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SPACECRAFT DYNAMICS SHOULD BE CONSIDERED IN KALMAN FILTER ATTITUDE ESTIMATION

Yaguang Yang* **and Zhiqiang Zhou**†

Kalman filter based spacecraft attitude estimation has been used in some high-profile missions and has been widely discussed in literature. While some models in spacecraft attitude estimation include spacecraft dynamics, most do not. To our best knowledge, there is no comparison on which model is a better choice. In this paper, we discuss the reasons why spacecraft dynamics should be considered in the Kalman filter based spacecraft attitude estimation problem. We also propose a reduced quaternion spacecraft dynamics model which admits additive noise. Geometry of the reduced quaternion model and the additive noise are discussed. This treatment is more elegant in mathematics and easier in computation. We use some simulation example to verify our claims.

INTRODUCTION

The Kalman filter found its earliest applications in some high-profile missions in the aerospace industry, such as the Apollo project [1]. Spacecraft attitude estimation has been a major research area since the Kalman filter was invented [2]. Although many different methods have been proposed, most models suggest using only quaternion kinematics equations of motion for the attitude estimation without considering spacecraft dynamics. See for example, some widely cited survey papers [2, 3] and references therein. This model reduces the problem size but discards useful spacecraft attitude information available in the spacecraft dynamics equation. The drawbacks of this simplified model are (a) when gyros measurements have significant noise, the spacecraft dynamics information is not used to prevent the degradation of the attitude estimation, and (b) when gyro measurements are not available (as a matter of fact, gyros are not used in most small spacecraft, for example, [4]), the simplified model cannot be used to estimate the spacecraft attitude.

There are papers that consider models including the spacecraft dynamics in Kalman filter designs, for example, [5, 6]. But to our best knowledge, there is no discussion of which model is a better fit of the application of spacecraft attitude estimation and there is no performance comparison for Kalman filters using the two different models.

In this paper, we will discuss the importance of the spacecraft dynamics to the attitude estimation problem and examine the performance difference between models that incorporate spacecraft dynamics and models that do not. As it is well-known that the models for the attitude estimation and for spacecraft dynamics are nonlinear, some natural choices for solving the estimation problem are either extended Kalman filter (EKF) or unscented Kalman filter (UKF).

*Instrumentation and Control Engineer, Office of Research, NRC, 21 Church Street, Rockville, 20850.
†Aerospace Technologist, Senior Member AIAA, NASA Langley Research Center, M/S 308, 8 Langley Blvd., Hampton, VA 23681.

We recogonize the recent trend of using an unscented Kalman filter instead of the extended Kalman filter in spacecraft attitude estimation problem [7, 8, 9], however we are also aware of some simulation comparison between the two methods and different opinions about the potential advantages of unscented Kalman filter [10]. Therefore, to simplify our presentation and simulation comparison, without loss of our focus, we will consider only the extended Kalman filter in this paper.

A special feature of the spacecraft attitude estimation problem is that the quaternion has a norm constraint, and many methods have been proposed to deal with this constraint [11, 12, 13, 14]. These methods are more complicated in concept and more expensive in computation than traditional EKF without the norm constraint. Therefore, we suggest using a reduced quaternion model which does not need the norm constraint [15, 16]. Though, there exists a singular point in this reduced quaternion model, no one has really compared the effectiveness and the performance of these two methods based on different (full and reduced) quaternion models. This comparison will be the second topic of this paper.

The remainder of the paper is organized as follows. Section 2 provides a description of the extended Kalman filter for spacecraft attitude estimation that follows common practice, i.e., using a model without spacecraft dynamics. Section 3 provides a parallel description of the extended Kalman filter for spacecraft attitude estimation that is our vision, i.e., using a model with spacecraft dynamics. The merits of the proposed model over commonly used models are discussed. Simulations and results for these two methods are presented in Section 4 to demonstrate the superiority of using a model with spacecraft dynamics. The conclusions are summarized in Section 5.

EXTENDED KALMAN FILTER WITHOUT SPACECRAFT DYNAMICS

This type of model is widely used in literature [2] for spacecraft attitude estimation and can be expressed as follows. Let $q_0 = \cos(\frac{\alpha}{2})$, $q = [q_1, q_2, q_3]^{\mathrm{T}} = \hat{e}\sin(\frac{\alpha}{2})$, and

$$\bar{q} = [q_0, q^{\mathrm{T}}]^{\mathrm{T}} \tag{1}$$

be the quaternion that represents the rotation of the body frame relative to the inertial frame, where $\hat{e}$ is the unit vector of the rotational axis and α is the rotational angle; the rate of change of the quaternion is given by [17]

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{2}$$

where $\omega = [\omega_1, \omega_2, \omega_3]^{\mathrm{T}}$ is the body rotational rate with respect to the inertial frame represented in the body frame. However, using this full quaternion model introduces a singularity in the covariance matrix [2]. Therefore, we suggest using a reduced representation derived in [16] given as follows.

$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_1) \tag{3}$$

where ϕ_1 is the process noise, and Ω is a matrix given by

$$\Omega = \begin{bmatrix} g(q) & -q_3 & q_2 \\ q_3 & g(q) & -q_1 \\ -q_2 & q_1 & g(q) \end{bmatrix}, \tag{4}$$

with $g(q) = \sqrt{1 - q_1^2 - q_2^2 - q_3^2}$. The reduced model embeds the unit length requirement in $g(q)$ which means that there is no need to consider the unit length constraint in EKF as it was treated in [12]. This model therefore significantly simplifies the problem, and the model has some other merits discussed in [16]. Assuming that three rate gyros and quaternion measurement sensors are installed on board, the measurement equation can be written as [9]

$$\dot{\beta} = \phi_2, \tag{5a}$$

$$\omega_y = \omega + \beta + \psi_1, \tag{5b}$$

$$q_y = q + \psi_2, \tag{5c}$$

where β is a drift in the angular rate measurement, ϕ_2 is the process noise, ω_y is the angular rate measurement obtained from gyros, q_y is the quaternion measurement (which can be obtained by using QUEST method [18] or analytic method [19] for measurements of astronomical vectors, such as sun sensor, magnetometer, gravitometer, and star trackers), and ψ_1 and ψ_2 are measurement noise.

Remark 0.1 *The reduced quaternion geometry of q_y can be seen from the following argument. For small noise ψ_2 and a quaternion $q = e\sin(\frac{\alpha}{2})$ which is bounded away from a singular point ($\|q\| < 1$), if q is small, q is a rotation whose Euler angles are half of the elements of q and q_y is a rotation whose Euler angles are half of the elements of $q + \psi_2$; if q is not small, $q + \psi_2$ is a rotation whose rotational axis is a perturbation of q satisfying $\|q_y\| \leq \|q\| + \|\psi_2\|$ and $\|q_y\| \leq 1$ (where $\|\psi_2\|$ is small), and the rotational angle around q_y is $\frac{\alpha}{2} + \delta$ and $|\delta|$ is small. Therefore, the quaternion perturbation model described in this paper is more general than the widely used multiplicative perturbation [11] because the former may have different rotational axes in the original and perturbed quaternion and the latter must have the same rotational axis in the original and perturbed quaternion.*

Let

$$x = \begin{bmatrix} q \\ \beta \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ \omega \end{bmatrix}.$$

We can rewrite the system in a compact form

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Omega\omega \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\Omega\phi_1 \\ \phi_2 \end{bmatrix} := f(x,u) + g(x,u)\phi, \tag{6a}$$

$$y = \begin{bmatrix} q_y \\ \omega_y \end{bmatrix} = \begin{bmatrix} q \\ \omega + \beta \end{bmatrix} + \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} := h(x,u) + \psi, \tag{6b}$$

where

$$f(x,u) = \begin{bmatrix} \frac{1}{2}\Omega\omega \\ 0 \end{bmatrix}, \quad g(x,u) = \begin{bmatrix} \frac{1}{2}\Omega & 0 \\ 0 & I_3 \end{bmatrix}, \quad h(x,u) = \begin{bmatrix} q \\ \omega + \beta \end{bmatrix}.$$

The simplest discrete version of (6) can be obtained by explicit Euler's method. However, the discrete formula obtained by this method is normally not stable for stiff differential equations [20]. In [12], the trapezoidal implicit method was proposed. But this method involves the solution of nonlinear system of equations which can be very expensive in computation [20]. We suggest using the linearly implicit Euler method described in [21, 22]. Let dt be the sampling time period and

$$X = \left(I_3 - dt \begin{bmatrix} -\frac{q_1\omega_1}{2g(q)} & \frac{\omega_3}{2} - \frac{q_2\omega_1}{2g(q)} & -\frac{\omega_2}{2} - \frac{q_3\omega_1}{2g(q)} \\ -\frac{\omega_3}{2} - \frac{q_1\omega_2}{2g(q)} & -\frac{q_2\omega_2}{2g(q)} & \frac{\omega_1}{2} - \frac{q_3\omega_2}{2g(q)} \\ \frac{\omega_2}{2} - \frac{q_1\omega_3}{2g(q)} & -\frac{\omega_1}{2} - \frac{q_2\omega_3}{2g(q)} & -\frac{q_3\omega_3}{2g(q)} \end{bmatrix} \right) \tag{7}$$

The discrete version of (6) is therefore given as follows:

$$q_{k+1} = q_k + X^{-1}\left(\frac{1}{2}\Omega_k\omega_k + \frac{1}{2}\Omega_k\phi_{1_k}\right)dt, \tag{8a}$$

$$\beta_{k+1} = \beta_k + \phi_{2_k}dt, \tag{8b}$$

$$y_k = \begin{bmatrix} q_{y_k} \\ \omega_{y_k} \end{bmatrix} = \begin{bmatrix} q_k \\ \beta_k \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_k \end{bmatrix} + \begin{bmatrix} \psi_{1_k} \\ \psi_{2k} \end{bmatrix} := x_k + u_k + \psi_k = H(x_k, u_k) + \psi_k, \tag{8c}$$

where $H(x_k, u_k) = x_k + u_k$. As always, we assume that ϕ_k and ψ_k are white noise signals and the following relations hold:

$$E(\phi_k) = 0, \quad E(\psi_k) = 0, \quad \forall k, \tag{9a}$$

$$E(\phi_k\phi_k^{\mathrm{T}}) = Q_k, \quad E(\psi_k\psi_k^{\mathrm{T}}) = R_k, \quad E(\psi_j\phi_i^{\mathrm{T}}) = 0, \quad \forall i, j, k, \tag{9b}$$

$$E(\phi_j\phi_i^{\mathrm{T}}) = 0, \quad E(\psi_j\psi_i^{\mathrm{T}}) = 0, \quad \forall i \neq j. \tag{9c}$$

We need some explicit expression of (8a) to obtain the formulas of the extended Kalman filter. Note that (7) can be simplified as

$$X(q,\omega,dt) = dt\begin{bmatrix} \frac{1}{dt} + \frac{q_1\omega_1}{2g} & -\frac{\omega_3}{2} + \frac{q_2\omega_1}{2g} & \frac{\omega_2}{2} + \frac{q_3\omega_1}{2g} \\ \frac{\omega_3}{2} + \frac{q_1\omega_2}{2g} & \frac{1}{dt} + \frac{q_2\omega_2}{2g} & -\frac{\omega_1}{2} + \frac{q_3\omega_2}{2g} \\ -\frac{\omega_2}{2} + \frac{q_1\omega_3}{2g} & \frac{\omega_1}{2} + \frac{q_2\omega_3}{2g} & \frac{1}{dt} + \frac{q_3\omega_3}{2g} \end{bmatrix} := \begin{bmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{d} & \bar{e} & \bar{f} \\ \bar{g} & \bar{h} & \bar{i} \end{bmatrix} dt.$$

Therefore,

$$X^{-1}(q,\omega,dt) = \frac{1}{dt(\bar{a}\bar{A} + \bar{b}\bar{B} + \bar{c}\bar{C})}\begin{bmatrix} \bar{A} & \bar{D} & \bar{G} \\ \bar{B} & \bar{E} & \bar{H} \\ \bar{C} & \bar{F} & \bar{I} \end{bmatrix},$$

where

$$\begin{aligned} \bar{A} &= (\bar{e}\bar{i} - \bar{f}\bar{h}), \quad \bar{B} = -(\bar{d}\bar{i} - \bar{f}\bar{g}), \quad \bar{C} = (\bar{d}\bar{h} - \bar{e}\bar{g}), \\ \bar{D} &= -(\bar{b}\bar{i} - \bar{c}\bar{h}), \quad \bar{E} = (\bar{a}\bar{i} - \bar{c}\bar{g}), \quad \bar{F} = -(\bar{a}\bar{h} - \bar{b}\bar{g}), \\ \bar{G} &= (\bar{b}\bar{f} - \bar{c}\bar{e}), \quad \bar{H} = -(\bar{a}\bar{f} - \bar{c}\bar{d}), \quad \bar{I} = (\bar{a}\bar{e} - \bar{b}\bar{d}), \end{aligned}$$

which leads to

$$\tfrac{1}{2}X^{-1}\Omega\omega dt := \begin{bmatrix} \bar{w}(q,\omega) \\ \bar{u}(q,\omega) \\ \bar{v}(q,\omega) \end{bmatrix}$$

$$= \frac{1}{2(\bar{a}\bar{A}+\bar{b}\bar{B}+\bar{c}\bar{C})}\begin{bmatrix} (\bar{A}g + \bar{D}q_3 - \bar{G}q_2)\omega_1 + (-\bar{A}q_3 + \bar{D}g + \bar{G}q_1)\omega_2 + (\bar{A}q_2 - \bar{D}q_1 + \bar{G}g)\omega_3 \\ (\bar{B}g + \bar{E}q_3 - \bar{H}q_2)\omega_1 + (-\bar{B}q_3 + \bar{E}g + \bar{H}q_1)\omega_2 + (\bar{B}q_2 - \bar{E}q_1 + \bar{H}g)\omega_3 \\ (\bar{C}g + \bar{F}q_3 - \bar{I}q_2)\omega_1 + (-\bar{C}q_3 + \bar{F}g + \bar{I}q_1)\omega_2 + (\bar{C}q_2 - \bar{F}q_1 + \bar{I}g)\omega_3 \end{bmatrix} \tag{10}$$

and

$$\tfrac{1}{2}X^{-1}\Omega\phi_1 dt := \begin{bmatrix} \bar{x}(q,\omega,\phi_1) \\ \bar{y}(q,\omega,\phi_1) \\ \bar{z}(q,\omega,\phi_1) \end{bmatrix}$$

$$= \frac{1}{2(\bar{a}\bar{A}+\bar{b}\bar{B}+\bar{c}\bar{C})}\begin{bmatrix} (\bar{A}g + \bar{D}q_3 - \bar{G}q_2)\phi_{11} + (-\bar{A}q_3 + \bar{D}g + \bar{G}q_1)\phi_{12} + (\bar{A}q_2 - \bar{D}q_1 + \bar{G}g)\phi_{13} \\ (\bar{B}g + \bar{E}q_3 - \bar{H}q_2)\phi_{11} + (-\bar{B}q_3 + \bar{E}g + \bar{H}q_1)\phi_{12} + (\bar{B}q_2 - \bar{E}q_1 + \bar{H}g)\phi_{13} \\ (\bar{C}g + \bar{F}q_3 - \bar{I}q_2)\phi_{11} + (-\bar{C}q_3 + \bar{F}g + \bar{I}q_1)\phi_{12} + (\bar{C}q_2 - \bar{F}q_1 + \bar{I}g)\phi_{13} \end{bmatrix} \tag{11}$$

Let

$$F_{k-1} = \begin{bmatrix} I_3 + \frac{\partial}{\partial q}\begin{bmatrix} \bar{w}(q,\omega) \\ \bar{u}(q,\omega) \\ \bar{v}(q,\omega) \end{bmatrix} & 0_3 \\ 0_3 & I_3 \end{bmatrix}_{\hat{x}_{k-1|k-1}}, \tag{12}$$

$$L_{k-1} = \begin{bmatrix} \frac{\partial}{\partial \phi_1}\begin{bmatrix} \bar{x}(q,\omega,\phi_1) \\ \bar{y}(q,\omega,\phi_1) \\ \bar{z}(q,\omega,\phi_1) \end{bmatrix} & 0_3 \\ 0_3 & dtI_3 \end{bmatrix}_{\hat{x}_{k-1|k-1},u_{k-1}} = \begin{bmatrix} \frac{1}{2}X^{-1}\Omega_{k-1}dt & 0_3 \\ 0_3 & dtI_3 \end{bmatrix}_{\hat{x}_{k-1|k-1},u_{k-1}}, \tag{13}$$

and

$$H_k = \frac{\partial H}{\partial x}|_{\hat{x}_{k|k-1},u_{k-1}} = I. \tag{14}$$

The extended Kalman filter iteration is as follows:

$$\hat{x}_{k|k-1} = F(\hat{x}_{k-1|k-1}, u_{k-1}) \tag{15a}$$
$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^{\mathrm{T}} + L_{k-1}Q_{k-1}L_{k-1}^{\mathrm{T}} \tag{15b}$$
$$\tilde{y}_k = y_k - H(\hat{x}_{k|k-1}) \tag{15c}$$
$$S_k = P_{k|k-1} + R_k \tag{15d}$$
$$K_k = P_{k|k-1}S_k^{-1} \tag{15e}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\tilde{y}_k \tag{15f}$$
$$P_{k|k} = (I - K_k)P_{k|k-1}. \tag{15g}$$

Since u_k is not available, it is suggested in [2] to set $u_k = \hat{\omega}_{k|k} = \omega_{y_k} - \hat{\beta}_{k|k}$ and $\hat{\omega}_{k|k-1} = \hat{\omega}_{k-1|k-1}$.

Remark 0.2 *Clearly, the extended Kalman filter using this model cannot be updated without three dimensional gyro measurements ω_{y_k}. In the next section, we will show that even if the gyro measurements are available, using this model is not as good as using a model which incorporates the spacecraft dynamics. In section 4, we will use simulation to compare the performance of two different methods to support our claim.*

Remark 0.3 *To improve the estimation accuracy of $\hat{x}_{k|k-1}$, we can reduce the step size of dt. But in some applications, the measurements may be available only after several sampling period. In this case,a multi-rate Kalman filter should be considered [23], which is not the focus of this paper.*

EXTENDED KALMAN FILTER WITH SPACECRAFT DYNAMICS

This type of model can be expressed as follows [15, 16].

$$\dot{\omega} = -J^{-1}\omega \times (J\omega) + J^{-1}u + \phi_1, \tag{16a}$$
$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_2), \tag{16b}$$
$$y = h(\omega, q) + \psi, \tag{16c}$$

where $[\omega, q]^{\mathrm{T}}$ is the state vector, y is the measurement vector, $\phi = [\phi_1, \phi_2]^{\mathrm{T}}$ is the process Gaussian noise which models various disturbance torques, ψ is the measurement Gaussian noise, J is the inertia matrix of the spacecraft, and Ω is defined in (4). Depending on the design, we may have angular rate measurements ω_y and quaternion measurement q_y; or we may have only quaternion measurement q_y. Assuming that three gyros and quaternion measurement sensors are installed on board, then the measurement equation can be written as [9]

$$\dot{\beta} = \phi_3, \tag{17a}$$
$$\omega_y = \omega + \beta + \psi_1, \tag{17b}$$
$$q_y = q + \psi_2, \tag{17c}$$

where β is a drift in the angular rate measurement, ϕ_3 is the process noise, ω_y is the angular rate measurement, q_y is the quaternion measurement, and ψ_1 and ψ_2 are measurement noise. The overall system equations are given as follows:

$$\dot{\omega} = -J^{-1}\omega \times (J\omega) + J^{-1}u + \phi_1, \tag{18a}$$
$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_2), \tag{18b}$$
$$\dot{\beta} = \phi_3, \tag{18c}$$
$$\omega_y = \omega + \beta + \psi_1, \tag{18d}$$
$$q_y = q + \psi_2, \tag{18e}$$

which can be rewritten as a standard state space model as follows:

$$\dot{x} = f(x, u) + \phi, \tag{19a}$$
$$y = Hx + \psi, \tag{19b}$$

where $x = [\omega^{\mathrm{T}}, q^{\mathrm{T}}, \beta^{\mathrm{T}}]^{\mathrm{T}}$, $y = [\omega_y^{\mathrm{T}}, q_y^{\mathrm{T}}]^{\mathrm{T}}$, $\phi = [\phi_1^{\mathrm{T}}, \phi_2^{\mathrm{T}}, \phi_3^{\mathrm{T}}]^{\mathrm{T}}$, $\psi = [\psi_1^{\mathrm{T}}, \psi_2^{\mathrm{T}}]^{\mathrm{T}}$, and

$$H = \begin{bmatrix} I_3 & 0_3 & I_3 \\ 0_3 & I_3 & 0_3 \end{bmatrix}.$$

The discrete version of (18) is given by

$$\begin{aligned} \begin{bmatrix} \omega_{k+1} \\ q_{k+1} \\ \beta_{k+1} \end{bmatrix} &= \left(\begin{bmatrix} \omega_k \\ q_k \\ \beta_k \end{bmatrix} + \begin{bmatrix} -J^{-1}\omega_k \times (J\omega_k) + J^{-1}u_k \\ \frac{1}{2}\,\Omega_k\omega_k \\ 0 \end{bmatrix} dt \right) + \begin{bmatrix} \phi_{1_k} \\ \frac{1}{2}\,\Omega_k\phi_{2_k} \\ \phi_{3_k} \end{bmatrix} dt \\ &= F(x_k, u_k) + G(x_k, u_k)\phi_k, \end{aligned} \tag{20a}$$

$$\begin{bmatrix} \omega_{y_k} \\ q_{y_k} \end{bmatrix} = \begin{bmatrix} I_3 & 0_3 & I_3 \\ 0_3 & I_3 & 0_3 \end{bmatrix} \begin{bmatrix} \omega_k \\ q_k \\ \beta_k \end{bmatrix} + \begin{bmatrix} \psi_{1_k} \\ \psi_{2_k} \end{bmatrix} = Hx_k + \psi_k, \tag{20b}$$

where

$$\Omega_k = \begin{bmatrix} \sqrt{1 - q_{1_k}^2 - q_{2_k}^2 - q_{3_k}^2} & -q_{3_k} & q_{2_k} \\ q_{3_k} & \sqrt{1 - q_{1_k}^2 - q_{2_k}^2 - q_{3_k}^2} & -q_{1_k} \\ -q_{2_k} & q_{1_k} & \sqrt{1 - q_{1_k}^2 - q_{2_k}^2 - q_{3_k}^2} \end{bmatrix}. \tag{21}$$

Note that for two vectors $w = [w_1, w_2, w_3]^{\mathrm{T}}$ and $v = [v_1, v_2, v_3]^{\mathrm{T}}$, the cross product of $w \times v$ can be written as the product of matrix $w^{\times}$ and vector v where

$$w^{\times} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

We also assume ϕ_k and ψ_k are white noise signals satisfying equations (9). For

$$F_1(x,u) = \left(-J^{-1}\omega_k \times (J\omega_k) + J^{-1}u_k\right) dt + \omega_k,$$

we have

$$\frac{\partial F_1}{\partial x} = \begin{bmatrix} I - J^{-1}(\omega^{\times}J - (J\omega)^{\times})dt & 0_3 & 0_3 \end{bmatrix}.$$

For $F_2(x,u) = \frac{1}{2}\,\Omega_k\omega_k dt + q_k$, we have

$$\frac{\partial F_2}{\partial x} = \begin{bmatrix} \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial q} & 0_3 \end{bmatrix},$$

with

$$\frac{\partial F_2}{\partial \omega} = \begin{bmatrix} \frac{g}{2} & -\frac{q_3}{2} & \frac{q_2}{2} \\ \frac{q_3}{2} & \frac{g}{2} & -\frac{q_1}{2} \\ -\frac{q_2}{2} & \frac{q_1}{2} & \frac{g}{2} \end{bmatrix} dt = \frac{1}{2}\Omega dt, \tag{22}$$

and

$$\frac{\partial F_2}{\partial q} = \begin{bmatrix} \frac{1}{dt} - \frac{q_1\omega_1}{2g(q)} & \frac{\omega_3}{2} - \frac{q_2\omega_1}{2g(q)} & -\frac{\omega_2}{2} - \frac{q_3\omega_1}{2g(q)} \\ -\frac{\omega_3}{2} - \frac{q_1\omega_2}{2g(q)} & \frac{1}{dt} - \frac{q_2\omega_2}{2g(q)} & \frac{\omega_1}{2} - \frac{q_3\omega_2}{2g(q)} \\ \frac{\omega_2}{2} - \frac{q_1\omega_3}{2g(q)} & -\frac{\omega_1}{2} - \frac{q_2\omega_3}{2g(q)} & \frac{1}{dt} - \frac{q_3\omega_3}{2g(q)} \end{bmatrix} dt. \tag{23}$$

For $F_3(x,u) = \beta_k$, we have

$$\frac{\partial F_3}{\partial x} = \begin{bmatrix} 0_3 & 0_3 & I_3 \end{bmatrix}.$$

Therefore,

$$F_{k-1} := \frac{\partial F}{\partial x}|_{\hat{x}_{k-1|k-1}, u_{k-1}} = \begin{bmatrix} I - J^{-1}(\omega^{\times}J - (J\omega)^{\times})dt & 0_3 & 0_3 \\ \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial q} & 0_3 \\ 0_3 & 0_3 & I_3 \end{bmatrix}_{\hat{x}_{k-1|k-1}}. \tag{24}$$

Let

$$L_{k-1} = \frac{\partial G}{\partial \phi_k}|_{\hat{x}_{k-1|k-1}, u_{k-1}} = \begin{bmatrix} I_3 & 0_3 & 0_3 \\ 0_3 & \frac{1}{2}\,\Omega_{k-1} & 0_3 \\ 0_3 & 0_3 & I_3 \end{bmatrix} dt. \tag{25}$$

The extended Kalman filter iteration is as follows:

$$\hat{x}_{k|k-1} = F(\hat{x}_{k-1|k-1}, u_{k-1}) \tag{26a}$$

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^{\mathrm{T}} + L_{k-1}Q_{k-1}L_{k-1}^{\mathrm{T}} \tag{26b}$$

$$\tilde{y}_k = y_k - H\hat{x}_{k|k-1} \tag{26c}$$

$$S_k = HP_{k|k-1}H^{\mathrm{T}} + R_k \tag{26d}$$

$$K_k = P_{k|k-1}H^{\mathrm{T}}S_k^{-1} \tag{26e}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\tilde{y}_k \tag{26f}$$

$$P_{k|k} = (I - K_kH)P_{k|k-1} \tag{26g}$$

Remark 0.4 *The beauty of the Kalman filter using spacecraft dynamics can be seen from (26f). The best estimation is composed of two parts. The first part is a prediction $\hat{x}_{k|k-1}$ which is based on the spacecraft dynamics and the inertia matrix information for the specific spacecraft. The second part is a correction $\tilde{y}_k$ which is based on observations. The filter gain K_k is constantly adjusted such that (a) if the noise is higher, the gain is reduced so that the estimation depends more on the information of the system dynamics, and (b) if the noise is lower, the gain is increased so that the estimation depends more on the measurement. That is the reason why spacecraft dynamics should be included in the attitude estimation problem even if angular rate measurements are available.*

As mentioned before, the Kalman filter with spacecraft dynamics works without the (gyro) measurement of spacecraft angular velocity vector with respect to the inertial frame. In this case, gyro measurement drift β does not exist. Therefore, the continuous system (18) is reduced to

$$\dot{\omega} = -J^{-1}\omega \times (J\omega) + J^{-1}u + \phi_1, \tag{27a}$$

$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_2), \tag{27b}$$

$$q_y = q + \psi_2. \tag{27c}$$

We still use (19) for this system but $x = [\omega^{\mathrm{T}}, q^{\mathrm{T}}]^{\mathrm{T}}$, $y = q_y$, $\phi = [\phi_1^{\mathrm{T}}, \phi_2^{\mathrm{T}}]^{\mathrm{T}}$, $\psi = \psi_1$ and $C = \begin{bmatrix} 0_3 & I_3 \end{bmatrix}$. The discrete version of (27) is given by

$$\begin{aligned} \begin{bmatrix} \omega_{k+1} \\ q_{k+1} \end{bmatrix} &= \left(\begin{bmatrix} \omega_k \\ q_k \end{bmatrix} + \begin{bmatrix} -J^{-1}\omega_k \times (J\omega_k) + J^{-1}u_k \\ \frac{1}{2}\,\Omega_k\omega_k \end{bmatrix} dt \right) + \begin{bmatrix} \phi_{1_k} \\ \frac{1}{2}\,\Omega_k\phi_{2_k} \end{bmatrix} dt \\ &= F(x_k, u_k) + G(x_k, u_k)\phi_k, \end{aligned} \tag{28a}$$

$$q_{y_k} = \begin{bmatrix} 0_3 & I_3 \end{bmatrix} \begin{bmatrix} \omega_k \\ q_k \end{bmatrix} + \psi_k = Hx_k + \psi_k, \tag{28b}$$

where Ω_k is the same as in (21). We also assume ϕ_k and ψ_k are white noise signals satisfying equations (9). For

$$F_1(x, u) = \left(-J^{-1}\omega_k \times (J\omega_k) + J^{-1}u_k\right) dt + \omega_k,$$

we have

$$\frac{\partial F_1}{\partial x} = \begin{bmatrix} I - J^{-1}(\omega^{\times}J - (J\omega)^{\times})dt & 0_3 \end{bmatrix}.$$

For $F_2(x, u) = \frac{1}{2}\,\Omega_k\omega_k dt + q_k$, we have

$$\frac{\partial F_2}{\partial x} = \begin{bmatrix} \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial q} \end{bmatrix},$$

with $\frac{\partial F_2}{\partial \omega}$ and $\frac{\partial F_2}{\partial q}$ the same as (22) and (23). Therefore,

$$F_{k-1} := \frac{\partial F}{\partial x}|_{\hat{x}_{k-1|k-1}, u_{k-1}} = \begin{bmatrix} I - J^{-1}(\omega^{\times}J - (J\omega)^{\times})dt & 0_3 \\ \frac{\partial F_2}{\partial \omega} & \frac{\partial F_2}{\partial q} \end{bmatrix}_{\hat{x}_{k-1|k-1}}. \tag{29}$$

Let

$$L_{k-1} = \frac{\partial G}{\partial \phi_k}|_{\hat{x}_{k-1|k-1}, u_{k-1}} = \begin{bmatrix} I_3 & 0_3 \\ 0_3 & \frac{1}{2}\,\Omega_{k-1} \end{bmatrix} dt. \tag{30}$$

The extended Kalman filter will be the same as (26).

SIMULATION TEST

Extended Kalman filters with and without spacecraft dynamics have been implemented in Simulink to assess their performances. The inertia matrix J of the spacecraft in the simulation has the following values taken from [24]:

$$\begin{bmatrix} 1200 & 100 & -200 \\ 100 & 2200 & 300 \\ -200 & 300 & 3100 \end{bmatrix}$$

The unit of J is $kg \cdot m^2$. The state and measurement noise variance matrices Q_k and R_k are positive definite and represent the noise magnitudes of the angular and angular rate in state dynamics and measurement instruments. While the dimensions of Q_k in the extended Kalman filters (with or without spacecraft dynamics) are different, R_k is the same for both filters and given by

$$R_k = 0.1 I_6$$

where I_6 is a 6×6 identity matrix. State dynamics noise Q_k for the filter without spacecraft dynamics is given by

$$Q_k = \begin{bmatrix} 0.4 I_3 & -0.004 I_3 \\ -0.004 I_3 & 0.4 I_3 \end{bmatrix}$$

For the filter with spacecraft dynamics, Q_k is given by a similar but different dimensional matrix

$$Q_k = \begin{bmatrix} 0.4 I_3 & -0.004 I_3 & 0_3 \\ -0.004 I_3 & 0.4 I_3 & 0_3 \\ 0_3 & 0_3 & 0.00005 I_3 \end{bmatrix}$$

0_3 is 3×3 matrix of zeroes. The initial values of the states $\hat{x}_{0|0}$ and the covariance $P_{0|0}$ are set to zeroes. The true and estimated quaternions for the Kalman filters with and without spacecraft dynamics are shown in Figure 1 through Figure 4.

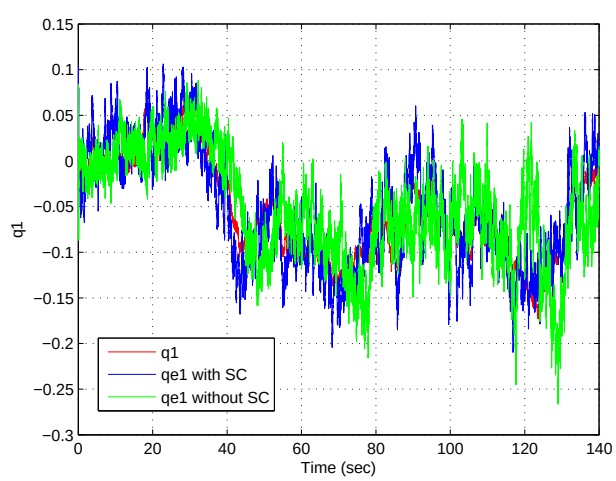

Figure 1. The first component of the estimated and true quaternion.

These figures show that the estimated attitudes for both filters follow the true attitude, but the estimation using spacecraft dynamics is clearly better than the estimation without using spacecraft

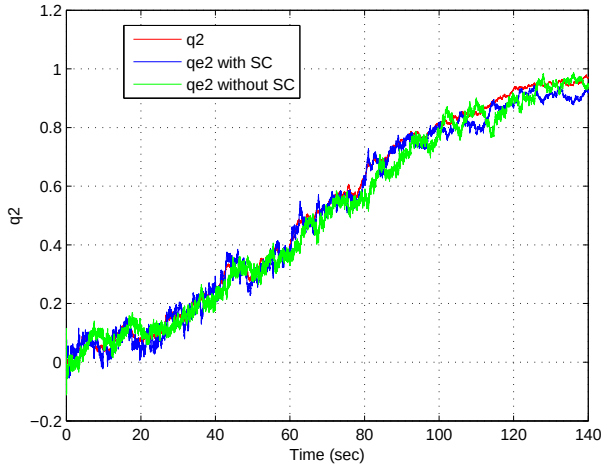

Figure 2. The second component of the estimated and true quaternion.

dynamics. The attitude errors between the estimated and true attitude are represented by the Euler angles, roll, pitch and yaw angle errors. The attitude errors of the extended Kalman filters with and without spacecraft dynamics are compared. The mean and standard deviation of the attitude errors with and without SC dynamics are summarized in Table 1.

Table 1. Mean and standard deviation of the attitude errors with and without SC dynamics

Euler angles	Attitude error mean (deg)	Attitude error standard deviation (deg)
Roll with SC dynamics	-0.4869	0.4145
Roll without SC dynamics	3.1075	4.8599
Pitch with SC dynamics	1.0194	1.1230
Pitch without SC dynamics	3.3738	7.7963
Yaw with SC dynamics	0.0889	0.3741
Yaw without SC dynamics	3.1770	4.9706

CONCLUSIONS

In this paper, we compared two different models that can be used for spacecraft attitude estimation. One model does not use spacecraft dynamics and is more popular in the guidance, navigation, and control community; the other one involves the spacecraft dynamics and has not been investigated as much as the first model. We adopted a reduced quaternion spacecraft dynamics model which admits additive noise. Geometry of the reduced quaternion model and the additive noise was discussed. This treatment is more elegant in mathematics and easier in computation. Our analysis and simulation results show that the second model and the corresponding extended Kalman filter is a better choice in attitude determination because the method uses more information and gives more accurate attitude estimation.

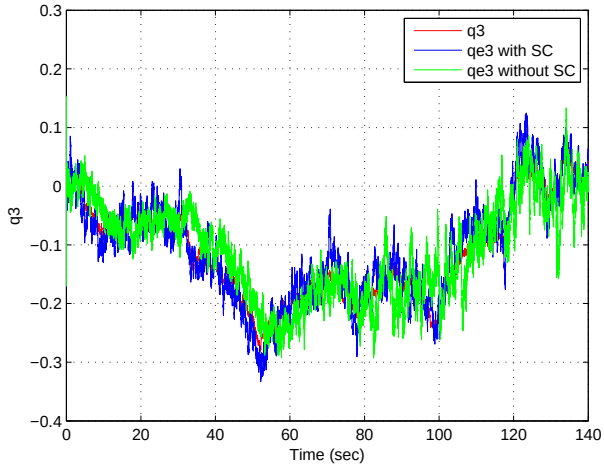

Figure 3. The third component of the estimated and true quaternion.

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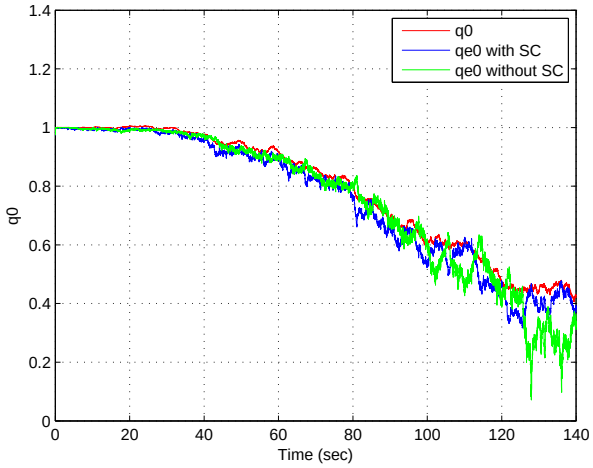

Figure 4. The fouth component of the estimated and true quaternion.

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