Conjunction Assessment Risk Analysis

Time Dependence of Collision Probabilities During Satellite Conjunctions

Doyle T. Hall\textsuperscript{1}, Matthew D. Hejduk\textsuperscript{2}, and Lauren C. Johnson\textsuperscript{1}

The 27\textsuperscript{th} AAS/AIAA Space Flight Mechanics Meeting
San Antonio TX, 2017 Feb 5-9

\textsuperscript{1}Omitron Inc. \quad \textsuperscript{2}Astrorum Consulting LLC
Outline

• Motivation and objectives

• Overview of collision probability theory

• Analysis of well-studied conjunctions

• Analysis of archived conjunctions

• Conclusions
Motivation and Objectives

• Motivation: The probability of collision, $P_c$, between two Earth-orbiting satellites can often *but not always* be approximated adequately using the “2D $P_c$” formulation.

• Objective: Implement an improved method to estimate collision probabilities
  – Use Coppola’s analytical “3D $P_c$” formulation*
  – Validate using well-studied test cases and Monte Carlo methods
  – Compare 2D and 3D $P_c$ for archived conjunctions

---

• Motivation and objectives

• Overview of collision probability theory
  Monte Carlo methods, 3D $P_c$ theory, 2D approximations
  • Analysis of well-studied conjunctions
  • Analysis of archived conjunctions

• Conclusions
Monte Carlo $P_c$ Estimation

• Collision probabilities can be estimated using Monte Carlo simulations
  – Computationally intensive, especially for low probability events

• Alfano* analyzes twelve conjunctions in detail using Monte Carlo simulations
  – Benchmark test cases that can be used for validation of the 3D $P_c$ software
  – Includes cases where the 2D $P_c$ method both succeeds and fails

Coppola’s 3D $P_c$ Formulation

• Coppola* provides an analytically-derived formulation to calculate $P_c$ and its time derivative

$$P_c = P_0 + \int_{t_0}^{t_0+T} \left( \frac{dP_c}{dt} \right) dt$$

1D time integral

$$\frac{dP_c}{dt} = \int C \left( \hat{\mathbf{r}} \right) d^2\hat{\mathbf{r}}$$

2D unit - sphere integral

• These integrals must be calculated numerically

Coppola provides an analytically-derived formulation to calculate $P_c$ and its time derivative.

$$P_c = P_0 + \int_{t_0}^{t_0+T} \left( \frac{dP_c}{dt} \right) dt$$

1D time integral

$$\frac{dP_c}{dt} = \int_{4\pi} I(\hat{\mathbf{r}}, t) d^2\hat{\mathbf{r}}$$

2D unit - sphere integral

Analyzing the probability rate* provides new insight into the time dependence of conjunction risks

Coppola’s Conjunction Time Bounds

• Coppola* also provides estimates for the bounding times of a conjunction
  – These often bracket the nominal time of closest approach (TCA), but not always

• These bounds only provide a first-cut approximation for the limits of the numerical integration over time
  – These limits sometimes need to be expanded to bracket sufficiently the time(s) when $dP_c/dt$ peaks

The 3D $P_c$ method is general enough to use:
- Gaussian Mixture Model state distributions*
- Complex dynamical motion models
- Full 6x6 time-dependent state covariances

CARA’s current implementation uses:
- Single Gaussian ECI state distributions
- The Keplerian two-body motion model
- Full 6x6 ECI-state covariances, propagated using an analytically-derived state transition matrix#

Future plans include more advanced approaches

Schematic Illustration of the Encounter Region

Illustration based on Alfano’s* test case #2

NOTE: Actual $1\sigma$ surfaces are much larger and thinner

2D $P_C$ assumptions:
- Presumes straight trajectory (green)
- Presumes static covariances (blue)

3D $P_C$ assumptions:
- Trajectories are curvilinear (black)
- Covariances vary throughout the encounter (pink, orange)

Approximations Used for 2D $P_c$ Estimation

- In terms of the relative position/velocity state vector and the associated 6x6 covariance matrix:

$$
\mathbf{x}(t) = \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix} \quad \approx \quad \begin{bmatrix} \mathbf{r}(t_{ca}) + (t - t_{ca})\mathbf{v}(t_{ca}) \\ \mathbf{v}(t_{ca}) \end{bmatrix}
$$

$$
\mathbf{P}(t) = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t)^T \\ \mathbf{B}(t) & \mathbf{C}(t) \end{bmatrix} \quad \approx \quad \begin{bmatrix} \mathbf{A}(t_{ca}) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}
$$

- Here $t_{ca} = TCA = \text{the time of closest approach}$
Using CARA’s current 3D $P_c$ software, the three approximations used in the 2D $P_c$ method can be relaxed in a step-by-step manner:

<table>
<thead>
<tr>
<th>Step</th>
<th>Linear motion</th>
<th>Keplarian 2-body motion</th>
<th>Time-varying position covariances</th>
<th>Position+velocity covariances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coppola 1</td>
<td>$A = A(TCA)$, $B = C = 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coppola 2</td>
<td>$A = A(TCA)$, $B = C = 0$</td>
<td>$A = A(t)$, $B = C = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coppola 3</td>
<td>$A = A(t)$, $B = C = 0$</td>
<td>$A = A(t)$, $B = C = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coppola 4</td>
<td>$A = A(TCA)$, $B = C = 0$</td>
<td>$A = A(t)$, $B = C = 0$</td>
<td></td>
<td>$P = P(t)$</td>
</tr>
</tbody>
</table>

Step “Coppola 1” employs all of the 2D $P_c$ assumptions
Step “Coppola 2” introduces Keplarian 2-body motion
Step “Coppola 3” introduces time-varying position covariances
Step “Coppola 4” introduces position+velocity covariances
Outline

• Motivation and objectives

• Overview of collision probability theory

• Analysis of well-studied conjunctions
  Benchmark test cases analyzed by S. Alfano*

• Analysis of archived conjunctions

• Conclusions

Validation using Alfano’s* Benchmarks

• **Benchmark case #3:**
  – “Linear” case where 2D $P_c$ is known to be accurate
  – The 3D $P_c$ software correctly reproduces the 2D $P_c$ approximation, and Alfano’s benchmark $P_c$ value

• **Benchmark case #10:**
  – “Nonlinear” case where 2D $P_c$ is known to be inaccurate
  – The 3D $P_c$ software correctly reproduces Alfano’s benchmark $P_c$ value

• **Other benchmark cases also analyzed (but not shown)**

These plots validate that the 3D $P_c$ software correctly reproduces both the Monte Carlo and 2D $P_c$ estimates, when using the 2D $P_c$ approximations.

Alfano’s* “Linear” Test Case #3

These plots validate that the 3D $P_c$ software correctly reproduces the 2D $P_c$ estimate, even when the 2D $P_c$ approximations are fully relaxed

Alfano’s* “Nonlinear” Test Case #10

These plots validate that the 3D $P_c$ software correctly yields different results as the 2D $P_c$ approximations are relaxed in a step-by-step fashion.

Alfano’s* “Nonlinear” Test Case #10

These plots validate that the 3D $P_c$ software correctly reproduces the Monte Carlo simulation, and that the $dP_c/dt$ profile has two blended peaks.

Alfano’s* “Nonlinear” Test Case #10

These plots validate that the 3D $P_c$ software correctly reproduces Alfano’s benchmark Monte Carlo results.
• Motivation and objectives

• Overview of collision probability theory

• Analysis of well-studied conjunctions

• Analysis of archived conjunctions
  2D vs. 3D results, repeating events, small-$P_c$ screening

• Conclusions
Analysis of Archived Conjunctions

• The 3D $P_c$ method has been applied to 80,453 archived conjunctions
  – Actual events that occurred between 2016 April 1 and 2016 June 1

• Relatively few have appreciable 3D $P_c$ values
  – Only 11,211 (14%) have $P_c \geq 10^{-15}$
  – Only 5,761 (7.2%) have $P_c \geq 10^{-7}$
  – Only 2,674 (3.3%) have $P_c \geq 10^{-5}$
Analysis of Archived Conjunctions

• The 3D $P_c$ method has been applied to 80,453 archived conjunctions
  – Actual events that occurred between 2016 April 1 and 2016 June 1

• Relatively few have appreciable 3D $P_c$ values
  – Only 11,211 (14%) have $P_c \geq 10^{-15}$
  – Only 5,761 (7.2%) have $P_c \geq 10^{-7}$
  – Only 2,674 (3.3%) have $P_c \geq 10^{-5}$

This is the most important set for the CARA team
Archived Conjunctions with 3D $P_c \geq 10^{-5}$

- For these, most 2D and 3D estimates were found to be relatively close to one another
  - 71% have 2D $P_c$ and 3D $P_c$ within 10% of one another
  - 85% have 2D $P_c$ and 3D $P_c$ within 30% of one another

- But smaller subsets were found to differ significantly
  - 5.6% have 2D $P_c$ and 3D $P_c$ separated by a factor of 3 or more
  - 2.4% have 2D $P_c$ and 3D $P_c$ separated by a factor of 10 or more

- The cases where 3D $P_c >> 2D P_c$ are of significant concern to the CARA team
  - Threatening conjunctions could be overlooked when using the 2D $P_c$ approximation
Repeating Events for Close Proximity Satellites

- Objects persistently orbiting in close proximity can make repeated close approaches to one another
  - Satellites within formations or clusters
  - These conjunctions can often be identified by their long durations
  - This can create multiple, blended peaks in $dP_c/dt$

- These types of conjunctions explain some *but not all* of the archived cases that have $3D P_c >> 2D P_c$

Archived conjunction involving two satellites flying in close proximity

- Coppola bounds for $\gamma = 1e^{-16}$
- Coppola 1: Linear motion, $A=A(TCA), B=C=0$
- Coppola 2: Kep2Body, $A=A(TCA), B=C=0$
- Coppola 3: Kep2Body, $A=A(t), B=C=0$
- Coppola 4: Kep2Body, $P=P(t)$

![Graph showing $dP_c/dt$ over time](image)
Isolated Conjunctions with 3D $P_c >> 2D P_c$

Archived conjunction where the 3D $P_c$ estimate exceeds the 2D $P_c$ estimate by a factor of about four.
Isolated Conjunctions with 3D $P_c \gg 2D P_c$

Archived conjunction where the 3D $P_c$ estimate exceeds the 2D $P_c$ estimate by several orders of magnitude.
A Screening Test for Small-$P_c$ Values

- Conjunctions with large relative-position Mahalanobis distances have small 3D $P_c$ values
- This correlation provides the basis for an efficient small-$P_c$ screening test
- Applying this screening test eliminates the need to calculate 3D $P_c$ for $\approx 80\%$ of all conjunctions

About 80\% of the archived conjunctions have $(M_D)_{min} > 10$ and 3D $P_c < 3 \times 10^{-17}$
Outline

• Motivation and objectives
• Overview of collision probability theory
• Analysis of well-studied conjunctions
• Analysis of archived conjunctions
• Conclusions

Conclusions

• The CARA team has implemented Coppola’s 3D $P_c$ formulation into software
  – Validated using Alfano’s benchmark test cases
  – Provides estimates for both $P_c$ and $dP_c/dt$
  – Provides insight into the time dependence of risk

• Archived conjunction analysis indicates that
  – Occasionally the 2D $P_c$ approximation can be very inaccurate
  – An efficient small-$P_c$ screening test can be used to speed processing for large numbers of conjunctions
Illustration of relative position trajectories for Alfano’s (2009) “nonlinear” example #2
Schematic Illustration of 2D $P_c$ Assumptions

Actual nonlinear nominal relative trajectory

Linear approximation

Illustration based on Alfano’s test case #2
Schematic Illustration of 2D $P_c$ Assumptions

Light blue dots show the nominal relative positions at Coppola’s conjunction time bounds

Illustration based on Alfano’s test case #2
Relative position PDFs evolve in time

NOTE: Actual $1\sigma$ surfaces are *much* larger and thinner

Illustration based on Alfano’s test case #2
2D $P_c$ approximates the PDFs as constant, and places them along the linearized trajectory.
The 2D $P_c$ approximation will be inaccurate if these PDF differences become too large during the conjunction.

Illustration based on Alfano’s test case #2
The *Mahalanobis Distance* measures the difference between the positions of the primary and secondary objects, relative to the scale of their combined covariance:

$$M_D(t) = \left( r^T A^{-1} r \right)^{1/2}$$

where

$$r = r(t) = r_s - r_p$$  \hspace{1cm}  $$A = A(t) = A_s + A_p$$

(relative position)  \hspace{1cm}  (combined covariance)
• The Mahalanobis distance varies as a function of time during a conjunction.

• The minimum value \((M_D)_{\text{min}}\) often occurs near the conjunction midpoint, but not always.

• \((M_D)_{\text{min}}\) values vary significantly for different conjunction events.
Monte Carlo $P_c$ Estimation Procedure

1. Sample the state PDFs for both the primary and secondary satellites

2. Propagate the sampled states over the desired time span, checking if the separation becomes less than the combined hard-body radii

3. If so, register a collision at the time the spheres defined by the hard-body radii make first contact

4. Repeat steps 1-3 to improve statistical estimation accuracy