Column Number Density Expressions Through M = 0 and M = 1 Point Source Plumes Along Any Straight Path

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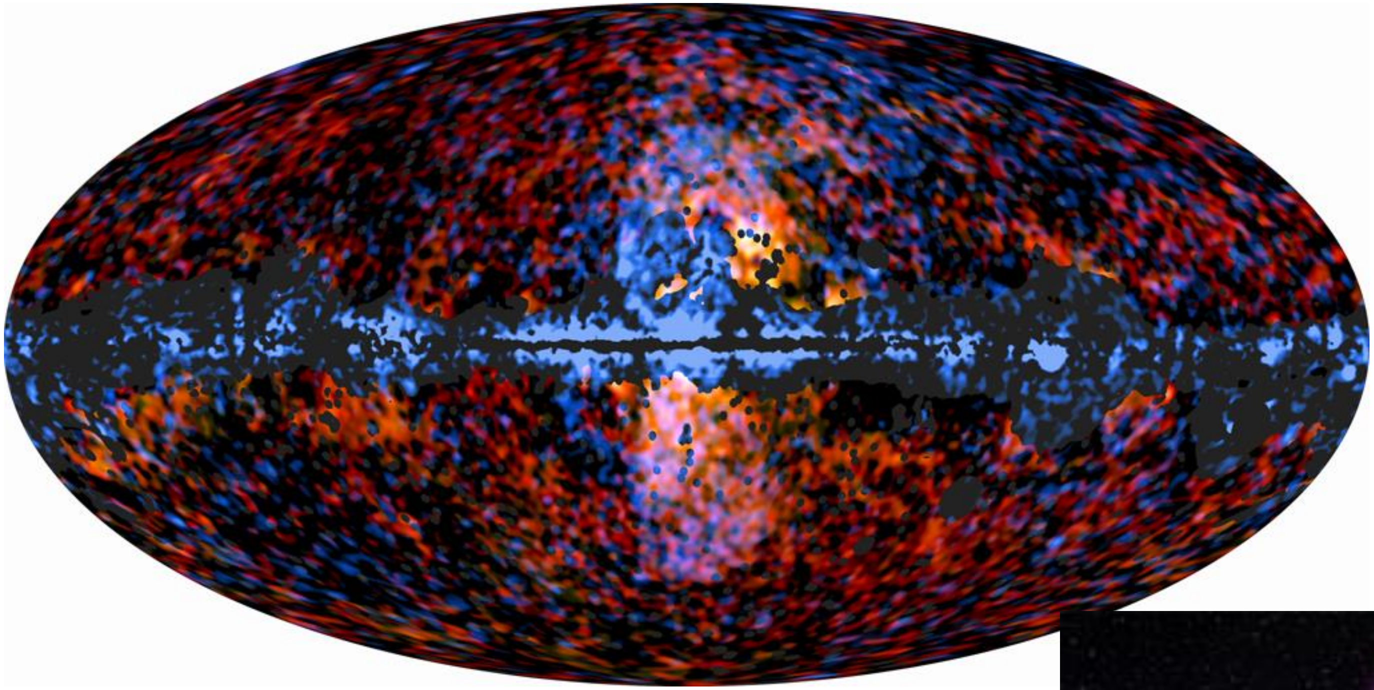
  - Results for $M = 0$ Cases
- Concluding Remarks

Introduction

- Providers of externally-mounted scientific payloads at the International Space Station (ISS) are required to evaluate column number density (CND, σ) associated with various gas releases and demonstrate that they fall below some maximum requirement
  - Must be considerate of other payloads
  - Since this includes unknown future additions, becomes a search for maximum CND along any path
- Occasionally astrophysicists are interested in estimating the amount of gas released by some event or process by evaluating light attenuation of a distant star having known properties due to this release
  - Milky Way center, black hole, "Fermi Bubbles"

ATV Edoardo Amaldi Approaches ISS

“Fermi Bubbles”



ESA/Planck Collaboration (microwave); NASA/DOE/Fermi LAT/Dobler et al./Su et al. (gamma rays)

Objective

- Develop analytical CND expressions for general paths that intercept various common point sources under high vacuum conditions
  - Effusion/low rate evaporation/outgassing ($M = 0$)
  - Venting via sonic orifice ($M = 1$)
  - Spherically-symmetric, radial expansion ($M = 0$)

Venting (Directed) Source Behavior

- External neutral gas phase sources on ISS result from a number of different physical mechanisms
  - Supersonic expansion through thruster nozzles
  - Pressure-driven acceleration to sonic conditions across an orifice
  - Surface evaporation, desorption (may or may not have bulk velocity)
  - Effusion--low-rate, high-*Kn* venting ($M = 0$)
  - Diffusion-limited outgassing ($M = 0$)
- This study assumes that, for these applications, the point source may be described using free molecule flow model approximations
  - Density levels fall rapidly with distance from source location
  - Existence of self-scattering collisions may not substantially alter plume distribution from free molecule flow description

Directed Source—Steady Density

- Can compute many different types of local quantities at receiver position $\boldsymbol{x}$ relative to source
  - Steady number density n from a directed axisymmetric source given by

$$n(\boldsymbol{x},t) = \frac{\beta \dot{N} \cos\theta}{A_1 \pi r^2} e^{w^2 - s^2} \left\{ w e^{-w^2} + \left(\frac{1}{2} + w^2 \right) \sqrt{\pi} \left(1 + \operatorname{erf} w \right) \right\}$$

  - Speed ratio $s \equiv \beta u_\mathrm{e} = \dfrac{u_\mathrm{e}}{\sqrt{2RT_\mathrm{e}}}; \quad w \equiv s \cos\theta$
  - A_1: normalization factor, function of s

Column Number Density (CND, σ)

- Integrated effect of molecules encountered across a prescribed path l
  - When unbounded,

$$\sigma = \int_0^{\infty} n \, dl$$

- For ISS application, the requirement not to exceed σ_{crit} allows one to determine the physical envelope around the source where the limit is violated
  - With a singularity at the source origin, the model will always predict some critical envelope
    - Not consequential for low $\dot{N}$

Effusive CND Expressions

- For low rate, high-*Kn* venting through an orifice with thermal effusion, no bulk motion, plume model density simplifies to

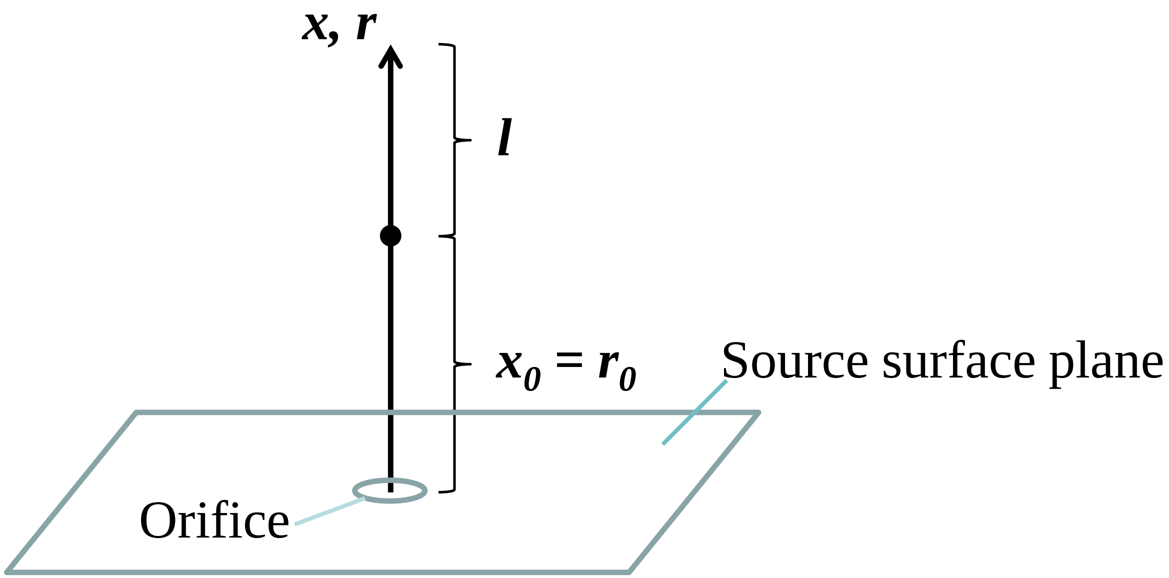
$$n(r,\theta)=\frac{\dot{N}\cos\theta}{r^2\sqrt{8\pi RT}}$$

  – Also describes density field due to outgassing or low rate volatile evaporation from a planar surface viewed from a distance

- Column number density given by

$$\sigma=\frac{\dot{N}}{\sqrt{8\pi RT}}\int_0^\infty\frac{\cos\theta}{r^2}dl$$

1-D Centerline Path



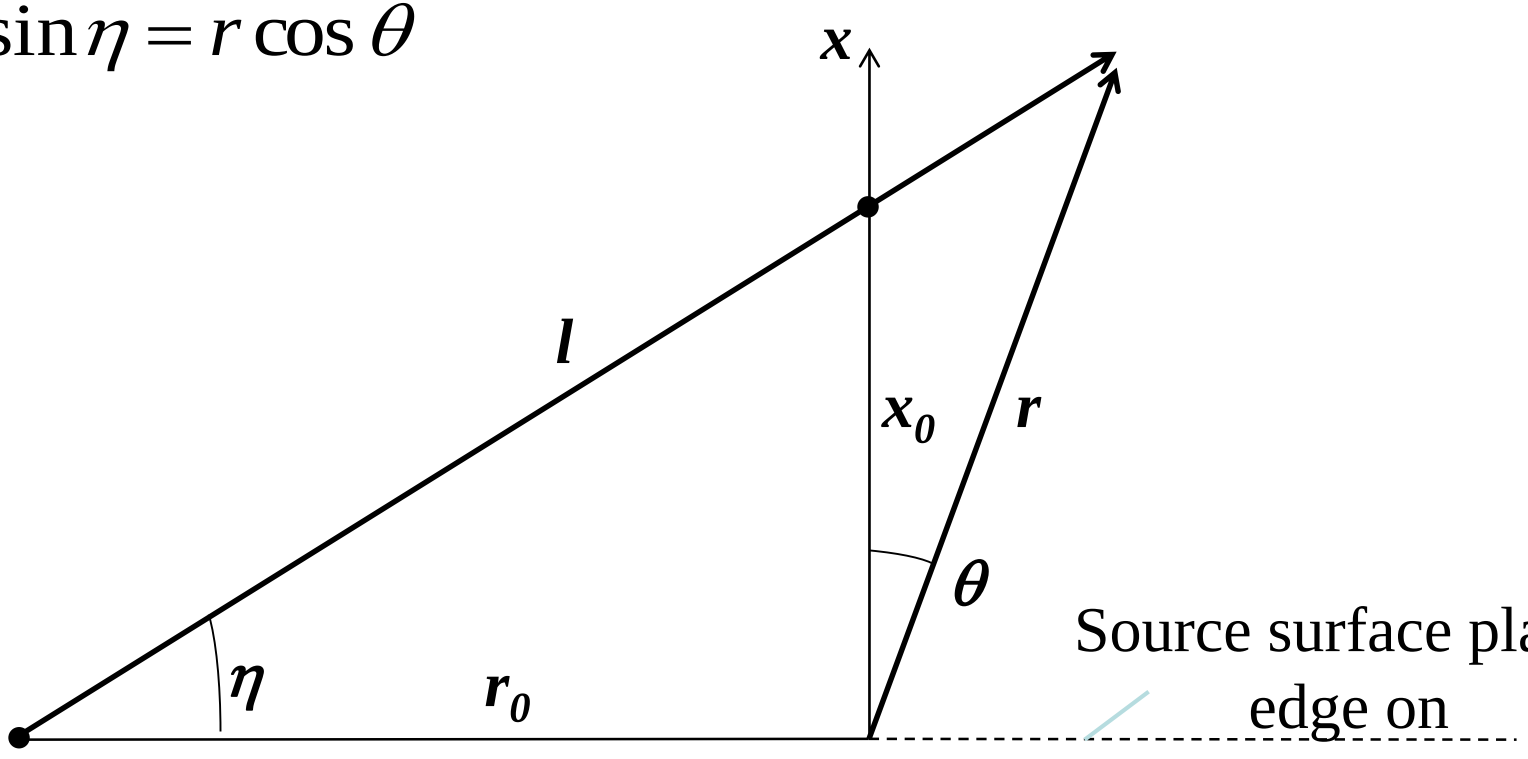
$$l = r - x_0$$

- For effusion, the centerline result is simply $\sigma_{cl,e}(x_0) = \dfrac{\dot{N}}{x_0\sqrt{8\pi RT}}$

- Since density is maximized along centerline, tempting to consider this path produces the highest CND. However, this is not so!

2-D Path, Surface Plane Intersection

$$l \sin \eta = r \cos \theta$$



2-D Path, Effusion

- Solution for effusion becomes

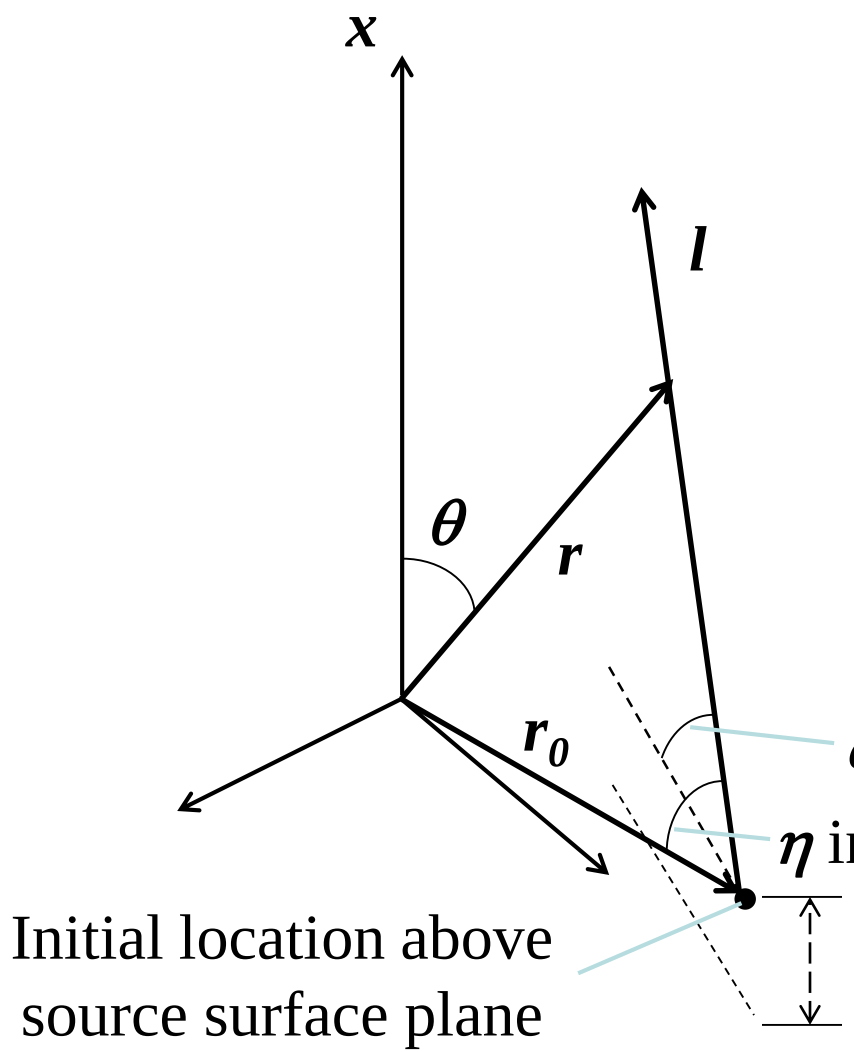
$$\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \frac{\sin\eta}{r_0(1-\cos\eta)} = \sigma_{cl,e} \frac{\tan\eta \sin\eta}{1-\cos\eta}$$

- In the limit where $r_0 \to \infty, \eta \to 0$
  - Vanishingly small distortion of triangle to describe a path parallel to source plane at height x_0, find

$$\sigma(\eta \to 0) \to 2\,\sigma_{cl,e}$$

  - Special case may be confirmed by evaluating σ along horizontal path at height x_0 directly
  - This case provides the maximum CND for effusion

3-D General Path



$$\boldsymbol{r}\cos\theta = \boldsymbol{l}\sin\omega + \boldsymbol{r_0}\cos\theta_0$$

3-D Path, Effusion

- Effusive gas solution:

$$\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \frac{\sin\omega - \cos\theta_0}{r_0\left(1-\cos\eta\right)}$$

- Solution still maximized for distant points along paths parallel to source plane separated by x_0
  - Collapses to previous solution

$$\sigma_{\max,e} \rightarrow 2\,\sigma_{cl,e}$$

Sonic Orifice Model

- When bulk fluid motion is involved ($s > 0$), plume model behavior becomes too complex to handle directly ($w = s\cos\theta$)

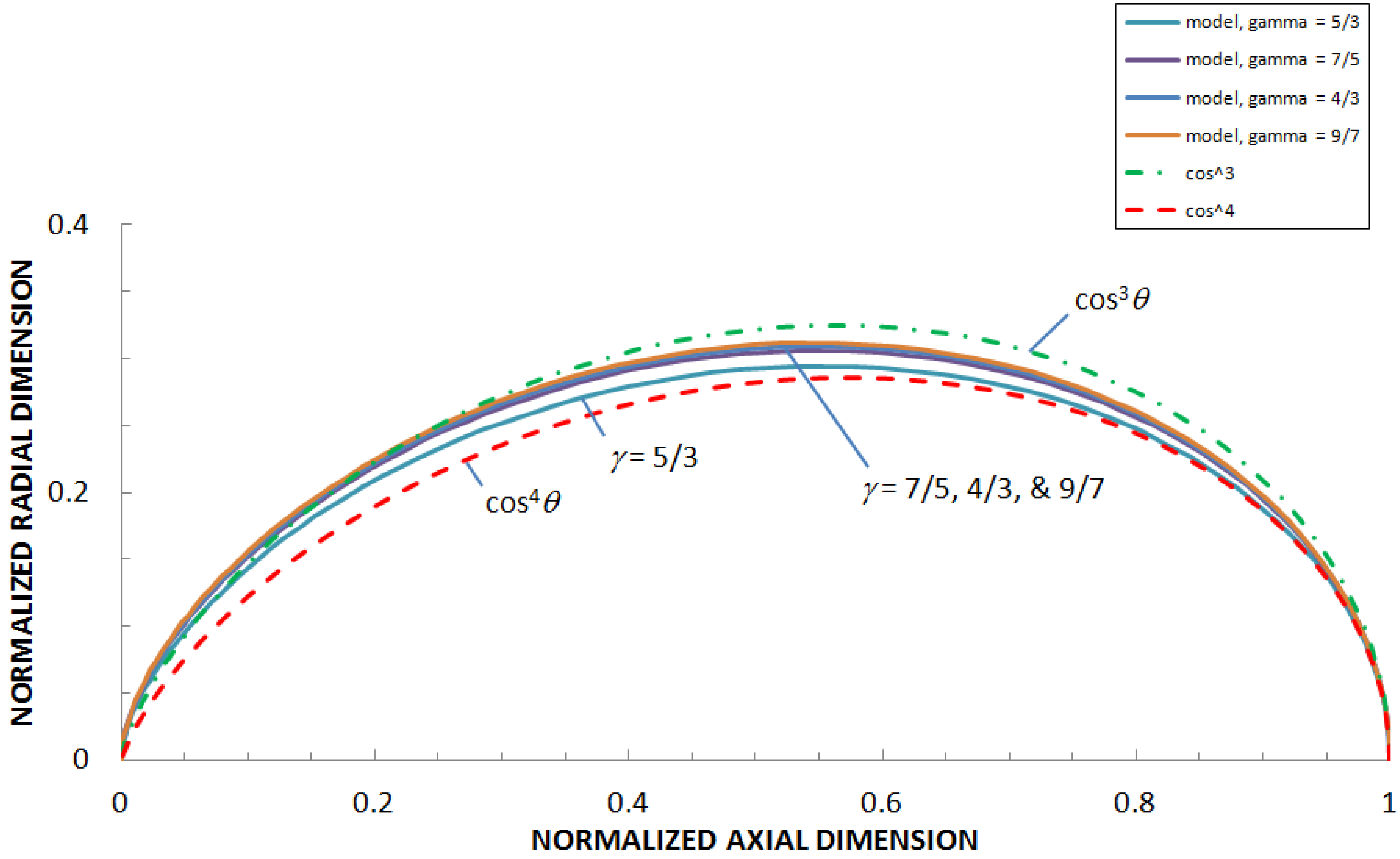
$$n(\mathbf{x},t) = \frac{\beta \dot{N} \cos\theta}{A_1 \pi r^2} e^{w^2 - s^2} \left\{ w e^{-w^2} + \left(\frac{1}{2} + w^2 \right) \sqrt{\pi} \left(1 + \operatorname{erf} w \right) \right\}$$

- Decided to approximate the model behavior, replacing angular distribution by $\cos^3\theta$

$$n_s(r,\theta) \approx K \frac{\cos^3\theta}{r^2}$$

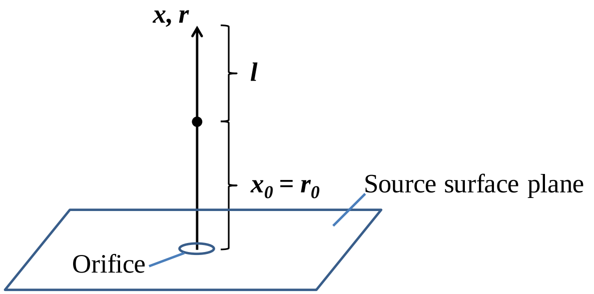
  - Good approximation for many species, different types

Sonic Angular Distribution Comparison



Some Sonic Model CNDs

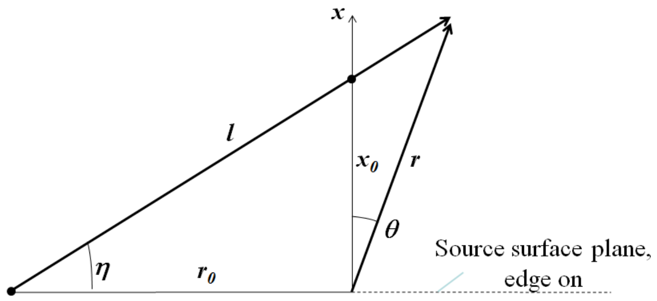
- 1-D centerline case: $\sigma_{cl,s} = \dfrac{K}{x_0}$



- 2-D, ∩ centerline & source surface plane:

$$\sigma_s = \frac{K}{3r_0 \sin\eta}\left[2(1+\cos\eta)+\frac{1}{2}\sin\eta\sin 2\eta\right]$$

$$= \frac{\sigma_{cl,s}}{3\cos\eta}\left[2(1+\cos\eta)+\frac{1}{2}\sin\eta\sin 2\eta\right]$$



  – Maximum effect: $\sigma(\eta \to 0) \to \dfrac{4}{3}\sigma_{cl,s}$

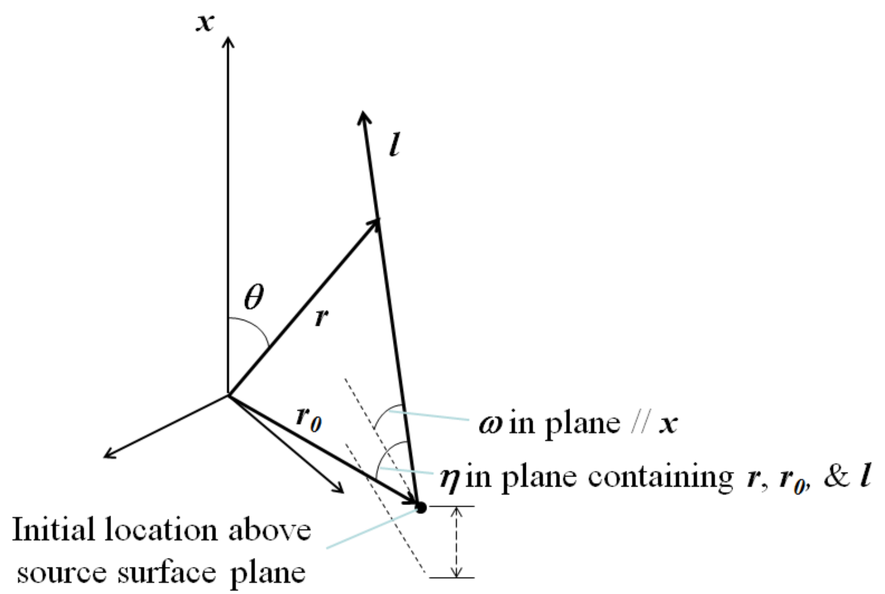
3-D Path, Sonic Approximation

- Generally,

$$\sigma = \frac{\sigma_{cl,s} \tan\eta}{3(1-\cos\eta)^2}\left\{\frac{\sin^3\omega(2+3\cos\eta-\cos^3\eta)}{(1+\cos\eta)^2} - 3\sin^2\omega\cos\theta_0 + 3\sin\omega\cos^2\theta_0 - \cos^3\theta_0(2-\cos\eta)\right\}$$

- Maximum effect when

$$\sigma_{\max,s} \to \frac{4}{3}\sigma_{cl,s}$$



Higher *M* CND Observations

- Assume adequate fit for our purposes using $n(r,\theta) \approx \tilde{K}\dfrac{\cos^m \theta}{r^2}$
- For axial, centerline case, find $\sigma_{cl} = \dfrac{\tilde{K}}{x_0}$
- From previous results, might think limiting transverse case becomes

$$\sigma_{xverse} \stackrel{?}{=} \frac{m+1}{m}\sigma_{cl}$$

  – Always larger than axial

- Actually

$$\sigma_{xverse} = \sigma_{cl} B\left(\frac{1}{2}, \frac{m+1}{2}\right) = \sqrt{\pi}\sigma_{cl}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

  – Axial case is larger for $m > 5$

Radial Point Source

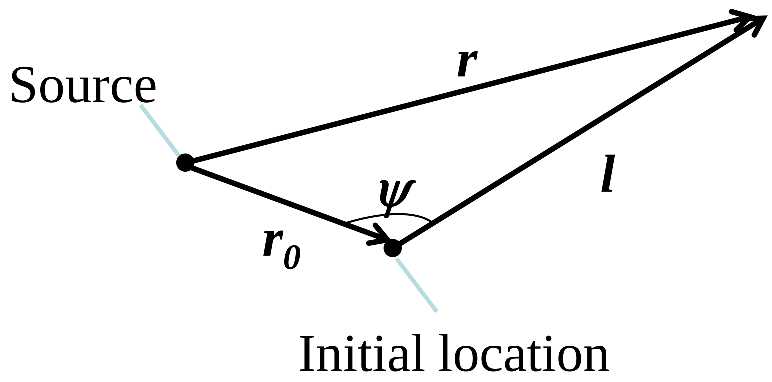
- Model spherically-symmetric expansion
  - No directional constraints
  - No bulk velocity (thermal expansion, s = 0)
- Use solution due to Narasimha

$$n_r(r) = \frac{\dot{N}}{\pi r^2 \sqrt{8\pi RT}}$$

Radial Point Source CND Expressions

- Generally,

$$\sigma = \frac{\dot{N}}{\pi r_0 \sqrt{8\pi RT}} \frac{\pi - \psi}{\sin\psi}$$



  - Notice $r_0 \sin\psi$ acts like x_0 in venting cases
- When $\psi = \pi$ (path along source radial line)

$$\sigma_r = \frac{\dot{N}}{\pi r_0 \sqrt{8\pi RT}}$$

- When $\psi = \pi/2$, path begins at right angles to source, $r_0 = x_0$, and

$$\sigma\left(\psi = \frac{\pi}{2}\right) = \frac{\pi}{2}\sigma_r$$

- Maximum CND found for path that extends to infinity in both directions:

$$\sigma_{\max,r} = \pi\,\sigma_r$$ (The $m = 0$ result!)

Concluding Remarks

- Undertook a study to determine closed form analytical solutions for a number of frequently encountered CND configurations
- For low-rate effusive venting and higher-rate sonic discharges, maximum CNDs should occur along paths parallel to the source plane that intersect the plume axis
- Maximum CNDs for paths immersed in the presence of an unconstrained radial source do not lie along radial trajectories
- For source angular distributions ~$\cos^m \theta$, it was shown for integer values of $m > 5$, maximum CND values switched from transverse to axial paths
  - Likely associated with spacecraft thruster plumes
- These analytical solutions and associated observations should greatly reduce the amount of effort needed to assess CNDs for a variety of space-related applications

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Backup Slides

Plume Model Formulation—Source

- Find particular solution to collisionless Boltzmann equation for source Q_1:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{g} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = Q_1$$

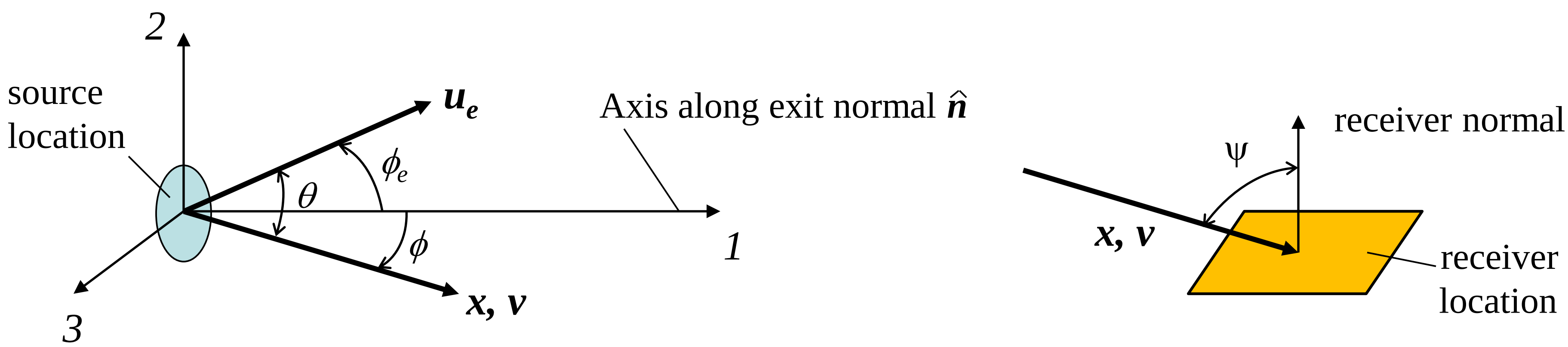
where Q_1 represents a Lambertian source superimposed on a bulk velocity

$$Q_1 \equiv \frac{2\beta^4}{A_1 \pi} \delta(\boldsymbol{x}) \dot{m}(t) |\boldsymbol{v} \cdot \hat{\boldsymbol{n}}| \exp\left(-\beta^2 (\boldsymbol{v} - \boldsymbol{u}_{\mathrm{e}})^2\right)$$

and the normalization factor is given by

$$A_1 \equiv e^{-s^2 \cos^2 \phi_{\mathrm{e}}} + \sqrt{\pi}\, s \cos\phi_{\mathrm{e}} \left(1 + \operatorname{erf}(s \cos\phi_{\mathrm{e}})\right)$$

Plume Model Formulation—Definitions



- Subscript *e* represents exit conditions from source
- Simplifies for axisymmetric conditions
  - $\phi_\mathrm{e} = 0$
  - $\phi = \theta$
- other definitions: $s \equiv \beta u_\mathrm{e} = \dfrac{u_\mathrm{e}}{\sqrt{2RT_\mathrm{e}}};\; w \equiv s\cos\theta$