CONSTRAINED CHEBYSHEV APPROXIMATIONS TO SOME ELEMENTARY FUNCTIONS SUITABLE FOR EVALUATION WITH FLOATING-POINT ARITHMETIC

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D.C. • MARCH 1972
Approximations which can be evaluated with precision using floating-point arithmetic are presented. The particular set of approximations thus far developed are for the function TAN and the functions of USASI FORTRAN excepting SQRT and EXPONENTIATION. These approximations are, furthermore, specialized to particular forms which are especially suited to a computer with a small memory, in that all of the approximations can share one general purpose subroutine for the evaluation of a polynomial in the square of the working argument.
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CONSTRAINED CHEBYSHEV APPROXIMATIONS TO SOME ELEMENTARY
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FLOATING-POINT ARITHMETIC

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SUMMARY

Approximations which can be evaluated with precision using floating-point arithmetic are presented. The particular set of approximations thus far developed are for the function TAN and the functions of USASI FORTRAN excepting SQRT and EXPONENTIATION. These approximations are, furthermore, specialized to particular forms which are especially suited to a computer with a small memory, in all that of the approximations can share one general purpose subroutine for the evaluation of a polynomial in the square of the working argument.

INTRODUCTION

The need for approximations of known quality to the mathematical functions commonly found in the function libraries of higher level computer languages, such as FORTRAN, has existed for some time. Approximations from the recent collection in the SIAM Series in Applied Mathematics (ref. 1) fill a large part of this need. These approximations have been somewhat optimized for speed, but they generally require that their evaluations be performed with some amount of precision beyond that which is required of the result.

In situations where it is desirable, for whatever reason, to evaluate the approximations using floating-point arithmetic with the precision of the result, the approximations of reference 1 prove to be not well conditioned for the minimization of the errors inherent in floating-point arithmetic.

It is the purpose of this report to present a family of approximations which can be evaluated with good precision using floating-point arithmetic. The particular set of approximations thus far developed are for the function TAN and the functions of USASI
FORTRAN excepting SQRT and EXPONENTIATION. These approximations are, furthermore, specialized to particular forms which are thought to be especially suited to a computer with a small memory, but which has an efficient method of reference to subroutines.

GENERAL CONSIDERATIONS

In general, these approximations are designed so that when the coefficients of a selected approximation are expressed in the floating-point representation of any computer and the given algebraic form is evaluated using the floating-point arithmetic of that computer then the accuracy of the implemented approximation is limited by the given nominal value of relative error or by the precision of the floating-point arithmetic used. Hence, these approximations are designed to avoid certain important sources of error that are inherent in the use of floating-point arithmetic where recourse to an occasional step of arithmetic with greater than nominal precision is overly difficult or slow. This is usually the situation when "double precision" versions of the approximations are being implemented.

The most pervasive source of these errors is a property of floating-point multiplication and division. It can be shown that these operations cannot produce ONTO mapping in the sense of Matula (ref. 2). This has two relevant consequences. The first, and probably more important, occurs when a change of scale is used to facilitate argument reduction. This situation is illustrated for the sine function when the argument is changed to "circle measurement" by multiplying by $4/\pi$.

For every argument $x$ and number base $\beta$ such that $\pi\beta^{-n}/4 < x < \beta^{-n}$ the value of the multiplied argument lies in the interval $\beta^{-n} < y < 4\beta^{-n}/\pi$. The effect is that the exponent part of $y$ is one unit greater than the exponent part of $x$ and an average of $\pi\beta/4$ successive values of $x$ are represented by a single value of $y$. Necessarily then, the same result is generated for each of these successive values of $x$. For at least one of these successive values the magnitude of the error in the result cannot be less than one-half the difference of the correct values of the sine function at the extremes of this small interval or approximately $1/2 \cos(x) \text{Ceil}(\pi\beta/4)$ units of the value of the least significant bit of the result even with no other sources of error. The symbol $\text{Ceil}(t)$ denotes the smallest integer greater than $t$; hence, for a base sixteen computer this error is approximately 6.3 units ($2\pi$). Examples of this large an error have been observed in a case where a change in scale of the argument was used during argument reduction. For this reason, a change in scale of the argument during argument reduction should be avoided.

The second consequence of this defect occurs when a floating-point multiplication or division is used as the final step of any evaluation. Small but systematic reduction in
error is achieved by writing all odd functions, the logarithm function, and the noncon-
stant terms of the exponential function as \( y + yf(y) \) rather than \( y(1 + f(y)) \). Sometimes
an extra step of arithmetic is added to the algorithm by this organization. If a method
of argument reduction which changes the scale of the independent variable is used, the
benefits of this organization will be negligible.

The approximations to be described are all some form of the Chebyshev approxima-
tion constrained to algebraic forms that terminate with an operation of addition or sub-
traction. It is typical of previously reported Chebyshev approximations of these ele-
mentary functions with relative error weight functions for extremes of relative error to
occur at the end points of the domain of derivation and for the relative error to increase
very rapidly outside this domain of derivation. This property of the previously reported
approximations imposes quite severe restrictions on the choice of integer multiplier for
the argument reduction. Each of the current approximations is constrained to take on
the value of the function at the end point of the domain of the approximation. This has
the effect of widening the valid domain somewhat beyond the nominal domain used for
derivation of the coefficients; hence, the restrictions on the correct choice of integer
multiplier for argument reduction are relieved. The details of the precision require-
ments for a reduced argument to stay well within this extended domain are discussed in
the appendix.

This constraint on the approximation's value at the boundary of its nominal domain
has also been imposed when no argument reduction is required. The effect of this con-
straint is that weak monotonicity can easily be achieved and continuity satisfactorily
simulated at a point where two different approximation segments must be joined. This
is realizable even for approximations whose accuracy is low compared to the nominal
precision of the floating-point arithmetic in use.

A further source of errors arises from the impossibility of representing arbitrary
real numbers in any finite length floating-point notation. Algebraic forms for the ap-
proximations presented here were selected so that those coefficients in which truncation
could produce sizable error in the final approximation would, if unconstrained, be very
nearly equal to integers or half integers. These more important coefficients are con-
strained to these generally representable integer or half integer values, and the remain-
ing coefficients are calculated subject to these constraints. Specific details of these
constraints as applied to each approximation are given in the DISCUSSION OF SPECIFIC
APPROXIMATIONS section.

These absence of optionally rounded floating-point arithmetic or the failure of weak
monotonicity or "continuity" can in some cases be compensated for by modification of
the values of selected coefficients. Such "fudges" are machine, word length, and num-
ber base dependent and no attempt has been made to include any.

Given some approximation \( R \) to a function \( f \), the relative error function for this
approximation is defined by
\[
ER(x) = \left| \frac{R(x) - f(x)}{f(x)} \right|
\]

wherever \( f(x) \neq 0 \). If within the domain of validity of the approximation \( f(x) = 0 \), the relative error can be defined for that point by

\[
ER(x) = \lim_{t \to x} \left| \frac{R(t) - f(t)}{f(t)} \right|
\]

One measure of the quality of an approximation is its extremal relative error; that is the least upper bound of the magnitude of \( ER(x) \) for all values \( x \) from the domain of validity of the approximation:

\[
\overline{ER} = \operatorname{lub} \left| ER(x) \right| \quad x \in D
\]

A term often used in describing the quality of an approximation is its precision; this is taken to be the negative of the logarithm of the extremal relative error:

\[
\text{Precision} = -\log_\beta(\overline{ER})
\]

Its value is very nearly equal to the minimum of the number of correct digits in the base \( \beta \) representation of the value of \( R(x) \) for any argument \( x \) from the domain of validity of the approximation.

**CONSEQUENT RESTRICTIONS ON FORMS USED**

The current set of Chebyshev approximations was developed to avoid serious errors from the previously mentioned sources. Hence, each approximation incorporates these characteristics:

1. The final arithmetic operation is always the addition of an exact term to an approximate term of smaller magnitude.
2. The coefficients are jointly constrained so that the approximation takes on the value of the approximated function at the boundary points of its nominal (reduced) domain.
3. The coefficients with most the influence on error are constrained to values that can be exactly represented in any computer's floating-point number system.

Because of a specific interest in their use in a computer which has a small memory, the forms used for these approximations are limited to those involving the use of a single polynomial in the square of an appropriately reduced argument.
It is expected that the theoretical value of extremal relative error of each approximation will be increased by observing all these constraints. Empirically this effect is small and fortuitously has not required the use of more elaborate approximations in any case that has been implemented.

**CURVE FIT**

The rational form used for any approximation presented is formally equivalent to one of the following: \( P, yP, (P + y)/(P - y), \) or \( y^3/P \). The symbol \( P \) represents a polynomial of degree \( N \) whose independent variable \( y^2 \) is the square of the reduced argument; the symbol \( Q \) will also be used. Some of the coefficients of \( P \) (or \( Q \)) are constrained to given values; all are constrained to give the theoretically correct value for the joining point. The coefficients are computed subject to these constraints by a slightly modified version of the second algorithm of Remes (ref. 3) using especially constructed error weighting functions so that each resulting approximation is uniform throughout the nominal domain. A known restriction on the use of such rational approximations is that they be pole-free. All the approximations, as generated, turned out to be so without specific attention to the problem. The coefficients presented in this report were computed on an IBM 7094-II computer using floating-point arithmetic with 140 binary digits in the fractional part of the floating-point number. Subroutines to perform this extended precision arithmetic and to evaluate many of the elementary functions using it have been provided by C. L. Lawson (ref. 4).

**DISCUSSION OF SPECIFIC APPROXIMATIONS**

**Logarithm**

For any \( x > 0 \) the natural logarithm can be defined in terms of its values over a limited domain as

\[
\ln(x) = n \ln(2) + \ln(y); \quad \frac{\sqrt{2}}{2} < y < \sqrt{2}
\]

The form of equation (1) implies the use of base two arithmetic in that the values of \( n \) and \( y \) are then obtained without error from the representation of the argument \( x \). The rational approximation selected for \( \ln(y) \) in the basic domain is
\[
\ln(y) \approx 2v + \frac{v^3}{Q(v^2)} \quad (2)
\]

\[
v = \frac{y - 1}{y + 1}; \quad \frac{\sqrt{2}}{2} < y < \sqrt{2}
\quad (3)
\]

When floating-point arithmetic is used the term \( y + 1 \) cannot be calculated exactly if the representation of \( y \) has a low order digit of one. The multiplier of any error in \( v \) is reduced from 2.0 to at most 0.395 by the use of the identity \( 2v = (y - 1) + v(1 - y) \) to convert equation (2) to the recommended form

\[
\ln(y) \approx (y - 1) + v \left[ 1 - y + \frac{v^2}{Q(v^2)} \right] \quad (4)
\]

As far as is known, further reduction in error can come only from using extended precision arithmetic.

The quantity \( n \ln(2) \) should be calculated and used in two parts: The more significant part, \( A \), is calculated using only that number of leading digits of \( \ln(2) \) that give an exact product with any value of \( n \) which can occur in an implementation; the less significant part, \( B \), is calculated using the best representation of the remainder of \( \ln(2) \). The various terms of the approximation should be summed starting from the right in approximation (5):

\[
\ln(x) \approx A + (y - 1) + B + v \left[ (1 - y) + \frac{v^2}{Q(v^2)} \right] \quad (5)
\]

Optimal use of rounding is quite difficult to achieve because of the large number of changing criteria. For most values of \( n \neq 0 \), the most important operation to be rounded is the left-most (final) addition of approximation (5). For \( n = 0 \), the second addition from the left is most important.

A change of scale of the independent variable to use logarithms of other than the natural base is not recommended because of the floating-point multiplication property unless the implementer is prepared to use somewhat extended precision arithmetic in the evaluation. In that case, an approximation from reference 1 should be applicable.

Coefficients for the approximations (2), (4), or (5) are identified according to the degree \( M \) of the polynomial \( Q(v^2) \) involved as \( \text{LOG}(\sqrt{2}, 0, M) \).
For any argument \( x \) the exponential function can be defined as
\[
ex = 2^n e^y
\]
in terms of its values over a base domain. Ideally, the integer \( n \) and the working argument \( y \) are selected so that
\[
y = x - n \ln(2) \quad |y| \leq \frac{\ln(2)}{2}
\]
A rational approximation
\[
e^y \approx 1 + \frac{2y}{2 - y + y^2 p(y^2)}
\]
is then used within the basic domain. The approximation described here is best implemented in base two arithmetic; the multiplication by \( 2^n \) in equation (6) can be done exactly, and the final addition of approximation (8) leaves a digit that can be used for rounding.

Because \( \ln(2) \) is irrational it is not possible to guarantee computing the correct integer \( n \), as defined by relation (7), except by completing the indicated reduction and verifying the containment \( |y| \leq \ln(2)/2 \). The need for such care is avoided because the approximations for \( e^y \) are constrained to take on as nearly as possible the correct values at the joining points, \( y = \pm \ln(2)/2 \). This insures that the attainable, weaker, containment \( |y| < \ln(2)/2 + \Delta \) is sufficient. (See the appendix for details.)

For negative values of the reduced argument the approximation (8) is not weakly monotonic. This is an artifact of floating-point representation in any number base \( \beta \) and is very similar to a situation discussed by D. W. Matula in reference 5. He pointed out the nonmonotone behavior of any floating-point implementation of \( f(y) = y/(2 + y) \) for arguments \( y \) approaching 1.0 from below. The behavior is similarly nonmonotone for arguments that approach many of the positive fractions \( \beta^{-k} \). In a floating-point implementation of approximation (8) the ratio \( 2y/[2 + y^2 p(y^2)] - y \) exhibits a similar failure of weak monotonicity for negative arguments. As the representation of \( y \) increases from some negative value to the next available value this ratio increases instead of decreasing.

This increase is sometimes sufficient to cause the sum to decrease producing a failure of weak monotonicity. The approximation can be restated in the algebraically
equivalent form

\[ e^y \approx 1 + y + \frac{y[y - y^2 P(y^2)]}{2 - [y - y^2 P(y^2)]} \]  

(9)

The use of expression (9) is recommended whenever high accuracy is required; it avoids the previously described computational difficulty at the cost of one extra storage operation and one operation of addition.

Coefficients for the polynomial \( P(y^2) \) of degree \( N \) used in approximation (8) are given the identification \( \text{EXP(ln}(2)/2, 0, N + 1) \).

Hyperbolic Sine and Hyperbolic Cosine

The formal definition

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]  

(10)

of the hyperbolic sine function suggests the implementation as

\[ \sinh(x) = \frac{\text{sgn}(x)}{2} \left( e^t - \frac{1}{e^t} \right) \quad t = |x| \]  

(11)

Direct use of equation (11) is computationally unstable for small arguments because of the addition of values with opposite signs and nearly equal magnitudes.

For small arguments the rational approximation

\[ \sinh(x) \approx x + \frac{x^3}{Q(x^2)} \quad |x| < b \]  

(12)

is used. The joining point \( b \) is selected to satisfy precision requirements of the approximation related to (11) which is used for large arguments.

A different difficulty exists for some large arguments. For any number base \( \beta \) direct implementation of approximation (11) is somewhat unstable whenever \( \sinh(t) < \beta^n < e^t/2 \) because the significance of one or more digits is lost by cancellation during the subtraction. Since \( \sinh(t) = s \geq 0 \) is equivalent to \( t = \ln \left( s + \sqrt{s^2 + 1} \right) \) we have this instability occurring whenever
\[
\ln(2\beta^n) \leq t < \left( \ln \beta^n + \sqrt{\beta^{2n} + 1} \right) 
\]

(13)

The most elegant known resolution of this difficulty was obtained from Mr. Hirondo Kuki in a private communication. Choose a value \( v \) large enough so that if \( t \) is any magnitude from one of the intervals (13) then, for \( y = t - v \), \( e^{y/2} \) has the same exponent part as \( \sinh(t) \). From this point of view suitable values are given by

\[
v \geq \ln \left( \beta^n + \sqrt{\beta^{2n} + 1} \right) - \ln(2\beta^n) = \ln \left( \frac{1 + \sqrt{1 + \beta^{-2n}}}{2} \right)
\]

(14)

The value of \( v \) is further selected to have a sufficient number of zero low order digits in its machine representation that no error is introduced in the subtraction \( t - v \) for any magnitude \( t \) such that \( \sinh(t) \) can be represented. An algebraic restatement of equation (10) leads to the approximation

\[
\sinh(x) \approx \text{sgn}(x) \left[ e^{y} + \left( \frac{e^{v}}{2} - 1 \right) e^{y} - \frac{e^{-v}}{2} e^{-y} \right] \quad y = |x| - v
\]

(15)

In a situation where rounding is available the condition \( (e^{y/2}) - 1 < 1/\beta \) is desirable in order that the addition provide a nearly correct rounding digit.

Another possible difficulty with the direct use of approximation (11) would occur for any magnitude \( t \) near the upper limit for which the value \( \sinh(t) \) can be represented in whatever floating-point number system is used. The required value \( e^{t} \) fails to be representable and a machine error condition would result from attempting its calculation. The computational scheme of approximation (14) is found to prevent this whenever \( v > \ln(2) \) without requiring any test except that the value \( \sinh(x) \) be itself representable.

At the joining points of the approximation segments, \( x = \pm b \), the rational approximations are constrained to take on the values obtained by evaluation of the formal definition (10) using high precision arithmetic. It may be necessary for an implementation that the coefficients of the rational approximation be adjusted so that its values at the joining points match the values actually produced by the approximation (14) used for large arguments. A reasonable selection of the joining point is the end of the first positive interval (13) for which the instability of a direct implementation of approximation (11) is avoided. For base two this means \( n = -1 \) and \( b = \ln[(1 + \sqrt{5})/2] \); for any larger base use \( n = 0 \) and \( b = \ln(1 + \sqrt{2}) \).

Polynomials \( Q(x^2) \) for use in the rational approximation (12) and tailored to base two arithmetic are valid in the domain \( |x| < \ln[(1 + \sqrt{5})/2] \). The coefficients for the polynomial of degree \( M \) are identified as \( \text{SINH} \{ \ln[(1 + \sqrt{5})/2], 0, M \} \) and the value selected for \( v \) of approximation (15) must satisfy \( \ln(2) \leq v < \ln(3) \). Approximations
using the coefficients identified as \( \text{SINH} \left[ \ln(1 + \sqrt{2}), 0, M \right] \) are valid in the domain \( |x| < \ln(1 + \sqrt{2}) \). These are given for use with number bases other than two; the associated value of \( v \) must satisfy \( \ln(2) \leq v < \ln(2.125) \).

The hyperbolic cosine function is defined as

\[
\cosh(x) = \frac{e^x + e^{-x}}{2}
\]  

A straightforward implementation would be valid for small and most large arguments. For arguments whose magnitude is near the upper limit for which \( \cosh(x) \) can be represented \( \cosh(x) \approx |\sinh(x)| \). The approximation

\[
\cosh(x) \approx e^y + \left( \frac{e^y}{2} - 1 \right) e^y + \frac{e^{-y}}{2} - y
\]

\[ y = |x| - v \]  

which is similar to approximation (15) and uses the same value of \( v \) is effective for all arguments for which \( \cosh(x) \) is representable.

**Hyperbolic Tangent**

The hyperbolic tangent function is defined as

\[
\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\]

This equation is not suitable as the basis for an evaluating algorithm: both numerator and denominator contain exponential terms that must be approximations, neither the sum nor the difference required can be precisely calculated and finally the computation ends with a division. The form

\[
\tanh(x) = \text{sgn}(x) \left( 1 - \frac{2}{e^{2y} + 1} \right)
\]

\[ y = |x| \]

is algebraically equivalent to (18). It is sufficiently well adapted to floating-point arithmetic to be used as the basis for an approximation to \( \tanh(x) \) for large arguments (\( |x| > b \)). The value of \( b \) is selected so that precision requirements of the approximation (19) can be satisfied. For small values of the argument \( x \) both equations (18) and (19) require the addition of values with opposite signs and nearly equal magnitudes;
hence, neither is satisfactory. The rational approximation

\[
\tanh(x) \approx x - \frac{x^3}{3.0 + x^2 Q(x^2)}
\]

(20)
is used therefore when \(|x| < b\).

It is desirable to round the result of the final arithmetic operation of either approximation; hence, a rounding digit must be generated during that final operation. This is assured if the floating-point exponent of the smaller term is less than that of the result. For large arguments using equation (19) this requires

\[
\frac{2}{e^{2b} + 1} < \frac{1}{\beta}
\]

which gives

\[
b > \ln\left(\frac{2\beta - 1}{2}\right)
\]

(21)

For small arguments using approximation (20) the rounding digit is generated if the floating-point exponent of \(x^3/[3.0 + x^2 Q(x^2)]\) is smaller than the floating-point exponent of \(x\) for every \(x \leq b\). Only for \(\beta = 2\) can both requirements be satisfied; with any other number base the floating-point representation of the value of the smaller term will not extend far enough to include the needed rounding digits.

The accuracy of the rational term of approximation (20) can be marginal near the limits of its domain; hence, the constant term of the denominator is constrained to the precisely representable value 3.0 which eliminates error from one important source. An equally important source of possible error is the calculation of \(x^3\); any available error reducing steps, such as rounding, should be used here.

When an implementation is for a number base greater than two, the floating-point representation of the value \(2y\) can be in error, whether calculated as \(y + y\) or as \(2y\), hence the form

\[
\tanh(x) = \text{sgn}(x) \left[ 1 - \frac{2}{(e^y)^2 + 1} \right]
\]

(22)

should be used for equation (19) to avoid an unnecessary loss of accuracy due to the representation of \(2y\).

Coefficients for the approximation (20) are identified according to the degree \(M\) of the denominator polynomial involved as \(\text{TANH}[\ln(3)/2, 0, M]\).
Sine and Cosine

The sine and cosine functions can be defined by Maclaurin series as

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots 
\]

(23)

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots 
\]

(24)

for all values of the argument \( x \). Direct implementations of equations (23) and (24) are not satisfactory as approximations because the functions are periodic and have repeated zeros for large arguments.

This difficulty is overcome by limiting the nominal domain of definition of the approximations to \(|x| < \pi/4\). The evaluation algorithms then become

\[
x = (4n + j) \frac{\pi}{2} + y \quad |y| \leq \frac{\pi}{4} 
\]

(25)

\[
\sin(x) = \begin{cases} 
\sin(y) & \text{if } j = 0 \\
\cos(y) & \text{if } j = 1 \\
-sin(y) & \text{if } j = 2 \\
-cos(y) & \text{if } j = 3 
\end{cases} 
\]

(26)

\[
\cos(x) = \begin{cases} 
\cos(y) & \text{if } j = 0 \\
-sin(y) & \text{if } j = 1 \\
-cos(y) & \text{if } j = 2 \\
\sin(y) & \text{if } j = 3 
\end{cases} 
\]

(27)

The polynomial approximations used for \( \sin(y) \) and \( \cos(y) \) are

\[
\sin(y) \approx y + y^3 P(y^2) 
\]

(28)

\[
\cos(y) \approx 1.0 + y^2 \left[ -0.5 + y^2 P_1(y^2) \right] 
\]

(29)

In approximation (28) the term \( y^3 P(y^2) \) has several sources of computational error: the value of \( y^2 \), the multiplication of \( y \) by \( y^2 \), and the truncated values of the coeffi-
Rounding can help reduce these errors. When the implementation uses floating-point arithmetic with small number base (\(\beta \leq 12\)), the alignment shift prior to the final addition of approximation (28) both attenuates the effects of these computational rounding errors in the rational term and produces a rounding digit.

Coefficients for the polynomial \(P(y^2)\) of degree \(N - 1\) used in approximation (28) are identified as \(\text{SIN}(\pi/4, N, 0)\). These approximations for \(N = 2, 3, \ldots, 7\) are comparable to approximations 3040, 3041, \ldots, 3045 of reference 1. The loss of nominal precision of the approximations (28) caused by imposing the boundary point value constraint is less than 0.14 decimal digit in all cases.

In approximation (29) for the cosine series the term \(y^2[-0.5 + y^2P_1(y^2)]\) can have a magnitude somewhat greater than 0.25; hence, only use of base two arithmetic ensures that the floating-point exponent of this term is less than that of the result. Even so, reduction in the effect of computational errors in that term may be marginal as may the accuracy of the rounding digit. The leading coefficients are constrained to precisely 1.0 and -0.5 so that no error is introduced by truncating their values for storage. The use of appropriate rounding is recommended.

Coefficients for the polynomial of degree \(N - 2\) used as approximation (29) are identified as \(\text{COS}(\pi/4, N, 0)\). These approximations for \(N = 3, 4, \ldots, 8\) are comparable to approximations 3820, 3821, \ldots, 3825 of reference 1. The loss of nominal precision of the approximations (29) caused by imposing the boundary point value constraint and the coefficient constraint is not overly large: in all cases it is less than 0.49 decimal digit.

**Tangent and Cotangent**

The tangent function can be defined in continued fraction form as

\[
\tan(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \ldots}}}}
\]

for any value of the argument. The tangent function is periodic, but any direct implementation of equation (30) valid for the entire cycle about the origin is impractical because of the large number of terms that would be required near the poles at \(\pm \pi/2\). The identity

\[
\tan(x) = \frac{1}{\tan(\frac{\pi}{2} - x)}
\]
is used to construct an evaluation algorithm in terms of the values of the tangent from the domain $|x| \leq \pi/4$.

$$x = (2k + j) \frac{\pi}{2} + y \quad |y| \leq \frac{\pi}{4}$$  \hfill (32)

$$\tan(x) = \begin{cases} 
\tan(y) & \text{if } j = 0 \\
\frac{-1}{\tan(y)} & \text{if } j = 1 \text{ and } y \neq 0
\end{cases}$$  \hfill (33)

The rational form used for the basic approximation is

$$\tan(y) \approx y + \frac{y^3}{3.0 + y^2 Q(y^2)}$$  \hfill (34)

Because the cotangent function is the reciprocal of the tangent, the same argument reduction and basic approximation can be used, with trivial modifications to equation (33), to evaluate the cotangent.

The magnitude of the rational term of approximation (34) can be almost 0.25; hence, only with the use of arithmetic of base four or less will an alignment shift occur before the final addition. When the implementation must use arithmetic of some larger number base, computational error in the rational term will not have its effect on the final result attenuated and no digit will be available for rounding. Because the accuracy of the rational term can be marginal, its constant term is constrained to the precisely representable value 3.0 so that no error is introduced by truncating that constant for storage. Another important source of error is the calculation of the numerator $y^3$; any possible error reducing steps, such as rounding, should be included in an implementation.

Coefficients for the approximation (34) are identified according to the degree $M$ of the denominator polynomial involved as $\text{TAN}(\pi/4, 0, M + 1)$. The approximation using $\text{TAN}(\pi/4, 0, 2)$ is comparable to approximation 4283 of reference 1.

Inverse Tangent

For any argument $x$ the principal value of the inverse tangent function can be defined as

$$\arctan(x) = \frac{x}{1 + \frac{x^2}{3 + \frac{4x^2}{5 + \ldots + \frac{k^2x^2}{(2k + 1) +}}} \ldots}$$  \hfill (35)
This continued fraction is not an economical computational algorithm for arguments with large magnitudes because of the number of terms required in the computation. The transformation

\[
\arctan(x) = \frac{\pi}{2} \text{sgn}(x) - \arctan(y) \quad y = \frac{1}{x}
\]  

(36)

can be used whenever \(|x| > 1\) to reduce the domain for which the basic approximation used need be valid. Further reduction can be obtained by applying

\[
\arctan(x) = \text{sgn}(x) \left[ \frac{\pi}{6} + \arctan(y) \right] \quad y = \frac{|x|\sqrt{3} - 1}{|x| + \sqrt{3}}
\]

(37)

whenever \(\tan(\pi/12) < |x| \leq 1\). The use of transformation (36) or (37) can introduce error both in calculating \(y\) and in subsequently calculating \(\arctan(x)\) using the value \(\arctan(y)\). For some arguments both must be used. Implementing the following elaborated scheme can avoid the cascading of these effects:

\[
\arctan(x) = \begin{cases}
\arctan(y) & \text{if } |x| < \tan\left(\frac{\pi}{12}\right) \\
\text{sgn}(x) \left[ \frac{\pi}{6} + \arctan(y) \right] & \text{if } \tan\left(\frac{\pi}{12}\right) < |x| \leq 1 \\
\text{sgn}(x) \left[ \frac{\pi}{3} - \arctan(y) \right] & \text{if } 1 < |x| < \frac{1}{\tan\left(\frac{\pi}{12}\right)} \\
\frac{\pi}{2} \text{sgn}(x) - \arctan(y) & \text{if } |x| > \frac{1}{\tan\left(\frac{\pi}{12}\right)}
\end{cases}
\]

(38)

The form selected for the basic approximation is

\[
\arctan(y) \approx y - \frac{y^3}{Q(y^2)}
\]

(39)

This approximation need be valid only for the domain \(|y| \lesssim \tan(\pi/12)\) and is in fact quite stable there even when implemented in floating-point arithmetic of any commonly used number base.

Coefficients for the polynomial \(Q(y^2)\) of degree \(M\) used by approximation (39) are identified as \(\text{ATAN}[	an(\pi/12), 0, M]\). The approximation using \(\text{ATAN}[	an(\pi/12), 0, 1]\)
is comparable to approximation 5050 of reference 1. The imposition of the boundary point value constraint causes a loss of 0.19 decimal digit of nominal precision.

**Inverse Sine and Inverse Cosine**

For any argument \( x \) with \( |x| < 1 \) the principal value of the inverse sine function is defined as

\[
\arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots
\]  

(40)

Various numerical problems associated with implementing this definition for arguments with magnitudes near 1.0 can be avoided by using the transformation

\[
\arcsin(x) = \text{sgn}(x) \left[ \frac{\pi}{2} - 2 \arcsin(y) \right]
\]

\[
y = \sqrt{\frac{1 - |x|}{2}}
\]

(41)

wherever \( |x| > 0.5 \). The rational approximation

\[
\arcsin(y) \approx y + \frac{y^3}{Q(y^2)}
\]  

(42)

is then used in either case.

Any errors that may be introduced by the argument transformation of (41) are preserved through the approximation; hence, all possible error reducing steps should be used. Implementation in base two arithmetic eases this problem somewhat because then neither the calculation of \((1 - |x|)/2\) nor the multiplication in \(2 \arcsin(y)\) can introduce error.

A suitable evaluation algorithm for the principal value of the inverse cosine function can be built around the identity

\[
\arccos(x) = \frac{\pi}{2} - \arcsin(x)
\]  

(43)

transformation (41) and approximation (42).

Coefficients for the polynomial \( Q(y^2) \) of degree \( M \) used in approximation (42) are identified as \( \text{ARSIN}(0.5, 0, M) \). The approximation using \( \text{ARSIN}(0.5, 0, 1) \) is comparable to approximation 4691 of reference 1; a loss of 0.19 decimal digit of precision is caused by the imposition of the boundary point value constraint.
The precision obtainable from approximation (42) increases only slowly with the
degree $M$ of the polynomial used. This may limit the utility of these approximations
where high precision is required.

RESULTS

Coefficients for use in implementing any of the approximations that have been dis­
cussed are presented herein. Note that these coefficients are for the polynomial $P(y^2)$
or $Q(y^2)$ required in the description of each approximation. Any specifically constrained
coefficients that may be needed were presented with that description. The coefficients
are listed in order of increasing powers of the square of the appropriate variable; for­
mally,

$$P(y^2) = P_{00} + P_{01}y^2 + P_{02}y^4 + \ldots$$

(44)

For each function considered the functional form and nominal interval of its approxi­
mations are presented as page headings to the lists of coefficients. Each set of coeffi­
cients is identified by an index number and the precision for which that approximation
is adequate. The precision is expressed as the number of binary digits (bits) and
the number of decimal digits. The coefficients are given in both binary (octal) and deci­
mal notation; in each radix system ($\beta = 2$ or $\beta = 10$) the coefficient is expressed as
$(n)F$ where $n$ is an integer and $F$ is a signed fraction whose magnitude is bounded by
$1/\beta$ and 1. The value of the numeral is $F*\beta^n$. Both parts of the binary numeral are,
for convenience, written in the common pseudo-octal representation.

The extreme values of the relative error function $ER(x)$ for each approximation
covered by this report are given in separate lists, indexed according to the same sys­
tem used for the sets of coefficients. With each value is displayed a set of points from
the nominal domain at which the relative error function attains its extreme magnitude.
The sign of the relative error at each point is indicated by a mark (+) or (-) attached to
the point. The natural symmetries of the various relative error functions are indicated;
this allows the identification of all the remaining extremal points of the approximation
and the corresponding signs.
\[
\log(x) \quad \sqrt{2} < x < \sqrt{2}, \quad y = (x-1)/(x+1), \quad \log(\sqrt{2}, 0, M) = 2y + y^3/q(y^2)
\]

**Binary Coefficients**

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**Decimal Coefficients**

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**Note:** The table entries are the binary and decimal coefficients for various values of \(M\), representing the precision in bits or digits. The values are formatted for readability, with each coefficient aligned for easier comparison and use in computations.
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LOG(X) \[ \sqrt{2}/2 < X < \sqrt{2} \], \[ Y = (X-1)/(X+1) \], \[ \log(\sqrt{2}, 0, M) = 2Y + Y^3/Q(Y^2) \]
\[ \log(x) \quad \sqrt{e} < x < \sqrt{e}, \quad y = (x-1)/(x+1), \quad \log(\sqrt{e}, 0, M) = 2y + y^3/6(q^2) \]

### Binary Coefficients

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### Binary Coefficients

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\[ \exp(y) \quad |y| < \ln(2)/2, \quad \exp(\ln(2)/2, N, 0) = 1 + 2y/(2 - y + y^2\exp(y^2)) \]

### Binary Coefficients

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**Decimal Coefficients**

#### N = 2

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**Decimal Coefficients**

#### N = 3

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**Decimal Coefficients**

#### N = 4

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**Decimal Coefficients**

#### N = 5

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**Decimal Coefficients**

#### N = 6

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**Notes:**

- Precision refers to the number of significant digits.
- Bits indicate the number of binary digits used for precision.
- Decimal coefficients are provided for each precision level.
\[
\begin{array}{ll}
\text{EXP}(Y) & |Y| < \ln(2)/2, \quad \text{EXP}(\ln(2)/2, N, 0) = 1 + 2Y/(2 - Y + Y^2 P(Y))
\end{array}
\]

### BINARY COEFFICIENTS

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<th>(N = 8)</th>
<th>PRECISION 93.4 BITS</th>
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### PRECISION

- **N = 7**: PRECISION 83.0 BITS
- **N = 8**: PRECISION 93.4 BITS
- **N = 9**: PRECISION 103.8 BITS
- **N = 10**: PRECISION 123.8 BITS
- **N = 11**: PRECISION 143.8 BITS
- **N = 12**: PRECISION 163.8 BITS

### DIGITS

- **N = 7**: 24.99 DIGITS
- **N = 8**: 28.12 DIGITS
- **N = 9**: 31.25 DIGITS
- **N = 10**: 34.37 DIGITS
- **N = 11**: 37.50 DIGITS
- **N = 12**: 40.62 DIGITS
\[ \sinh(y) \quad |y| < \ln((1 + \sqrt{5})/2), \quad \sinh(\ln((1 + \sqrt{5})/2, 0, M) = y + y^3/(6y^2) \]

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**BINARY COEFFICIENTS**

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\[ \text{SINH}(Y) \quad |Y| < \ln((1+\sqrt{c})/2), \quad \text{SINH}(\ln((1+\sqrt{c})/2), 0, M) = Y + \frac{3}{4}(Y^2) \]
\[
\sinh(y) \quad |y| < \ln(1+\sqrt{2}) \quad \sinh(\ln(1+\sqrt{2}), 0, M) = y + \frac{y^3}{3} \frac{1}{2}
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SINH(Y) = \frac{Y^3}{3} + \frac{Y^5}{45} + \frac{Y^7}{294} + \frac{Y^9}{16710} + \ldots

where |Y| < ln(1 + \sqrt{2}), \quad \text{SINH}[\ln(1 + \sqrt{2})], \quad O, M = Y + Y^3/3

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\[
\sinh(y) \quad |y| < \ln(1 + \sqrt{2}), \quad \sinh(\ln(1 + \sqrt{2}), 0, M) = y + y^3/6\]

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\[ \tanh(Y) \quad |Y| < \ln(3)/2 \quad \tanh(\ln(3)/2, 0, M) = Y - Y^3/(3 + Y^2q(Y^2)) \]

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\( \text{TANH}(y) \quad |y| < \ln(3)/2 \quad \text{TANH}(\ln(3)/2, 0, M) = y - y^3/(3 + y^2 q(y^2)) \)

### Binary Coefficients

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\[ \sin(y) \quad |y| < \pi/4, \quad \sin(\pi/4, N, 0) = y + y^3 \sin(y^2) \]

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32
\[
\cos(y) \quad |y| < \pi/4, \quad \cos(\pi/4, N, 0) = 1 + y^2(-0.5 + y^2P(y^2))
\]

### BINARY COEFFICIENTS

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\[ \cos(Y) \quad |Y| < \pi/4, \quad \cos(\pi/4, N, 0) = 1 + Y^2(-.5 + Y^2P(Y^2)) \]

### Binary Coefficients

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### Binary Coefficients

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### Decimal Coefficients

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34
### TAN(Y) \[ |Y| < \pi/4, \quad \text{TAN}(\pi/4, 0, M) = Y + Y^3/(3 + Y^2Q(Y^2)) \]

#### Binary Coefficients

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### Binary Coefficients

| $M = 8$ | Precision 66.1 Bits | $M = 9$ | Precision 73.2 Bits | $M = 10$ | Precision 80.2 Bits |
|---|---|---|---|---|
| $\text{TAN}(Y)$ | $|Y| < \pi/4$, $\text{TAN}(\pi/4, O, M) = Y + Y^3/(3 + Y^2q(Y^2))$ | $\text{TAN}(\pi/4, O, M) = Y + Y^3/(3 + Y^2q(Y^2))$ | $\text{TAN}(\pi/4, O, M) = Y + Y^3/(3 + Y^2q(Y^2))$ |
| $M = 8$ | Precision 66.1 Bits | $M = 9$ | Precision 73.2 Bits | $M = 10$ | Precision 80.2 Bits |
| $Q00$ | Precision 19.90 Digits | $Q00$ | Precision 22.02 Digits | $Q00$ | Precision 24.15 Digits |

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*M = 9: 72 bits, 1 sign bit, 71 fraction bits, 1 implied leading 1.

*M = 10: 80 bits, 1 sign bit, 79 fraction bits, 1 implied leading 1.
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### Binary Coefficients

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ATAN(Y)  \quad |Y| < \tan(\pi/12), \quad ATAN(\tan(\pi/12), O, M) = Y - Y^2/2(Y^2)

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41
\( \text{ATAN}(y) \quad |y| < \tan(\pi/12) \), \( \text{ATAN}(\tan(\pi/12), O, W) = y - y^3/Q(y^2) \)

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\[
\text{ATAN}(y) \quad |y| < \tan(\pi/12), \quad \text{ATAN}(\tan(\pi/12), 0, M) = y - y^3/Q(y^2)
\]

**Binary Coefficients**

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ARSIN(Y) \quad |Y| < 0.5, \quad ARSIN(0.5, 0, M) = Y + \frac{Y^3}{3} \sqrt{1 - Y^2}

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AR\sin(y) \quad |y| < 0.5, \quad AR\sin(0.5, 0, M) = y + \frac{y^3}{3!}(y^2)

### Binary Coefficients

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### ARSIN(Y)

For $|Y| < 0.5$, 

\[
\text{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2)
\]

#### Binary Coefficients

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\[ \text{PRECISION} = 58.9 \text{ BITS} \]

\[ \text{PRECISION} = 17.74 \text{ DIGITS} \]

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46
ARSIN(\(y\)) for \(|y| < 0.5\), \(ARSIN(0.5, 0, M) = y + y^3/Q(y^2)\)

**Binary Coefficients**

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\text{ARSIN}(Y) \quad |Y| < 0.5, \quad \text{ARSIN}(0.5, 0, M) = Y + \frac{Y^3}{Q(Y^2)}
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### Binary Coefficients

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### Binary Coefficients

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<td>77457  35273  11000  41204</td>
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<td>( -13)</td>
<td>77716  55620  71763  25371</td>
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<td>( -14)</td>
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### Decimal Coefficients

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\[
\log(x) \quad \sqrt{2/3} < x < \sqrt{2}, \quad y = (x-1)/(x+1), \quad \log(\sqrt{2}, 0, M) = 2y + y^3/\sqrt{y^2}
\]

\[
\text{ER}(1) = \text{ER}(\sqrt{2}) = 0, \quad \text{ER}(1/x) = \text{ER}(x)
\]

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>M = 1</td>
<td>.29295*10^{-7}</td>
<td>1.1722(-), 1.3601(+)</td>
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<tr>
<td>M = 2</td>
<td>.99921*10^{-10}</td>
<td>1.1255(+), 1.2770(-), 1.3846(+)</td>
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<tr>
<td>M = 3</td>
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<td>1.0988(-), 1.2222(+), 1.3252(-), 1.3954(+)</td>
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<tr>
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<td>M = 6</td>
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<td>M = 7</td>
<td>.43943*10^{-21}</td>
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<tr>
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<tr>
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<td>1.0397(+), 1.0988(+), 1.1521(-), 1.2040(+), 1.2532(-)</td>
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<tr>
<td>M = 10</td>
<td>.1146*10^{-27}</td>
<td>1.0366(-), 1.0903(-), 1.1392(+), 1.1871(-), 1.2332(+)</td>
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<tr>
<td>M = 11</td>
<td>.70612*10^{-30}</td>
<td>1.0366(-), 1.0831(+), 1.1282(-), 1.1725(+), 1.2157(-)</td>
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</table>
\[ \text{EXP}(y) \quad \text{where} \quad |y| < \ln(2)/2, \quad \text{EXP}(\ln(2)/2, N, 0) = 1 + \frac{2y}{2 - y + y^{2}P(y^{2})} \]

\[ \text{ER}(0) = \text{ER}(\ln(2)/2) = 0, \quad \text{ER}(-x) = -\text{ER}(x) \]

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>N = 2</td>
<td>(63842 \times 10^{-9})</td>
<td>(0.18992 (+), 0.31550 (-))</td>
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<td>N = 3</td>
<td>(40152 \times 10^{-12})</td>
<td>(0.14745 (+), 0.26087 (-), 0.32913 (+))</td>
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<td>N = 4</td>
<td>(27364 \times 10^{-15})</td>
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<td>N = 5</td>
<td>(19349 \times 10^{-18})</td>
<td>(0.10241 (+), 0.18908 (-), 0.25697 (+), 0.30741 (-), 0.33864 (+))</td>
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<tr>
<td>N = 6</td>
<td>(13962 \times 10^{-21})</td>
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<tr>
<td>N = 7</td>
<td>(10202 \times 10^{-24})</td>
<td>(0.07799 (+), 0.14705 (-), 0.20516 (+), 0.25461 (-), 0.29457 (+), 0.32399 (-), 0.34202 (+))</td>
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<td>N = 8</td>
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<td>(0.06978 (+), 0.13219 (-), 0.18576 (+), 0.23281 (-), 0.27283 (+), 0.30530 (-), 0.32856 (+), 0.34294 (-))</td>
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<td>N = 9</td>
<td>(55720 \times 10^{-31})</td>
<td>(0.06313 (+), 0.12001 (-), 0.16951 (+), 0.21392 (-), 0.25303 (+), 0.28611 (-), 0.31257 (+), 0.33183 (-), 0.34360 (+))</td>
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</table>
\[ \sinh(y) = \ln((1 + \sqrt{1}})/2), \quad \sinh(\ln((1 + \sqrt{1}))/2, 0, M) = y + y^3/(6y^2) \]
\[ \er(0) = \er(\ln(1 + \sqrt{1}))/2) = 0, \quad \er(-x) = \er(x) \]

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<tr>
<th>INDEX</th>
<th>EXTREMA</th>
<th>ERROR</th>
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<tbody>
<tr>
<td>M = 1</td>
<td>1.2837*10^{-6}</td>
<td>2.2063 (+), 4.2706 (-)</td>
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<tr>
<td>M = 2</td>
<td>1.1396*10^{-8}</td>
<td>1.6449 (+), 3.3988 (-), 4.5198 (+)</td>
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<tr>
<td>M = 3</td>
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<td>M = 9</td>
<td>1.3215*10^{-31}</td>
<td>5.949 (+), 1.3168 (-), 1.9771 (+), 2.5899 (-), 3.1464 (+), 3.6372 (-), 4.0529 (+), 4.3853 (-), 4.6276 (+), 4.7750 (-)</td>
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</tbody>
</table>
\[ \sinh(Y) = \left| Y \right| \ln(1 + \sqrt{2}), \quad \sinh(\ln(1 + \sqrt{2}), 0, M) = Y + \frac{Y^3}{3!} \]  
\[ \text{ER}(0) = \text{ER}(\ln(1 + \sqrt{2})) = 0, \quad \text{ER}(-X) = \text{ER}(X) \]

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<thead>
<tr>
<th>INDEX</th>
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<tbody>
<tr>
<td>M = 1</td>
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<tr>
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<td>(6.2358 \times 10^{-28})</td>
<td>.09971(+), .22110(-), .33299(+), .43790(-), .53484(+), .62240(-), .69918(+), .76389(-), .81543(+), .85290(-)</td>
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</tbody>
</table>
\[
\text{TANH}(Y) \quad |Y| < \ln(3)/2 \quad \text{TANH(}\ln(3)/2, 0, M) = Y - \frac{Y^3}{3 + Y^2(\text{Q}(Y))}
\]
\[
\text{ER}(0) = \text{ER}(\ln(3)/2) = 0, \quad \text{ER}(-X) = \text{ER}(X)
\]

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<td>.13686 \times 10^{-10}</td>
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<td>M = 4</td>
<td>.43338 \times 10^{-13}</td>
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<td>.14459 \times 10^{-15}</td>
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<tr>
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<td>.49670 \times 10^{-18}</td>
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<td>M = 7</td>
<td>.17375 \times 10^{-20}</td>
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<td>.61526 \times 10^{-23}</td>
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<td>M = 9</td>
<td>.21976 \times 10^{-25}</td>
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<td>.79000 \times 10^{-28}</td>
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<tr>
<td>M = 11</td>
<td>.28537 \times 10^{-30}</td>
<td>.10302(-), .17910(+), .24468(-), .30451(+), .35891(-), .40753(+), .44984(-), .48531(+), .51344(-), .53384(+)</td>
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</table>

58
\[ \sin(y) \quad |y| < \pi/4, \quad \sin(\pi/4, n, 0) = y + y^3p(y^2) \]
\[ er(0) = er(\pi/4) = 0, \quad er(-x) = er(x) \]

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<tbody>
<tr>
<td>( N = 2 )</td>
<td>( .23205 \times 10^{-5} )</td>
<td>( .36559 (+), .69983 (-) )</td>
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<tr>
<td>( N = 3 )</td>
<td>( .44477 \times 10^{-8} )</td>
<td>( .27194 (+), .55879 (-), .73895 (+) )</td>
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<tr>
<td>( N = 4 )</td>
<td>( .58471 \times 10^{-11} )</td>
<td>( .21601 (+), .45934 (-), .64141 (+), .75607 (-) )</td>
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<td>( .18003 (+), .38819 (-), .55771 (+), .68555 (-), .76515 (+) )</td>
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<td>( N = 6 )</td>
<td>( .39562 \times 10^{-17} )</td>
<td>( .15403 (+), .33543 (-), .49007 (+), .61722 (-), .71204 (+), .77037 (-) )</td>
</tr>
<tr>
<td>( N = 7 )</td>
<td>( .21951 \times 10^{-20} )</td>
<td>( .13460 (+), .29498 (-), .43562 (+), .55714 (-), .65608 (+), .72920 (-), .77406 (+) )</td>
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<tr>
<td>( N = 8 )</td>
<td>( .97348 \times 10^{-24} )</td>
<td>( .11952 (+), .26308 (-), .39134 (+), .50572 (-), .60376 (+), .68288 (-), .74096 (+), .77645 (-) )</td>
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<tr>
<td>( N = 9 )</td>
<td>( .35273 \times 10^{-27} )</td>
<td>( .10749 (+), .23731 (-), .35483 (+), .46192 (-), .55679 (+), .63746 (-), .70214 (+), .74938 (-), .77815 (+) )</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>( .10634 \times 10^{-30} )</td>
<td>( .09765 (+), .21609 (-), .32427 (+), .42447 (-), .51527 (+), .59514 (-), .68262 (+), .71644 (-), .75562 (+), .77941 (-) )</td>
</tr>
</tbody>
</table>
\[
\cos(y) \quad |y| < \pi/4, \quad \cos(\pi/4, N, 0) = 1 + y^2(-.5 + y^2P(y^2))
\]

\[
\text{ER}(0) = \text{ER}(\pi/4) = 0, \quad \text{ER}(-X) = \text{ER}(X)
\]

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<tr>
<td>N = 3</td>
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<td>(49199(-), \ 73091(+))</td>
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<tr>
<td>N = 4</td>
<td>(13274 \times 10^{-9})</td>
<td>(39375(-), \ 62490(+), \ 75365(-))</td>
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<td>(32766(-), \ 53804(+), \ 67966(-), \ 76425(+))</td>
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<tr>
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<td>(10127 \times 10^{-15})</td>
<td>(28041(-), \ 46957(+), \ 60932(-), \ 70960(+), \ 77019(-))</td>
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<td>(60233 \times 10^{-19})</td>
<td>(24499(-), \ 41534(+), \ 54816(-), \ 65254(+), \ 72812(-), \ 77390(+))</td>
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<tr>
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<td>(28613 \times 10^{-22})</td>
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<tr>
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<td>(11081 \times 10^{-25})</td>
<td>(19548(-), \ 33610(+), \ 45224(-), \ 55198(+), \ 63533(-))</td>
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<tr>
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<td>(35616 \times 10^{-29})</td>
<td>(17752(-), \ 30651(+), \ 41483(-), \ 51014(+), \ 59263(-))</td>
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<td>(96433 \times 10^{-33})</td>
<td>(16257(-), \ 28159(+), \ 38278(-), \ 47338(+), \ 55377(-), \ 62336(+), \ 68143(-), \ 72727(+), \ 76036(-), \ 78036(+))</td>
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</tbody>
</table>
\[ \tan(y) \quad |y| < \pi/4, \quad \tan(\pi/4, 0, M) = y + \frac{y^3}{(3 + y^2q(y^2))} \]
\[ \text{ER}(0) = \text{ER}(\pi/4) = 0, \quad \text{ER}(-x) = \text{ER}(x) \]

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<tbody>
<tr>
<td>M = 2</td>
<td>(1.2158 \times 10^{-6})</td>
<td>(0.49264(-), \ 0.73112(+))</td>
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<tr>
<td>M = 3</td>
<td>(7.1323 \times 10^{-9})</td>
<td>(0.39416(+), \ 0.62527(-), \ 0.75375(+))</td>
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<td>(4.6965 \times 10^{-11})</td>
<td>(0.32799(-), \ 0.53839(+), \ 0.67987(-), \ 0.76430(+))</td>
</tr>
<tr>
<td>M = 5</td>
<td>(3.2665 \times 10^{-13})</td>
<td>(0.28065(+), \ 0.46988(-), \ 0.60956(+), \ 0.70973(-), \ 0.77022(+))</td>
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<tr>
<td>M = 6</td>
<td>(2.3421 \times 10^{-15})</td>
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<td>(1.7114 \times 10^{-17})</td>
<td>(0.21761(+), \ 0.37195(-), \ 0.49644(+), \ 0.59963(-), \ 0.68134(+))</td>
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<td>(1.2665 \times 10^{-19})</td>
<td>(0.19559(-), \ 0.33627(+), \ 0.45243(-), \ 0.55216(+), \ 0.63547(-))</td>
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<td>(0.17761(+), \ 0.30665(-), \ 0.41500(+), \ 0.51030(-), \ 0.59278(+))</td>
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<td>M = 10</td>
<td>(7.1075 \times 10^{-24})</td>
<td>(0.16265(-), \ 0.28172(+), \ 0.38293(-), \ 0.47353(+), \ 0.55392(-))</td>
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<tr>
<td>M = 11</td>
<td>(5.3688 \times 10^{-26})</td>
<td>(0.15001(+), \ 0.26046(-), \ 0.35523(+), \ 0.44119(-), \ 0.51886(+))</td>
</tr>
<tr>
<td>M = 12</td>
<td>(4.0712 \times 10^{-28})</td>
<td>(0.13919(-), \ 0.24214(+), \ 0.33112(-), \ 0.41265(+), \ 0.48734(-))</td>
</tr>
<tr>
<td>M = 13</td>
<td>(3.0967 \times 10^{-30})</td>
<td>(0.12982(+), \ 0.22619(-), \ 0.30996(+), \ 0.38735(-), \ 0.45900(+))</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
\text{ATAN}(Y) & \quad |Y| < \tan(\pi/12), \quad \text{ATAN}(\tan(\pi/12)) 0, M = Y - Y^3/Q(Y^2) \\
\text{ER}(O) & = \text{ER}(\tan(\pi/12)) = 0, \quad \text{ER}(-X) = \text{ER}(X)
\end{align*} \]

<table>
<thead>
<tr>
<th>INDEX</th>
<th>EXTREMAL ERROR</th>
<th>POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 1</td>
<td>( \times 36768 \times 10^{-6} )</td>
<td>.12101(+), .23714(-)</td>
</tr>
<tr>
<td>M = 2</td>
<td>( \times 28901 \times 10^{-8} )</td>
<td>.09071(+), .18822(-), .25135(+)</td>
</tr>
<tr>
<td>M = 3</td>
<td>( \times 29772 \times 10^{-10} )</td>
<td>.07247(+), .15453(-), .21727(+), .25753(-)</td>
</tr>
<tr>
<td>M = 4</td>
<td>( \times 35092 \times 10^{-12} )</td>
<td>.06038(+), .13064(-), .18857(+), .23287(-), .26080(+)</td>
</tr>
<tr>
<td>M = 5</td>
<td>( \times 44825 \times 10^{-14} )</td>
<td>.05176(+), .11298(-), .16560(+), .20931(-), .24224(+)</td>
</tr>
<tr>
<td>M = 6</td>
<td>( \times 60388 \times 10^{-16} )</td>
<td>.04530(+), .09944(-), .14720(+), .18877(-), .22891(+)</td>
</tr>
<tr>
<td>M = 7</td>
<td>( \times 84485 \times 10^{-18} )</td>
<td>.04028(+), .08877(-), .13227(+), .17128(-), .20493(+)</td>
</tr>
<tr>
<td>M = 8</td>
<td>( \times 12157 \times 10^{-19} )</td>
<td>.03627(+), .08014(-), .11998(+), .15643(-), .18899(+)</td>
</tr>
<tr>
<td>M = 9</td>
<td>( \times 17877 \times 10^{-21} )</td>
<td>.03298(+), .07302(-), .10971(+), .14376(-), .17476(+)</td>
</tr>
<tr>
<td>M = 10</td>
<td>( \times 26747 \times 10^{-23} )</td>
<td>.03024(+), .06706(-), .10102(+), .13288(-), .16234(+)</td>
</tr>
<tr>
<td>M = 11</td>
<td>( \times 40585 \times 10^{-25} )</td>
<td>.02792(+), .06199(-), .09358(+), .12345(-), .15141(+)</td>
</tr>
<tr>
<td>M = 12</td>
<td>( \times 62304 \times 10^{-27} )</td>
<td>.02593(+), .05763(-), .08714(+), .11523(-), .14176(+)</td>
</tr>
<tr>
<td>M = 13</td>
<td>( \times 96588 \times 10^{-29} )</td>
<td>.02421(+), .05384(-), .08152(+), .10800(-), .13318(+)</td>
</tr>
<tr>
<td>M = 14</td>
<td>( \times 15099 \times 10^{-30} )</td>
<td>.02270(+), .05052(-), .07657(+), .10159(-), .12554(+)</td>
</tr>
</tbody>
</table>

62
\[
\begin{align*}
\text{ARSIN}(Y) \quad |Y| < 0.5, & \quad \text{ARSIN}(0.5, O, M) = Y + Y^5/T(Y^2) \\
\text{ER}(0) = \text{ER}(0.5) = 0, & \quad \text{ER}(-X) = \text{ER}(X)
\end{align*}
\]

<table>
<thead>
<tr>
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<th>EXTREMAL ERROR</th>
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</tr>
</thead>
<tbody>
<tr>
<td>M = 1</td>
<td>+11955*10^{-4}</td>
<td>+24038(-), +44933(+)</td>
</tr>
<tr>
<td>M = 2</td>
<td>+38635*10^{-6}</td>
<td>+17844(+), +36192(-), +47235(+)</td>
</tr>
<tr>
<td>M = 3</td>
<td>+16363*10^{-7}</td>
<td>+14167(-), +29819(+), +41247(-), +48241(+)</td>
</tr>
<tr>
<td>M = 4</td>
<td>+79250*10^{-9}</td>
<td>+11739(+), +25196(-), +35967(+), +43921(-), +48778(+)</td>
</tr>
<tr>
<td>M = 5</td>
<td>+41582*10^{-10}</td>
<td>+10017(-), +21750(+), +31637(-), +39644(+), +45521(-)</td>
</tr>
<tr>
<td>M = 6</td>
<td>+23011*10^{-11}</td>
<td>+08735(+), +19104(-), +28125(+), +35834(-), +42033(+)</td>
</tr>
<tr>
<td>M = 7</td>
<td>+13226*10^{-12}</td>
<td>+07743(-), +17018(+), +25257(-), +32546(-), +38733(-)</td>
</tr>
<tr>
<td>M = 8</td>
<td>+78206*10^{-14}</td>
<td>+06953(+), +15334(-), +22889(+), +29731(-), +35748(+)</td>
</tr>
<tr>
<td>M = 9</td>
<td>+47270*10^{-15}</td>
<td>+06309(-), +13948(+), +20908(-), +27317(+), +33095(-)</td>
</tr>
<tr>
<td>M = 10</td>
<td>+29077*10^{-16}</td>
<td>+05774(+), +12790(-), +19232(+), +25238(-), +30749(+)</td>
</tr>
<tr>
<td>M = 11</td>
<td>+18144*10^{-17}</td>
<td>+05322(-), +11807(+), +17797(-), +23435(+), +28683(-)</td>
</tr>
<tr>
<td>M = 12</td>
<td>+11457*10^{-18}</td>
<td>+04936(+), +10963(-), +16556(+), +21860(-), +26843(+)</td>
</tr>
<tr>
<td>M = 13</td>
<td>+73083*10^{-20}</td>
<td>+04602(-), +10231(+), +15474(-), +20475(+), +25212(-)</td>
</tr>
<tr>
<td>M = 14</td>
<td>+47018*10^{-21}</td>
<td>+04311(+), +09590(-), +14523(+), +19250(-), +23756(+)</td>
</tr>
<tr>
<td>M = 15</td>
<td>+30475*10^{-22}</td>
<td>+04054(-), +09024(+), +13680(-), +18158(+), +22451(-)</td>
</tr>
<tr>
<td>M = 16</td>
<td>+19881*10^{-23}</td>
<td>+03826(+), +08521(-), +12928(+), +17181(-), +21275(+)</td>
</tr>
<tr>
<td>M = 17</td>
<td>+13045*10^{-24}</td>
<td>+03622(-), +08071(+), +12254(-), +16301(+), +20212(-)</td>
</tr>
</tbody>
</table>
\[ \text{ARSIN}(Y) \quad |Y| < 0.5, \quad \text{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2) \]

\[ \text{ER}(0) = \text{ER}(0.5) = 0, \quad \text{ER}(-X) = \text{ER}(X) \]

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<thead>
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<th>EXTREMAL ERROR</th>
<th>POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M = 18 )</td>
<td>( \approx 6028 \times 10^{-26} )</td>
<td>( 0.03439(+) ), ( 0.07666(-) ), ( 0.11646(+) ), ( 0.15505(-) ), ( 0.19266(+) ), ( 0.22854(-) ), ( 0.26308(+) ), ( 0.29590(-) ), ( 0.32680(+) ), ( 0.35561(-) ), ( 0.38216(+) ), ( 0.40630(-) ), ( 0.42791(+) ), ( 0.44685(-) ), ( 0.46304(+) ), ( 0.47640(-) ), ( 0.48685(+) ), ( 0.49435(-) ), ( 0.49886(+) )</td>
</tr>
<tr>
<td>( M = 19 )</td>
<td>( \approx 6991 \times 10^{-27} )</td>
<td>( 0.03271(-) ), ( 0.07299(+) ), ( 0.11095(-) ), ( 0.14782(+) ), ( 0.18366(-) ), ( 0.21834(+) ), ( 0.25169(-) ), ( 0.28353(+) ), ( 0.31372(-) ), ( 0.34207(+) ), ( 0.36845(-) ), ( 0.39272(+) ), ( 0.41475(-) ), ( 0.43444(+) ), ( 0.45169(-) ), ( 0.46642(+) ), ( 0.47856(-) ), ( 0.48806(+) ), ( 0.49487(-) ), ( 0.49897(+) )</td>
</tr>
</tbody>
</table>

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, December 4, 1971,  
APPENDIX - STRATEGY OF ARGUMENT REDUCTION

Within the scope of this report argument reduction is required only for the exponential function and for the circular functions. No argument reduction is required for the logarithm approximation in the sense that the working argument is obtained without error from the floating-point representation of the actual argument.

For these cases, given the related transcendental constant \( K \) (either \( \ln(2) \) or \( \pi/2 \)), the reduced argument \( y \) is defined in terms of \( K \) and the given argument \( x \) by

\[
y = x - nK
\]  

(A1)

where \( n \) is an integer. Because the approximations are constrained to have negligible error for \( y = \pm K/2 \), adequately small errors will result for a somewhat wider interval. We, therefore, require only that \( y \) lie in the interval

\[
-\left(\frac{K}{2} + \Delta\right) < y < \frac{K}{2} + \Delta
\]  

(A2)

Table I given at the end of this appendix shows the value of \( \Delta \) allowed by each of these approximations.

Given an upper bound \( N \) on the magnitude of the integers allowed for use in relation (A1) a value of \( n \) for which inequality (A2) is satisfied is given by

\[
n = [kx]
\]  

(A3)

The symbol \([Z]\) means the nearest integer to \( Z \) and the multiplier \( k \) satisfies the inequality

\[
\frac{1}{K + \frac{2\Delta}{2N + 1}} < k < \frac{1}{K - \frac{2\Delta}{2N - 1}}
\]  

(A4)

If \( 2\Delta/(2N + 1) \) is greater than \( \beta \) times the value of a one in the least significant digit of the machine precision representation of \( K \), then the numbers \( 1/\left(K + [2\Delta/(2N + 1)]\right) \), \( 1/K \), \( 1/\left(K - [2\Delta/(2N - 1)]\right) \) have distinct representations. The rounded for storage representation of the value \( 1/K \) is then a suitable value for \( k \).

In the case of the exponential function the bound \( N \) is typically determined by the limitations of exponent overflow or underflow on the representation of the computed result. For the circular functions which (except for poles) are defined and representable.
for all arguments the bound on $N$ must be somewhat arbitrary and is related to the details of the actual evaluation of the reduced argument $y$.

For any of these functions the required transcendental constant, $\ln(2)$ or $\pi/2$, cannot be exactly represented. It may, however, be represented to any required precision as a sequence of constants $K_1, K_2, \ldots$ of successively decreasing magnitude whose correct sum is very nearly equal to the desired $K$. At least three such constants are generally required. A minimum limitation on the lengths of the constants $K_1$ and $K_2$ is that the products $nK_1$ and $nK_2$ be exactly representable in the floating-point notation of the computer of implementation.

A further requirement of any implementation is that the difference $x - nK_1$ be computed exactly. This cannot be guaranteed for an arithmetic system in which no guard digits are provided for floating point addition unless the given argument $x$ is broken into shorter parts and the constant $K_1$ subject to more severe restrictions on its length. In any case, when $K_1$ is subjected only to the limitation that the product $nK_1$ be exactly representable the difference $x - nK_1$ is always exactly representable.

For any $n$ there is always some value of $x$ such that $x - nK_1$ equals zero. The reduced argument is then the negative of the correctly rounded sum of $nK_2 + nK_3$ which should cause a minimum of trouble.

If $K_1, K_2,$ and $K_3$ are of the same sign and the sign of $x - nK_1$ is opposite to that of $x$, the final calculation of the reduced argument requires the correct addition of three terms of like sign. No arithmetic trouble occurs in adding these terms in the order $(nK_3 + nK_2) + (x - nK_1)$ with rounding on the final addition. If $K_1, K_2,$ and $K_3$ are of the same sign and the sign of $x - nK_1$ is the same as the sign of $x$, which should happen in about one-half the cases, completion of the argument reduction can cause further cancellation of lead digits and result in an unrecoverable error. Greater care with regard to the details of the reduction is required to avoid unwanted loss of precision. In this situation the difficulty caused by mixed signs could be resolved by the use of a second set of constants $K'_1, K'_2, \ldots$, where $K'_1$ is just larger than $K_1$ and the $K'_2, \ldots$ are negative; therefore, the smaller terms $nK'_2, \ldots$ have the same sign as $x - nK'_1$. The small interval for which $x - nK_1$ has the same sign as $x$ but $x - nK'_1$ is opposite in sign remains unresolved. Assuming that this variant is implemented, difficulty with further cancellation can occur only for very small reduced arguments.
TABLE I. - VALUES OF $\Delta$ FOR VARIOUS APPROXIMATIONS

<table>
<thead>
<tr>
<th>J</th>
<th>exp($Y$) EXP[ln(2)/2, J, 0]</th>
<th>sin($Y$) SIN($\pi/4$, J, 0)</th>
<th>cos($Y$) COS($\pi/4$, J, 0)</th>
<th>tan($Y$) TAN($\pi/4$, 0, J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.01041</td>
<td>0.02881</td>
<td>-------</td>
<td>0.01780</td>
</tr>
<tr>
<td>3</td>
<td>0.00585</td>
<td>0.01561</td>
<td>0.01788</td>
<td>0.01015</td>
</tr>
<tr>
<td>4</td>
<td>0.00378</td>
<td>0.00983</td>
<td>0.01054</td>
<td>0.00702</td>
</tr>
<tr>
<td>5</td>
<td>0.00265</td>
<td>0.00677</td>
<td>0.00704</td>
<td>0.00505</td>
</tr>
<tr>
<td>6</td>
<td>0.00196</td>
<td>0.00495</td>
<td>0.00507</td>
<td>0.00382</td>
</tr>
<tr>
<td>7</td>
<td>0.00152</td>
<td>0.00378</td>
<td>0.00383</td>
<td>0.00300</td>
</tr>
<tr>
<td>8</td>
<td>0.00121</td>
<td>0.00298</td>
<td>0.00300</td>
<td>0.00242</td>
</tr>
<tr>
<td>9</td>
<td>0.00098</td>
<td>0.00241</td>
<td>0.00242</td>
<td>0.00199</td>
</tr>
<tr>
<td>10</td>
<td>-------</td>
<td>0.00199</td>
<td>0.00199</td>
<td>0.00167</td>
</tr>
<tr>
<td>11</td>
<td>-------</td>
<td>-------</td>
<td>0.00167</td>
<td>0.00142</td>
</tr>
<tr>
<td>12</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>0.00122</td>
</tr>
<tr>
<td>13</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
</tbody>
</table>
REFERENCES


"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—National Aeronautics and Space Act of 1958

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