^2 & 0 & 0 \\
0 & 1/\sigma_v^2 & 0 \\
0 & 0 & 1/\sigma_w^2
\end{bmatrix}
\]

- Resultant simple formula for chi-squared variables

\[
\epsilon C^{-1} \epsilon^T = \frac{\epsilon_u^2}{\sigma_u^2} + \frac{\epsilon_v^2}{\sigma_v^2} + \frac{\epsilon_w^2}{\sigma_w^2} = \chi^2_{3 \text{ dof}}
\]
Covariance Realism: Non-Diagonal Covariances

- Mahalanobis distance formulary naturally accounts for correlation terms
- Two-dimensional example:

\[ \varepsilon C^{-1} \varepsilon^T = \frac{1}{1 - \rho^2} \left( \frac{\varepsilon_x^2}{\sigma_x^2} + \frac{\varepsilon_y^2}{\sigma_y^2} - \frac{2\rho \varepsilon_x \varepsilon_y}{\sigma_x \sigma_y} \right) \]

- Conforms to intuition
  - As \( \rho \) approaches zero, diagonal case recovered
• Mahalanobis distance set should conform to 3-DoF $\chi^2$ distribution
• Expected value for each calculation is DoF, 3 in this case
• Each Mahalanobis point in principle produces a scale factor
  – $mCm$ sizes covariance such that $\varepsilon C^{-1}\varepsilon^T$ will have a value of 3
  – $m^2$ thus the proper factor by which to scale the covariance in order to produce the expected value
• However, not every Mahalanobis calculation expected to equal expected value
  – Instead, a chi-squared distribution with expected value of 3
• To set scale factor(s), choose factor that brings entire Mahalanobis distance set into conformity with expected distribution
Empirical Distribution Function (EDF) GOF: Exquisite Solution

• Sum of vertical differences between “ideal” and “real” behavior
  – Hypothetical graph at left

• Cramér – von Mises formulation the most appropriate for current situation
  – Equations at right
  – Weighting function ($\psi$) set to unity a better choice for outlier-infused situations

• $Q$ used to consult tables of $p$-values to determine likelihood of match between ideal and real distribution
  – Best approach is to be able to use $p$-value, as this has a clear statistical meaning

\[
Q = n \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \psi(x) \, dx
\]

$\psi(x) = 1$

• But what if we want a distribution of scale factors?
Covariance Realism: Distribution of Scale Factors

- Rank-ordering of results can give reasonable PDF of scale factors
  - Presume 100 squared Mahalanobis distance values (M2)
    - Derived from JSpOC covariance realism data
  - Rank order list
  - Align each entry with the 3-DoF $\chi^2$ value that corresponds to that percentile
  - Quotient of two terms is (square of) scale factor that produces the $\chi^2$ value expected for that particular percentile

- Examples:

<table>
<thead>
<tr>
<th>Percentile</th>
<th>$x^2$</th>
<th>Square of M Distance</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (0.01)</td>
<td>0.115</td>
<td>0.183</td>
<td>1.594</td>
</tr>
<tr>
<td>2 (0.02)</td>
<td>0.185</td>
<td>0.245</td>
<td>1.326</td>
</tr>
<tr>
<td>3 (0.03)</td>
<td>0.245</td>
<td>0.353</td>
<td>1.440</td>
</tr>
<tr>
<td>4 (0.04)</td>
<td>0.300</td>
<td>0.418</td>
<td>1.393</td>
</tr>
</tbody>
</table>
HARD-BODY RADIUS
• HBR is typically found by circumscribing both objects in spheres and combining the objects into one bounding sphere
  – Size of the secondary is typically not known, so added as a large estimate of debris object dimensions

  Secondary is conservative assessment of debris object dimensions

  Largest spacecraft dimension in sphere

• HBR uncertainties that follow represent a more realistic estimate of the area in the conjunction plane
  – The combined uncertainties are much smaller than the bounding sphere
Hard-Body Radius: Min and Max Using Approximation Equations

Could presume uniform distribution between these values as first-order approximation of PDF, but seems rather arbitrary.
Hard-Body Radius: Projected Area Approach

- Randomized orientation of primary satellite to capture the average area
  - Ball-and-stick model to be created for each primary asset
  - Includes rotating solar panels
- Projected radius
  - Actual hit area of the satellite expressed as a circular radius
  \[ r = \sqrt{\frac{A}{\pi}} \]
• NASA/JSC Orbital Debris Program Office (ODPO) has sophisticated satellite model and full Euler angle rotation software to generate projected area PDFs

• Comparison of results for Hubble Space Telescope between ODPO software and ball-and-stick model:

<table>
<thead>
<tr>
<th></th>
<th>Average Area [m²]</th>
<th>Average Effective HBR [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude HST model</td>
<td>60.3</td>
<td>4.3</td>
</tr>
<tr>
<td>(corresponding to chart)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sophisticated HST model (Matney*)</td>
<td>63.7</td>
<td>4.5</td>
</tr>
</tbody>
</table>

Good agreement

Hard-Body Radius: Projected Area Approach Implementation

• Assemble ball-and-stick model of primary satellite
• Rotate through all Euler angles and project into plane
• Create empirical PDF of projected areas
• Express as PDF of radii of circles of equivalent area

• If satellite orientation is known at TCA, then area can be projected directly into conjunction plane
  – Can then perform integration by means of a contour integral
  – Lingering problem of how to incorporate area for secondary object
Hard-Body Radius: Secondary Object HBR Uncertainty

• For intact spacecraft, possible to use published dimensions
  – For payloads, these are often not precise enough to be useful, and at least some canonical models would have to be imposed
    • Error in all of this great enough that approach is questionable
  – For rocket bodies, published dimensions are probably adequate
    • But many booster types lack published dimensions

• Most common secondaries are debris objects, for which no size information is available

• Thus, forced to estimate size from radar cross-section (RCS) value
  – Objects do not have single RCS value but PDF of values, depending on radar response and object aspect function
  – PDFs of individual objects’ RCS values not available, only averaged values
  – As proxy could use canonical distribution
    • Swerling III distribution is most common for debris, and also most conservative in terms of size*

Hard-Body Radius: Swerling Distribution Family

- Swerling distributions derive from the gamma distribution family
  - Location parameter ($\gamma$) = 0
  - Shape parameter ($m$) fixed
  - Scale parameter ($\beta$) estimated from sample (MLE)
- Swerling I/II is gamma with $m=1$
  - Exponential distribution
  - Presumes Rayleigh scattering
- Swerling III/IV is gamma with $m=2$
  - Erlang distribution
  - Presumes correlation with object orientation; more correct assumption
- S-notation is gamma with $m$ given
  - $S_{1.5} = \text{gamma with } m=1.5$ &c.

\[
f(x; \gamma, \beta, m) = \frac{1}{\beta^m \Gamma(m)} (x - \gamma)^{m-1} \exp \left( -\frac{x - \gamma}{\beta} \right) \]

\[
f(x; \beta) = \frac{1}{\beta} \exp \left( -\frac{x}{\beta} \right) \]

\[
f(x; \beta) = \frac{1}{\beta^2} x \exp \left( -\frac{x}{\beta} \right) \]
Hard-Body Radius: Radar OSEM Basic Rubric

• Simulated hyperkinetic destruction of satellite in vacuum chamber
• Collected pieces and subjected them to individual analysis
  – “Observed” each piece with radar in anechoic chamber
  – Articulated full range of aspect angles and full range of radar frequencies
  – Recorded resultant RCS of each aspect/frequency configuration
• Collected results and plotted in dimensionless format
  – RCS / $\lambda^2$; size / $\lambda$
  – Results follow basic theory of Rayleigh, Mie, and optical regions
• Begin with average RCS
• Produce RCS PDF using Swerling III distribution
  – Scale parameter estimated by mean RCS divided by shape parameter
• Send distribution through ODPO size estimator to generate size PDF
  – Certified only for objects smaller than 20cm, but this is most debris
PC CALCULATION
RESAMPLING
Pc Calculation Resampling

• Resampling/bootstrap methods often used to generate confidence intervals when calculation final distribution unknown
• Early attempts at this with Pc used resampling with invariant covariances
  – Take position draw on primary and secondary covariance at TCA
  – Find new TCA; this defines new nominal miss vector
  – Recompute Pc with this new miss vector and unaltered covariances
  – Problem: covariance is clearly correlated with conjunction geometry
    • Cannot produce new miss distance from covariance-based sampling and then recompute Pc using those same covariances
• Need approach that considers miss distance / covariance linkage
J.H. Frisbee proposed a resampling technique that would also address the correlation problem:

- Choose samples from the combined covariance to generate $m$ miss vectors.
- Take mean of $m$ miss vectors—this is new nominal miss.
- Take sample covariance of $m$ miss vectors—this is new combined covariance.
- Compute $P_c$ using this mean miss distance and sample combined covariance.
- Repeat procedure $n$ times—this produces bootstrap dataset.
Resampling Approach Issues

• In this framework, covariances are considered representatives of parent distributions, here further characterized by resampling.

• Issue: what should be the value of $m$?
  – In bootstrapping, want the bootstrap sample size to equal the single-sample size that would have been used (or was used) to estimate the parameter.
  – Thus, want the number of samples (DoF) of the bootstrap resampling ($m$) to equal the DoF that produced the covariance in the first place.
    • That is, the DoF of the generating OD.
• DoF is usually calculated as the number of data points minus the number of estimated parameters
  – JSpOC ODs calculated with SSN obs (usually have range, azimuth, and elevation—three observables)
  – Obs provided in “tracks”—group of obs taken during one tracking session
• Thus, tabulation issues arise
  – Each ob provides 3 DoF, minus the estimated parameters
  – However, rather little information content in interior obs of a track
    • JSpOC “track weighting” confirms this—all tracks weighted the same in the OD, regardless of length
  – Better tabulation to count each track as equivalent of one state estimate
    • Longish track about enough data to execute a single state estimate, to first order
    • Total estimated parameters in OD would thus be only one—one state estimated
  – DoF calculation is thus “# of tracks – 1”
    • Would need to be amended for DS, where obs report only two parameters, and needs more work in general
Resampling Approach Schematic

- Repeated thousands of times to calculate distribution of Pc values
- Benefits
  - Correlation of the miss vector and the covariance
  - Maintains an equivalent sampling level to the original OD
  - Naturally responds to variations in tracking density
PROCESS RESULTS
Example #1
Conclusions and Future Work

• **Proposed method**
  – Characterizes the PDF that can represent the Pc from a particular conjunction, given the uncertainties in covariances, HBR and natural variation in the Pc calculation
  – Gives a sense of the dynamic range of the Pc and allow maneuver decisions to be based on percentile points of this range rather than the nominal value alone
  – Provides a mechanism for obtaining a better expression of the calculation’s central tendency (here the median)

• **Future Work**
  – Refine DoF calculation and generate expansion for angles-only cases
  – Survey results from runs of large datasets
    • Stability studies of simplifying assumptions for faster processing
  – Examine potential as a Pc forecaster