A TIME-REGULARIZED MULTIPLE GRAVITY-ASSIST LOW-THRUST BOUNDED-IMPULSE MODEL FOR TRAJECTORY OPTIMIZATION

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The multiple gravity assist low-thrust (MGALT) trajectory model combines the medium-fidelity Sims-Flanagan bounded-impulse transcription with a patched-conics flyby model and is an important tool for preliminary trajectory design. While this model features fast state propagation via Kepler's equation and provides a pleasingly accurate estimation of the total mass budget for the eventual flight-suitable integrated trajectory it does suffer from one major drawback, namely its temporal spacing of the control nodes. We introduce a variant of the MGALT transcription that utilizes the generalized anomaly from the universal formulation of Kepler's equation as a decision variable in addition to the trajectory phase propagation time. This results in two improvements over the traditional model. The first is that the maneuver locations are equally spaced in generalized anomaly about the orbit rather than time. The second is that the Kepler propagator now has the generalized anomaly as its independent variable instead of time and thus becomes an iteration-free propagation method. The new algorithm is outlined, including the impact that this has on the computation of Jacobian entries for numerical optimization, and a motivating application problem is presented that illustrates the improvements that this model has over the traditional MGALT transcription.

INTRODUCTION

Bounded-impulse trajectory models are an integral part of many preliminary design methodologies. For the case of continuous-thrust trajectory design, the multiple gravity-assist low-thrust (MGALT) model has been incorporated into many tool chains such as the Evolutionary Mission Trajectory Generator (EMTG), Mission Analysis Low-Thrust Optimization (MALTO), Gravity-Assist Low-thrust Local Optimization Program (GALLOP) and the Parallel Global Multiobjective Optimizer (PaGMO). The MGALT transcription combines the medium-fidelity Sims-Flanagan bounded-impulse model, which divides a trajectory phase into \( N \) equal sized time segments with a bounded-impulse placed at the center of each segment, with a patched-conics flyby model. Its strength lies in generating accurate mass budgets for interplanetary missions. MGALT optimization runs generate medium-fidelity solutions and, therefore, are typically used as seeds for higher fidelity models such as the finite-burn low-thrust (FBLT) model and flight-grade tools such as GMAT and Mystic.

While MGALT is capable of accurately approximating many types of interplanetary trajectories, certain classes of missions such as those requiring multiple-revolutions of the central body, with no intermediate flybys, as well as those requiring large eccentricity changes present more of a challenge to this model. The more revolutions that the spacecraft requires to complete a transfer, the more control discretization nodes are required to ensure that the bounded-impulse trajectory will be suitable as a seed to a finite-burn model. For trajectories requiring large eccentricity changes, the original Sims-Flanagan model, which evenly spaces control nodes in time, places these nodes at precisely non-optimal locations. That is, for high-eccentricity orbits, the control nodes cluster around apoapsis and are sparse at periapsis where energy changes are more efficiently achieved and therefore where higher control authority is typically desirable. The only way to mitigate this shortcoming of MGALT is to add more control points, which increases the computational complexity of the problem. For this reason, a bounded-impulse transcription that requires fewer control nodes and,

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at a minimum, results in a geometrically even distribution of the control points around the trajectory phase would be desirable.

Redistribution of the control nodes is typically achieved using a time-regularization transformation such as a Sundman transformation. This has recently been investigated by Pellegrini et al.\textsuperscript{10} who applied generalized Sundman transformations to the Stark and Kepler (sub-case of Stark) problems. Their Stark solution method is based on a series solution, where the length of the series determines the accuracy of the solution. The Stark model was also extended for use with a Taylor integrator similar to the work by Yam et al.\textsuperscript{11} In this paper, we introduce an exact method for solving the time-regularized Kepler problem that does not rely on a series solution. Solving Kepler’s equation typically requires iterative techniques such as Newton’s method, the Laguerre-Conway method or bisection. We propose a strategy that transforms a universal Kepler propagator into an iteration-free method at the expense of the introduction of one additional decision variable per trajectory phase (to be chosen by an NLP optimizer). These new variables are the total generalized anomaly to propagate the spacecraft state by in a trajectory phase ($\chi_p$). The addition of these decision parameters results in an execution speed increase due to the elimination of numerical iteration in the propagator, and also a more favorable distribution of control nodes for highly eccentric trajectories. In addition, it is possible to compute analytic partial derivatives of the problem constraint vector $c$ with respect to the $\chi_p$ variables. The redistribution of nodes using this method is a Sundman transformation,\textsuperscript{10,12} as such we will hereafter refer to the modified model as the multiple gravity-assist low-thrust Sundman (MGALTS) transcription.

**ALGORITHM**

For the case of the original MGALT model, a numerical optimizer selects the phase flight time variable ($\Delta t_p$), and a Kepler solver then determines the corresponding change in eccentric anomaly, or generalized anomaly, for a universal variable formulation. When time-regularization is applied, where control nodes are distributed in a geometrically even sense along the trajectory, the angular variable ($E$ or $\chi$) becomes the decision variable in lieu of the flight time. This is a non-intuitive variable for the human mission designer to select, however, as any given angular displacement can result in dramatically different flight times depending on the geometry of the orbit. Requiring a mission designer to select angular displacements instead of flight times is really not feasible. An even more difficult task for a human would be to determine, \textit{a priori}, the temporal spacing between each control node required to achieve perfect geometrical distribution of the control nodes.

Instead of each of these undesirable jobs, in this method we remove the burden of having a mission designer select the angular displacement and shift the responsibility to the NLP solver. Then, the universal Kepler solver is used in an inverse sense to compute the time-of-flight required to satisfy an even distribution of control points, in addition to propagating the state without the need for iteration. The procedure for evaluating a trajectory phase, and for obtaining an inverse sense to compute the time-of-flight required to satisfy an even distribution of control points, in addition to propagating the state without the need for iteration. The procedure for evaluating a trajectory phase, and for obtaining an inverse sense to compute the time-of-flight required to satisfy an even distribution of control points, in addition to propagating the state without the need for iteration.

1. $\chi_p$ is an NLP decision variable and is equal to the total generalized anomaly required to propagate one trajectory phase. The size of a propagation segment is $\Delta \chi_k = \frac{2\pi}{N}$

2. Starting at the left boundary, propagate inwards to the phase center for $\frac{N}{2}$ segments of generalized anomaly

3. For each $k^{th}$ propagation step in the forward half-phase:
   
   (a) compute $\alpha_k$ to determine if the current conic orbit is elliptic, hyperbolic or parabolic (this affects the computation of the universal variables $U_{n_k}$ and their derivatives)
   
   (b) compute the universal functions $U_{n_k}$ $n = 0, 1, 2, 3$
   
   (c) compute $\frac{\partial U_{n_k}}{\partial \alpha_k}$ and $\frac{\partial U_{n_k}}{\partial \chi_k}$ $n = 0, 1, 2, 3$ (for use in computing step g).
   
   (d) compute the segment propagation time $\Delta t_k$ resulting from a propagation through $\Delta \chi_k$ of generalized anomaly as well as $\frac{\partial \Delta t_k}{\partial \chi_k}$
   
   (e) compute the Lagrange coefficients and their derivatives $F_k, \frac{\partial F_k}{\partial \chi_k}, \dot{F}_k, \frac{\partial \dot{F}_k}{\partial \chi_k}, G_k, \frac{\partial G_k}{\partial \chi_k}, \dot{G}_k$, and $\frac{\partial \dot{G}_k}{\partial \chi_k}$
   
   (f) compute the propagated state $X_k^- = [r_k^- \ v_k^-]^T$
   
   (g) compute $\frac{\partial X_k^-}{\partial \chi_{k-1}}$ and $\frac{\partial \Delta t_k}{\partial X_{k-1}^-}$, which form the basis for computing match-point derivatives
   
   (h) compute the maximum available thrust $T_{\text{max}_k}$
   
   (i) compute $\Delta v_{\text{max},k}$
In Equation (1), \( N_{\text{active}} \) is the number of active thrusters, \( D \) is the thruster duty cycle, \( T_{\text{max}} \) is the maximum available thrust for the current segment computed using the propulsion system power model and \( m_k \) is the mass of the spacecraft at the left-hand side of the segment. The spacecraft’s mass across the \( k^{th} \) bounded impulse is computed using the following equation:

\[
\Delta v_{\text{max}}^k = \frac{N_{\text{active}} D T_{\text{max}} \Delta t_k}{m_k}
\]

Application: Rendezvous with Comet 45P/Honda-Mrkos-Pajdušáková

The problem selected to illustrate the benefits of time regularization is a notional rendezvous with the comet 45P/Honda-Mrkos-Pajdušáková. This is a short-period comet with a period of approximately 5.25 years. We consider a direct flight to the comet, with no intermediate flybys. This rendezvous problem was solved once using the MGALT transcription (Figure 1) and then a second time using the MGALTS transcription (Figure 2) using the exact same configuration parameters. The NLP objective function was the minimization of the propellant consumed. Specific information about power modeling and throttle logic used is omitted here but is available, and adheres to Discovery class mission proposal requirements.13, 14
Table 1. 45P/Honda-Mrkos-Pajdušáková rendezvous mission parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum allowed initial mass</td>
<td>4000 kg</td>
</tr>
<tr>
<td>Earliest allowed launch date</td>
<td>August 27th, 2016</td>
</tr>
<tr>
<td>Launch window</td>
<td>5 years</td>
</tr>
<tr>
<td>Maximum flight time</td>
<td>unbounded</td>
</tr>
<tr>
<td>Latest allowed arrival date</td>
<td>January 1st, 2030</td>
</tr>
<tr>
<td>Launch vehicle</td>
<td>Atlas V (555) NLS-2</td>
</tr>
<tr>
<td>Maximum launch C3</td>
<td>21.7156 km²/s²</td>
</tr>
<tr>
<td>Launch declination bounds</td>
<td>[−28.5°, 28.5°]</td>
</tr>
<tr>
<td>Arrival type</td>
<td>rendezvous</td>
</tr>
<tr>
<td>Thruster</td>
<td>2 x NEXT TT11 high thrust</td>
</tr>
<tr>
<td>Throttle logic</td>
<td>minimum number of thrusters</td>
</tr>
<tr>
<td>Thruster duty cycle</td>
<td>0.95</td>
</tr>
<tr>
<td>Solar power coefficients</td>
<td>γ₀ = 1.32077, γ₁ = −0.10848, γ₂ = −0.11665, γ₃ = 0.10843, γ₄ = −0.01279</td>
</tr>
<tr>
<td>Spacecraft bus power requirement coefficients</td>
<td>aₜ/s = 1.0, bₜ/s = 0.0, cₜ/s = 0.0</td>
</tr>
<tr>
<td>Solar array BOL power (launch at 1 A.U.)</td>
<td>40.0 kW</td>
</tr>
<tr>
<td>Power margin (bₜ_launch)</td>
<td>15%</td>
</tr>
<tr>
<td>Post-launch checkout coast</td>
<td>60 days</td>
</tr>
<tr>
<td>Number of segments per phase</td>
<td>200</td>
</tr>
<tr>
<td>Ephemeris</td>
<td>SPICE</td>
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<tr>
<td>SNOPT feasibility tolerance</td>
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</tr>
<tr>
<td>SNOPT optimality tolerance</td>
<td>1.0e-4</td>
</tr>
<tr>
<td>Objective function</td>
<td>maximize final mass</td>
</tr>
</tbody>
</table>

Event #1: Launch Earth 11/21/2017
C₃ = 21.716 km²/s²
DLA = 20.1°
m = 4000 kg

Event #2: Lt rendezvous 45P 12/29/2027
m = 3238 kg

Figure 1. Rendezvous with 45P using MGALT with 200 segments, Δtmission = 3690 days

shorter time. The MGALT solution on the other hand is not able to achieve the same energy change per periapsis passage. This works out in a particularly unlucky way for this mission as it results in the spacecraft having to make an entire extra revolution of the sun in order to carry out the rendezvous. Since the MGALT node placement is less optimal, the optimizer adds an additional revolution in order to place the maneuvers more effectively, which in turn requires that the spacecraft launch earlier.

By increasing the number of control nodes for the MGALT model, a solution in the same family as the MGALTS solution can be obtained due to the increased number of nodes at periapsis. Figure 5 shows the new MGALT solution,
Event # 1:
Launch
Earth
11/10/2019
\( C_3 = 21.716 \text{ km}^2/\text{s}^2 \)
DLA = 16.6\(^\circ\)
m = 4000 kg

Event # 2:
Lt rendezvous
45P
10/28/2027
m = 3217 kg

Figure 2. Rendezvous with 45P using time-regularized MGALTS with 200 segments, \( \Delta t_{\text{mission}} = 2909 \text{ days} \)

Figure 3. MGALT throttle magnitude and distance from the sun as a function of time.

This time using 400 segments instead of 200. Increasing the segment count by degrees will eventually yield a trajectory with a more comparable flight time to the MGALTS solution.
Figure 4. MGALTS throttle magnitude and distance from the sun as a function of time.

Figure 5. Rendezvous using the time-regularized MGALT transcription with 400 segments

CONCLUSION
Time-regularization of the bounded-impulse Kepler low-thrust model has been achieved using an exact, non-iterative analytic algorithm. It is also possible to obtain analytic partial derivatives for the nonlinear constraints defining this
transcription. This methodology is particularly effective at solving trajectory problems requiring large eccentricity changes, especially those requiring multiple revolutions of the central body. By replacing time with the generalized anomaly as the independent propagation variable, the control nodes of the MGALT model can be distributed in an even geometric fashion, increasing the number of nodes near periapsis. This important feature of the model allows for a discrete control history that more closely matches the optimal one that could be found using the necessary conditions of the Calculus of Variations. This difference can lead to trajectory solutions with a similar cost function value to those produced by the MGALT model, but vary significantly in topology. This is illustrated by the example comet rendezvous problem that was solved in this paper where a nearly identical cost function (within 1%) was achieved with MGALT and MGALTS, however, the flight time of the MGALTS solution was slightly over two years shorter.

APPENDIX

Kepler Propagation with Universal Functions

The transcription described in this paper utilizes Keplerian two-body propagation using the Lagrange coefficients,\(^1^2\)

\[
\begin{bmatrix}
  r_k \\
  v_k
\end{bmatrix} =
\begin{bmatrix}
  F_k & G_k \\
  \dot{F}_k & \dot{G}_k
\end{bmatrix}
\begin{bmatrix}
  r_{k-1} \\
  v_{k-1}
\end{bmatrix},
\]

(3)

For use with a numerical optimizer, it is generally beneficial to use a two-body propagation method capable of robustly propagating any orbit initial condition (i.e. elliptical, parabolic or hyperbolic) initialized by a search method. For this reason, a propagation method based on universal orbit variables is employed. The universal variables are defined according to the energy regime of the orbit:

\[
\begin{align*}
\alpha_k &> 0 \\
U_{1k} &= \chi_k (1 - y_k S_k) \\
U_{2k} &= \chi_k C_k \\
U_{3k} &= \chi_k S_k \\
U_{0k} &= 1 - \alpha_k U_{2k}
\end{align*}
\]

where,

\[
\begin{align*}
\alpha_k &= \frac{1}{a_k} = \frac{2}{r_k} - \frac{v_k^2}{\mu} \\
y_k &= \alpha_k \chi_k^2
\end{align*}
\]

(5)

(6)

\[
C_k = \frac{1}{y_k} (1 - \cos(\sqrt{y_k}))
\]

(7)

\[
S_k = \frac{1}{y_k^3} (\sqrt{y_k} \sin(\sqrt{y_k}))
\]

(8)

\[
\begin{align*}
\alpha_k &< 0 \\
U_{0k} &= \cosh(\sqrt{-\alpha_k} \chi_k) \\
U_{1k} &= \frac{1}{\sqrt{-\alpha_k}} \sinh(\sqrt{-\alpha_k} \chi_k) \\
U_{2k} &= \frac{1}{\alpha_k} (1 - U_{0k}) \\
U_{3k} &= \frac{1}{\alpha_k} (\chi_k - U_{1k})
\end{align*}
\]

(9)

\[
\begin{align*}
\alpha_k &= 0 \\
U_{0k} &= 1 \\
U_{1k} &= \chi_k \\
U_{2k} &= \frac{1}{2} U_{1k} \chi_k \\
U_{3k} &= \frac{1}{3} U_{2k} \chi_k
\end{align*}
\]

(10)

The propagation time \(\Delta t_k\) associated with a corresponding change in generalized anomaly \(\chi_k\) is computed as follows:

\[
\Delta t_k = \frac{1}{\sqrt{\mu}} \left( r_{01} U_1 + \sigma_0 U_2 + U_3 \right)
\]

(11)

\[
\frac{\partial \Delta t_k}{\partial \chi_k} = \frac{1}{\sqrt{\mu}} \left( r_{01} \frac{\partial U_1}{\partial \chi_k} + \sigma_0 \frac{\partial U_2}{\partial \chi_k} + \frac{\partial U_3}{\partial \chi_k} \right)
\]

(12)
where,

\[ \sigma_k = \frac{r_k \cdot v_k}{\sqrt{\mu}} \quad (13) \]

The distance from the central body at the end of the propagation segment is:

\[ r_{k+1} = r_k U_{0_k} + \sigma_k U_{1_k} + U_{2_k} \quad (14) \]

The Lagrange coefficients and their derivatives are given by:

\[ F_k = 1 - \frac{U_{2_k}}{r_k} \quad (15) \]
\[ \frac{\partial F_k}{\partial \chi_k} = -\frac{1}{r_k} \frac{\partial U_{2_k}}{\partial \chi_k} \quad (16) \]
\[ \dot{F}_k = -\sqrt{\mu} \frac{U_{1_k}}{r_k} \quad (17) \]
\[ \frac{\partial \dot{F}_k}{\partial \chi_k} = \sqrt{\mu} \left( \frac{U_{1_k}}{r_k^2} \frac{\partial r_{k+1}}{\partial \chi_k} - \frac{1}{r_{k+1}} \frac{\partial U_{1_k}}{\partial \chi_k} \right) \quad (18) \]
\[ G_k = \frac{1}{\sqrt{\mu}} \left( r_k U_{1_k} + \sigma_k U_{2_k} \right) \quad (19) \]
\[ \frac{\partial G_k}{\partial \chi_k} = \frac{1}{\sqrt{\mu}} \left( r_k \frac{\partial U_{1_k}}{\partial \chi_k} + \sigma_k \frac{\partial U_{2_k}}{\partial \chi_k} \right) \quad (20) \]
\[ \dot{G}_k = 1 - \frac{U_{2_k}}{r_{k+1}} \quad (21) \]
\[ \frac{\partial \dot{G}_k}{\partial \chi_k} = \frac{U_{2_k}}{r_{k+1}^2} \frac{\partial r_{k+1}}{\partial \chi_k} - \frac{1}{r_{k+1}} \frac{\partial U_{2_k}}{\partial \chi_k} \quad (22) \]

where,

\[ \frac{\partial r_{k+1}}{\partial \chi_k} = r_k \frac{\partial U_{0_k}}{\partial \chi_k} + \sigma_k \frac{\partial U_{1_k}}{\partial \chi_k} + \frac{\partial U_{2_k}}{\partial \chi_k} \quad (23) \]
\[ \dot{r}_{k+1} = r_k \dot{U}_{0_k} + \sigma_k \dot{U}_{1_k} + \dot{U}_{2_k} \quad (24) \]
\[ \sigma_{k+1} = \sigma_k U_{0_k} + (1 - \alpha_k r_k) U_{1_k} \quad (25) \]
\[ \frac{\partial U_{0_k}}{\partial \chi_k} = -\alpha_k U_{1_k}; \quad \frac{\partial U_{n_k}}{\partial \chi_k} = U_{n-1_k} \quad n = 1, 2, \ldots \quad (26) \]

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REFERENCES


