Column Number Density Expressions Through $M = 0$ and $M = 1$ Point Source Plumes Along Any Straight Path

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Introduction

• Providers of externally-mounted scientific payloads at the International Space Station (ISS) are required to evaluate column number density (CND, $\sigma$) associated with various gas releases and demonstrate that they fall below some maximum requirement
  – Must be considerate of other payloads
  – Since this includes unknown future additions, becomes a search for maximum CND along any path

• Occasionally astrophysicists are interested in estimating the amount of gas released by some event or process by evaluating light attenuation of a distant star having known properties due to this release
  – Milky Way center, black hole, “Fermi Bubbles”
ATV Edoardo Amaldi Approaches ISS
“Fermi Bubbles”

ESA/Planck Collaboration (microwave); NASA/DOE/Fermi LAT/Dobler et al./Su et al. (gamma rays)
Objective

• Develop analytical CND expressions for general paths that intercept various common point sources under high vacuum conditions
  – Effusion/low rate evaporation/outgassing \((M = 0)\)
  – Venting via sonic orifice \((M = 1)\)
  – Spherically-symmetric, radial expansion \((M = 0)\)
Venting (Directed) Source Behavior

• External neutral gas phase sources on ISS result from a number of different physical mechanisms
  – Supersonic expansion through thruster nozzles
  – Pressure-driven acceleration to sonic conditions across an orifice
  – Surface evaporation, desorption (may or may not have bulk velocity)
  – Effusion—low-rate, high-$Kn$ venting ($M = 0$)
  – Diffusion-limited outgassing ($M = 0$)

• This study assumes that, for these applications, the point source may be described using free molecule flow model approximations
  – Density levels fall rapidly with distance from source location
  – Existence of self-scattering collisions may not substantially alter plume distribution from free molecule flow description
Directed Source—Steady Density

• Can compute many different types of local quantities at receiver position \( x \) relative to source
  
  – Steady number density \( n \) from a directed axisymmetric source given by

  \[
  n(x,t) = \frac{\beta \dot{N} \cos \theta}{A_1 \pi r^2} e^{w^2-s^2} \left\{ w e^{-w^2} + \left( \frac{1}{2} + w^2 \right) \sqrt{\pi} (1 + \text{erf} \ w) \right\}
  \]

  – Speed ratio \( s \equiv \beta u_e = \frac{u_e}{\sqrt{2RT_e}} \); \( w \equiv s \cos \theta \)

  – \( A_1 \): normalization factor, function of \( s \)
Column Number Density (CND, $\sigma$)

- Integrated effect of molecules encountered across a prescribed path $l$
  - When unbounded,
    $$\sigma = \int_{0}^{\infty} n \, dl$$

- For ISS application, the requirement not to exceed $\sigma_{\text{crit}}$ allows one to determine the physical envelope around the source where the limit is violated
  - With a singularity at the source origin, the model will always predict some critical envelope
    - Not consequential for low $\dot{N}$
Effusive CND Expressions

• For low rate, high-\(Kn\) venting through an orifice with thermal effusion, no bulk motion, plume model density simplifies to

\[
n(r, \theta) = \frac{\dot{N} \cos \theta}{r^2 \sqrt{8\pi RT}}
\]

– Also describes density field due to outgassing or low rate volatile evaporation from a planar surface viewed from a distance

• Column number density given by

\[
\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \int_0^\infty \frac{\cos \theta}{r^2} dl
\]
1-D Centerline Path

- For effusion, the centerline result is simply \( \sigma_{cl,e}(x_0) = \frac{\dot{N}}{x_0 \sqrt{8\pi RT}} \)

- Since density is maximized along centerline, tempting to consider this path produces the highest CND. However, this is not so!

\[ l = r - x_0 \]
2-D Path, Surface Plane Intersection

\[ l \sin \eta = r \cos \theta \]
2-D Path, Effusion

• Solution for effusion becomes

\[ \sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \frac{\sin \eta}{r_0(1 - \cos \eta)} = \sigma_{cl,e} \frac{\tan \eta \sin \eta}{1 - \cos \eta} \]

• In the limit where \( r_0 \to \infty, \eta \to 0 \)
  
  – Vanishingly small distortion of triangle to describe a path parallel to source plane at height \( x_0 \), find

  \[ \sigma(\eta \to 0) \to 2 \sigma_{cl,e} \]

  – Special case may be confirmed by evaluating \( \sigma \) along horizontal path at height \( x_0 \) directly

  – This case provides the maximum CND for effusion
3-D General Path

\[ r \cos \theta = l \sin \omega + r_0 \cos \theta_0 \]

Initial location above source surface plane

\( \omega \) in plane // \( x \)

\( \eta \) in plane containing \( r, r_0, \) & \( l \)
3-D Path, Effusion

• Effusive gas solution:

\[
\sigma = \frac{\dot{N}}{\sqrt{8\pi RT}} \frac{\sin \omega - \cos \theta_0}{r_0 (1 - \cos \eta)}
\]

• Solution still maximized for distant points along paths parallel to source plane separated by \(x_0\)
  – Collapses to previous solution

\[
\sigma_{\text{max},e} \rightarrow 2\sigma_{cl,e}
\]
Sonic Orifice Model

- When bulk fluid motion is involved \((s > 0)\), plume model behavior becomes too complex to handle directly \((w = s \cos \theta)\)

\[
n(x, t) = \frac{\beta \dot{N} \cos \theta}{A_1 \pi r^2} e^{w^2 s^2} \left\{ w e^{-w^2} + \left( \frac{1}{2} + w^2 \right) \sqrt{\pi} \left( 1 + \text{erf} \ w \right) \right\}
\]

- Decided to approximate the model behavior, replacing angular distribution by \(\cos^3 \theta\)

\[
n_s(r, \theta) \approx K \frac{\cos^3 \theta}{r^2}
\]

- Good approximation for many species, different types
Sonic Angular Distribution Comparison

![Diagram showing normalized radial and axial dimensions with various gamma values and angular distributions]

- Model, gamma = 5/3
- Model, gamma = 7/5
- Model, gamma = 4/3
- Model, gamma = 9/7
- cos^3
- cos^4
- γ = 5/3
- γ = 7/5, 4/3, & 9/7

Normalized Radial Dimension vs. Normalized Axial Dimension
Some Sonic Model CNDs

- 1-D centerline case: \( \sigma_{cl,s} = \frac{K}{x_0} \)

- 2-D, \( \cap \) centerline & source surface plane:

\[
\sigma_s = \frac{K}{3r_0 \sin \eta} \left[ 2(1 + \cos \eta) + \frac{1}{2} \sin \eta \sin 2\eta \right]
= \frac{\sigma_{cl,s}}{3 \cos \eta} \left[ 2(1 + \cos \eta) + \frac{1}{2} \sin \eta \sin 2\eta \right]
\]

- Maximum effect: \( \sigma(\eta \to 0) \to \frac{4}{3} \sigma_{cl,s} \)
3-D Path, Sonic Approximation

• Generally,

\[
\sigma = \frac{\sigma_{cl,s} \tan \eta}{3(1 - \cos \eta)^2} \left\{ \frac{\sin^3 \omega (2 + 3 \cos \eta - \cos^3 \eta)}{(1 + \cos \eta)^2} - 3 \sin^2 \omega \cos \theta_0 + 3 \sin \omega \cos^2 \theta_0 - \cos^3 \theta_0 (2 - \cos \eta) \right\}
\]

• Maximum effect when

\[
\sigma_{\text{max},s} \rightarrow \frac{4}{3} \sigma_{cl,s}
\]
Higher $M$ CND Observations

- Assume adequate fit for our purposes using $n(r, \theta) \approx \tilde{K} \frac{\cos^m \theta}{r^2}$

- For axial, centerline case, find $\sigma_{cl} = \frac{\tilde{K}}{x_0}$

- From previous results, might think limiting transverse case becomes

$$\sigma_{xverse} = \frac{m+1}{m} \sigma_{cl}$$

  - Always larger than axial

- Actually

$$\sigma_{xverse} = \sigma_{cl} B\left(\frac{1}{2}, \frac{m+1}{2}\right) = \sqrt{\pi} \sigma_{cl} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

  - Axial case is larger for $m > 5$
Radial Point Source

• Model spherically-symmetric expansion
  – No directional constraints
  – No bulk velocity (thermal expansion, \( s = 0 \))

• Use solution due to Narasimha

\[
n_r(r) = \frac{\dot{N}}{\pi r^2 \sqrt{8\pi RT}}
\]
Radial Point Source CND Expressions

- Generally,
  \[ \sigma = \frac{\dot{N}}{\pi r_0 \sqrt{8\pi RT}} \frac{\pi - \psi}{\sin \psi} \]
  - Notice \( r_0 \sin \psi \) acts like \( x_0 \) in venting cases
- When \( \psi = \pi \) (path along source radial line)
  \[ \sigma_r = \frac{\dot{N}}{\pi r_0 \sqrt{8\pi RT}} \]
- When \( \psi = \pi / 2 \), path begins at right angles to source, \( r_0 = x_0 \), and
  \[ \sigma \left( \psi = \frac{\pi}{2} \right) = \frac{\pi}{2} \sigma_r \]
- Maximum CND found for path that extends to infinity in both directions:
  \[ \sigma_{\text{max},r} = \pi \sigma_r \]
  (The \( m = 0 \) result!)
Concluding Remarks

- Undertook a study to determine closed form analytical solutions for a number of frequently encountered CND configurations
- For low-rate effusive venting and higher-rate sonic discharges, maximum CNDs should occur along paths parallel to the source plane that intersect the plume axis
- Maximum CNDs for paths immersed in the presence of an unconstrained radial source do not lie along radial trajectories
- For source angular distributions $\sim \cos^m \theta$, it was shown for integer values of $m > 5$, maximum CND values switched from transverse to axial paths
  - Likely associated with spacecraft thruster plumes
- These analytical solutions and associated observations should greatly reduce the amount of effort needed to assess CNDs for a variety of space-related applications
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Backup Slides
Plume Model Formulation—Source

• Find particular solution to collisionless Boltzmann equation for source $Q_1$:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{g} \cdot \frac{\partial f}{\partial \mathbf{v}} = Q_1$$

where $Q_1$ represents a Lambertian source superimposed on a bulk velocity

$$Q_1 \equiv \frac{2\beta^4}{A_1 \pi} \left[ \frac{\delta(x) m(t) |\mathbf{v} \cdot \hat{n}| \exp(-\beta^2 (\mathbf{v} - \mathbf{u}_e)^2)}{A_1 \pi} \right]$$

and the normalization factor is given by

$$A_1 \equiv e^{-s^2 \cos^2 \phi_e} + \sqrt{\pi} s \cos \phi_e (1 + \text{erf}(s \cos \phi_e))$$
• Subscript $e$ represents exit conditions from source
• Simplifies for axisymmetric conditions
  – $\phi_e = 0$
  – $\phi = \theta$
• other definitions: $s \equiv \beta u_e = \frac{u_e}{\sqrt{2RT_e}}$; $w \equiv s \cos \theta$