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AIRCRAFT RANGE OPTIMIZATION USING SINGULAR PERTURBATIONS

by
Joseph Taffe O'Connor
June 1973

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by

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by

JOSEPH TAFFE O'CONNOR

Submitted to the Department of Aeronautics and Astronautics on June 8, 1973, in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

ABSTRACT

An approximate analytic solution is developed for the problem of maximizing the range of an aircraft for a fixed end state. In general, this problem can not be solved analytically and is even very difficult to solve computationally. In this dissertation the problem is formulated as a singular perturbation problem and solved by means of matched inner and outer asymptotic expansions and the minimum principle of Pontryagin.

Attention is focused on cruise at constant Mach number in the stratosphere, and on transition to and from cruise at constant Mach number. The state vector includes altitude, flight path angle, and mass. Normal acceleration and maneuvering drag effects are included. Lift is the control variable. Since Mach number is constant, thrust is constrained to be a function of state and control variables and is not itself a control variable. Specific fuel consumption becomes a linear function of power setting in the vicinity of cruise values.

Cruise represents the outer solution. In cruise, altitude and flight path angle are essentially constant and only mass changes. In the inner solutions, corresponding to transitions between cruise and the specified initial and final conditions, mass is essentially constant and altitude and velocity vary.

A solution is developed which is valid for cruise but which fails to satisfy the initial and final conditions. The cruise solution is shown to yield the Breguet range equation. By transforming the independent variable near the initial and final conditions, we can seek solutions which are valid for the two inner solutions but not for cruise.

The inner solutions can not be obtained without simplifying the state equations. However, to linearize them would completely eliminate their dependence on altitude, as well as the dependence on altitude of any potential optimal control. The singular perturbation approach overcomes this difficulty by allowing us to make a quadratic approximation to some of the state equations under certain circum-

stances. The resulting problem is solved analytically, and the two inner solutions are matched to the outer solution. A modified Breguet range equation is developed which accounts for the changes in range due to starting from initial conditions not on a Breguet cruise and ending at final conditions not on a Bregeut cruise.

The optimal control policy for transition is compared to several alternate control policies for supersonic cruise using the Boeing SST and for transonic cruise using the Boeing 707 and the McDonnell Douglas F-4.

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GLOSSARY

a	defined in equation (3.37)
$c_{0,\ 1}$	coefficients in approximation to SFC (3.15)
C_D	total drag coefficient
C_{D_o}	profile drag coefficient
C_F	thrust coefficient
C_L	lift coefficient
D	aerodynamic drag force
f	subscript for final value
$f_{1,\ 2,\ 3}$	forcing functions (5.57, 5,59, 5.61)
g	gravitational acceleration, 32.17 ft/sec^2
h	altitude
ΔH	scaled altitude increment (3.18)
$\Delta\hat{H}$	(5.10)
$\Delta\tilde{H}$	(5.66)
$\mathscr{H}$	variational Hamiltonian
i	subscript for initial value
K	induced drag constant
L	aerodynamic lift force
m	mass
M	normalized mass (3.20)
M_{lb}	(4.53, 4.54)
$\tilde{M}$	(5.64)
$\mathscr{M}$	Mach number
p	time-derivative operator
q	dynamic pressure

Q	scaled dynamic pressure (3.19)
$\hat{Q}$	(5.8, 5.9)
$\tilde{Q}$	(5.66)
r	range
R	scaled range, the independent variable
R_L	transformed range in the left boundary layer (5.1)
R_R	transformed range in the right boundary layer (5.2)
S	wing area
SFC	specific fuel consumption
T	thrust
u	control variable (3.12)
v	true airspeed
W	weight
x	defined in equations (4.21, 4.32, 4.35)
y	defined in equation (3.16)
Z	(5.49)
β	atmospheric scale height $(20,800 \text{ ft})^{-1}$
γ	flight path angle
γ_{lb}	(5.15)
ϵ	perturbation parameter (3.35)
ζ	damping ratio
η	throttle setting, or T/T_{max}
λ_γ	costate for flight path angle
λ_H	costate for altitude
λ_M	costate for mass
$\lambda_{\tilde{M}}$	costate for mass (5.67)

ρ	atmospheric density
ω	frequency of oscillation
ω_n	natural frequency
$(\dot{\ })$	$d(\ )/dt$
$(\ )'$	$d(\ )/dR$

CHAPTER I

INTRODUCTION

1.1 Problem Statement

This thesis is an attempt to develop an analytic solution to an airplane performance optimization problem that has proven very difficult to solve by computational methods. It is the problem of maximizing the cruise range of a gas-turbine powered aircraft. The problem is stated as a fixed end-state optimal control problem with a three dimensional state vector (altitude, flight path angle, and mass) in which the independent variable (range) is to be maximized. Attention is focused on cruise at constant Mach number in the stratosphere, and on transition to and from cruise at constant Mach number. Simplified computational solutions have indicated the existence of transonic and supersonic Mach number limiting, or constant Mach number cruise, for range optimal trajectories [1]. By use of singular perturbation techniques and the minimum principle of Pontryagin, approximate analytic solutions are obtained as asymptotic expansions in three separate segments of the problem: cruise, and transitions to and from cruise, and these segments are matched to form a single solution, valid from initial through final conditions.

1.2 Maximum Range in Cruise

The problem of evaluating maximum range in cruise is of fundamental importance to the design of any aircraft. It can not be separated

from the basic choice of mating a power-plant, with its thrust-producing and fuel-consuming characteristics, to an airframe with its lift and drag characteristics in cruise and its fuel-carrying capability. An airframe-power-plant combination will have one best altitude for cruise and one best velocity for cruise. At that speed and altitude the rate of consumption of fuel with respect to range is minimized. The aircraft must, of course, be able to maintain equilibrium flight at that altitude and velocity. The choice of the optimum altitude-velocity combination requires an iterative approach. At a trail point drag can be calculated. Referring to the engine characteristics (maximum thrust as a function of altitude and Mach number) one can determine if there is enough thrust available to balance the drag in equilibrium flight. If there is not, the trial point is not valid. For a valid trial point the fuel consumption, in pounds per second, corresponding to the required level of thrust is divided by velocity to give the parameter, pounds of fuel per mile, which is to be minimized. The minimum value will correspond to an altitude that is a compromise between the altitude for the most efficient unpowered flight by the aircraft (maximum lift-drag ratio) and that for the most efficient operation of the power-plant at constant velocity.

Cruise velocity should be as large as possible but for gas-turbine powered aircraft it is limited by two considerations and hence will occur in one of two velocity regions, transonic or supersonic. For a transonic optimum the cruise velocity is limited by the beginning of the transonic drag rise. The rapid increase in drag associated with the transonic region translates to increased thrust required to balance the drag in cruise and to increased fuel consumption. For a supersonic optimum the cruise velocity is limited not by fuel consumption but by consideration

of the maximum temperature that the airframe can withstand [2].

Having determined a cruise condition, one can estimate the resulting range capability by means of the Breguet range equation. This classic relation equates range to the product of a powered flight efficiency factor and the natural logarithm of the ratio of the initial mass of the aircraft to the final mass of the aircraft after its fuel budget is used up. The powered flight efficiency factor is a product of the lift-drag ratio (airframe efficiency) and the ratio of cruise speed to specific fuel consumption (power-plant efficiency). This relation, of course, estimates range only for flight at the previously determined cruise condition and in no way accounts for flight to or from that condition.

The Breguet range equation can be said to represent a one-dimensional approach to range capability estimation. To derive it one need consider only the state differential equation for mass, together with the equilibrium flight assumptions of lift equals weight and thrust equals drag. A correction factor can be derived to account for the fact that altitude does not remain constant in Breguet cruise but must slowly increase as fuel usage causes weight to decrease [3]. Only initial and final mass can be specified. Altitude is essentially a control variable, chosen at a particular value of mass to maximize the derivative of range with respect to fuel.

Edelbaum has shown [4] that in the larger context of a maximum range cruise including initial transition to cruise and final transition from cruise, the Breguet cruise describes the optimal cruise portion for those

problems in which range is not so short as never to require a cruise portion. Edelbaum formulates a solution to the problem of range-optimal climb to cruise and descent from cruise in terms of the energy-state method, another one-dimensional approach. Use of the energy-state method, described by various authors [1, 4 through 9], permits changes in velocity and thereby a complete solution from sea level to cruise. In the energy-state method normal acceleration is neglected (lift equals weight). As a result drag is a non-linear function of altitude, velocity and mass.

A recent study by Teren and Daniele [10] has expanded the state vector of the range-optimal cruise problem to two dimensions, altitude and mass, while neglecting normal acceleration and holding velocity constant. Thrust coefficient is taken as the control variable. Again, lift equals weight and drag is a non-linear function of altitude, mass and constant velocity. The problem is formulated as a non-linear two-point boundary value problem and an approximate graphical method of solution is presented.

Kelley, Falco and Ball [11] used a four dimensional state vector (velocity, altitude, flight path angle and mass) in studying various airplane performance problems including the maximum range problem. They were investigating the usefulness of the method of gradients in obtaining computational solutions to these problems. For short range problems their optimal result was a boost-glide or bang-bang solution. They reported that attempts at solving long range problems, which would include a constant velocity cruise segment, were frustrated by convergence difficulties.

This thesis considers the range-optimal cruise problem with a three-dimensional state vector comprised of mass, altitude, and flight path angle, with lift becoming the control variable. Velocity remains constant, and so the problem is restricted to cruise and transitions to and from cruise at cruise velocity. Inclusion of normal acceleration (equation for flight path angle) means that the effect of lift, as well as altitude and velocity, on drag is included. Approximate analytic solutions are developed through the use of singular perturbation methods which, as will be shown in Chapter V, allow the drag force to be expressed as a quadratic function of altitude, mass, and maneuvering lift. The equation for mass then becomes a quadratic function of altitude, flight path angle, mass and lift. The other two state equations are linearized and the resulting optimal control problem is solvable.

1.3 Singular Perturbation Problems

A singular perturbation problem [12, 13, 14] can be described as a set of differential equations involving a small dimensionless parameter, say ϵ. The nature of the ϵ-dependence is such that if ϵ were to approach zero the order of the set of differential equations would be reduced. As a result the boundary conditions associated with the equations could not all be met simultaneously for a zero value of ϵ. Viewed in another way one could say that the method of ordinary perturbations, involving the expansion of the dependent variables in power series in ϵ, would fail to produce a solution that would be valid in the neighborhood of the boundary conditions.

Such problems are often solved by the method of matched asymptotic expansions, in which a stretching transformation applied to the independent variable in the neighborhood of the singularity transforms the problem to one that can be solved by ordinary perturbation methods in that vicinity. Then these solutions which are valid only in the neighborhood of the singularity can, by a choice of constants, be matched with those solutions that apply everywhere except in the neighborhood of the singularity to produce a single solution that will be valid throughout the region of interest of the problem.

The range-optimal cruise problem seems well suited to formulation as a singular perturbation problem. It is convenient to think of the problem as separable into a climb, a cruise and a descent. Certain variables, such as altitude and flight path angle, undergo their greatest variations during climb and descent, but remain nearly constant during cruise. For mass the reverse is true, it being nearly constant in climb and descent, but varying most during cruise. Thus the problem is largely describable in terms of mass variation at nearly constant altitude and flight path angle except in "boundary layers" near initial and final time. We may think of mass as having its own characteristic time which is different from that of altitude and flight path angle. This characteristic of the problem makes it likely to be describable as a singular perturbation problem and offers the hope of yielding an approximate analytic solution that is uniformly valid over the entire time interval of the problem.

Interest in singular perturbation methods as applied to problems in aircraft dynamics begins with Ashley [15]. Drawing on an earlier work by Kevorkian [16] on reduced-order modelling, Ashley was able to demonstrate the separation of aircraft longitudinal dynamics into the short period and phugoid modes on the basis of the wide separation of their characteristics times. Kelley and Edelbaum [6] explored the idea of using singular perturbation methods to obtain a first order improvement to the energy-state solution to some optimal performance problems for airplanes. They thereby avoid the unrealistic instantaneous changes in altitude and velocity that occur in energy state solutions. Kelley also has suggested the use of singular perturbations in two-point boundary value problems and in reduced order modelling of aircraft performance problems [17, 18, 19].

Kelley's objectives were to find reduced order approximations to certain airplane performance optimization problems that could still be related to the higher order computational solutions. These reduced order solutions could serve either to provide insights to improve the computational solution or as good approximations in themselves to the higher order computational solutions.

This thesis carries forward the ideas of Kelley and Edelbaum by setting up and solving the range optimal cruise problem as a singular perturbation problem. The perturbation parameter is developed naturally out of the parameters of the problem. The resulting solution is easily

related to lower order solutions and the nature of the solution gives indications of why the computational solutions are difficult to obtain.

1.4 Chapter Summary

Chapter II presents and solves a problem similar in form to the state equations of the range optimal cruise problem. The solution of this problem demonstrates the techniques of solving a singular perturbation problem by means of stretching transformations applied to the independent variable in the vicinity of singularities (boundary layers) and matched asymptotic expansions.

Chapter III shows that the range optimal cruise problem can be expressed as a singular perturbation problem with singularities occurring at the initial and final state.

In Chapter IV a solution is obtained to the problem of cruising flight which is valid everywhere except in the vicinity of the singularities. It is shown to be the Breguet solution.

In Chapter V a solution is developed that is valid in the vicinity of general initial and final conditions but fails to be valid elsewhere. The solution is obtained by applying the minimum principle after expressing the problem as a linear optimal control problem with a quadratic cost. An optimal control is obtained and the optimal state trajectories are matched asymptotically to the cruise solution. A corrected Breguet equation is developed, accounting for fuel penalties (or bonuses)

associated with achieving initial and final conditions.

Chapter VI presents a cost comparison of the optimal trajectory with trajectories using various non-optimal controls in the near vicinity of cruise. It also studies the control as a sub-optimal control over large changes in altitude. Applications to several different aircraft are discussed.

Chapter VII presents the conclusions and contributions of this thesis and suggests possible future work related to the thesis.

CHAPTER II

SINGULAR PERTURBATION PROBLEMS

Singular perturbation problems and the techniques of solving them are most easily presented by formulating and solving a demonstration problem. The demonstration problem used here is chosen for similarity to the state equations of the range optimization problem. It is adapted from O'Malley [13].

Consider the following set of differential equations in x, y, t and ϵ

$$\dot{x} = y \tag{2.1}$$

$$\epsilon\dot{y} = -x - y \tag{2.2}$$

where ϵ is a small parameter. Initial values of x and y are specified

$$x(t = 0) = a \tag{2.3}$$

$$y(t = 0) = b \tag{2.4}$$

It happens that this set of equations could be solved directly in terms of ϵ to give the result

$$x(t) = c_1 e^{+r_1 t} + c_2 e^{+r_2 t} \tag{2.5}$$

$$y(t) = r_1 c_1 e^{+r_1 t} + r_2 c_2 e^{+r_2 t} \tag{2.6}$$

where

$$r_1 = -\frac{1}{2\epsilon}\left[1 + \sqrt{1 - 4\epsilon}\right] \tag{2.7}$$

$$r_2 = -\frac{1}{2\epsilon}\left[1 - \sqrt{1 - 4\epsilon}\right] \tag{2.8}$$

$$c_1 = \frac{r_2 a - b}{r_2 - r_1} \tag{2.9}$$

$$c_2 = \frac{-r_1 a + b}{r_2 - r_1} = a - c_1 \tag{2.10}$$

In cases where the equations can not be solved directly a useful technique is to assume that the dependent variables can be expanded in power series in ϵ

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) \tag{2.11}$$

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) \tag{2.12}$$

where the x_i and y_i are functions of t only and are independent of ϵ.

At this point some terms will be defined as they relate to this thesis. More rigorous definitions of these terms can be found in [14]. Consider the sequence

$$\epsilon, \epsilon^2, \epsilon^3 \ldots \tag{2.13}$$

As $\epsilon \to 0$ each term is small relative to the one preceding it. Such a sequence is called an asymptotic sequence. Our expansion for $x(t, \epsilon)$ (and $y(t, \epsilon)$) is a series of functions of time weighted by successive terms of the asymptotic sequence in powers of ϵ. If, as $\epsilon \to 0$, each term of the expansion is small relative to the one preceding it, the series can be called an asymptotic expansion. If in some domain of interest, D, the series is an asymptotic expansion for all values of t within D, then the asymptotic expansion is said to be uniformly valid within D.

Now our solution technique calls for the asymptotic expansions for x and y to be substituted into the original equations. The resulting series expansion of the left hand side of each equation must equal the series expansion of the right hand side of the equation. Since the x_i and y_i are independent of ϵ and since the expanded equations are valid for all small ϵ, the coefficients of a given power of ϵ on both sides of an equation must be equal. The zero order problem (coefficients of ϵ^0) is solved first. The first order problem is then solved in terms of the zero order problem, etc. This is the approach of ordinary perturbations. For it to be successful the resulting solution must be uniformly valid in the time domain of interest.

Proceeding with our example, the expanded equations, to first order in ϵ, are

$$\dot{x}_0 + \epsilon\dot{x}_1 = y_0 + \epsilon y_1 \tag{2.14}$$

$$\epsilon\dot{y}_0 = -(x_0 + y_0) - \epsilon(x_1 + y_1) \tag{2.15}$$

The zero order problem is

$$\dot{x}_0 = y_0 \tag{2.16}$$

$$x_0 + y_0 = 0 \tag{2.17}$$

$$x_0(t = 0) = a \tag{2.18}$$

$$y_0(t = 0) = b \tag{2.19}$$

Its solution is

$$x_0 = c_3 e^{-t} \tag{2.20}$$

$$y_0 = -c_3 e^{-t} \tag{2.21}$$

The first order problem is

$$\dot{x}_1 = y_1 \tag{2.22}$$

$$\dot{y}_0 = -x_1 + y_1 \tag{2.23}$$

$$x_1(t = 0) = 0 \tag{2.24}$$

$$y_1(t = 0) = 0 \tag{2.25}$$

The result for y_0 is now used in solving the first order problem

$$x_1 + y_1 = -c_3 e^{-t} \tag{2.26}$$

Differentiating and substituting for $\dot{x}_1$

$$\dot{y}_1 + y_1 = c_3 e^{-t} \tag{2.27}$$

$$y_1 = c_4 e^{-t} + c_3 t e^{-t} \tag{2.28}$$

$$x_1 = -c_4 e^{-t} - c_3 e^{-t}(1+t) \tag{2.29}$$

Summarizing for x and y we have a solution to first-order in ϵ or to two terms of the expansions

$$x = c_3 e^{-t} - \epsilon[c_4 e^{-t} + c_3 e^{-t}(1+t)] \tag{2.30}$$

$$y = -c_3 e^{-t} + \epsilon[c_4 e^{-t} + c_3 t e^{-t}] \tag{2.31}$$

Now it is obvious that for $\epsilon = 0$ this solution cannot satisfy general initial conditions on both x and y. It can only satisfy one of them, and requires that the other be equal to it.

$$a = -b \tag{2.32}$$

For any other initial conditions, one of the variables, say y, must make an instantaneous jump at $t = 0+$ from its true initial condition to the value $-x(0)$ and that jump is not necessarily of order ϵ. That is, at $x = 0$ we must have $\dot{y} \to \infty$ so that we cannot say $\epsilon\dot{y} \to 0$ as $\epsilon \to 0$. Borrowing a term from similar problems in fluid mechanics we refer to this singular region as a boundary layer. The width of this boundary layer is of order ϵ. Outside of this boundary layer, that is, for

$$t > \epsilon \geq 0 \tag{2.33}$$

the solution is valid.

Such problems, for which ordinary perturbations fail, comprise a large class of singular perturbation problems. A direct way of identifying this kind of singular perturbation problem is by the fact that for $\epsilon = 0$ the order of the system of first order differential equations is reduced. Instead of two differential equations we have one algebraic and one differential equation. That means a reduction in the number of constants available for meeting boundary conditions and hence a failure to be able simultaneously to satisfy all boundary conditions.

In order to analyze the region in which our solution fails to be valid it is useful to "stretch" the independent variable by the transformation

$$\tau = \frac{t}{\epsilon} \tag{2.34}$$

with the differential relationship

$$\frac{1}{\epsilon}\frac{d}{d\tau} = \frac{d}{dt} = (\dot{\ }) \tag{2.35}$$

We can think of this device as allowing us to view the problem on a faster time-scale as, for example, one would change the time-base on an oscilloscope to reveal an initial transient in what had appeared as a square wave. The transformed equations are

$$\frac{d}{d\tau} x = \epsilon y \tag{2.36}$$

$$\frac{d}{d\tau} y = -x - y \tag{2.37}$$

Again we expand dependent variables in powers of ϵ and group corresponding powers of ϵ

$$\frac{d}{d\tau} x_0 = 0 \tag{2.38}$$

$$\frac{d}{d\tau} y_0 = -x_0 - y_0 \tag{2.39}$$

$$\frac{d}{d\tau} x_1 = y_0 \tag{2.40}$$

$$\frac{d}{d\tau} y_1 = -x_1 - y_1 \tag{2.41}$$

The initial conditions are

$$x_0(\tau = 0) = a \tag{2.42}$$

$$y_0(\tau = 0) = b \tag{2.43}$$

$$x_i(\tau = 0) = y_i(\tau = 0) = 0 \qquad i = 2, 3, \ldots \tag{2.44}$$

Notice that the transformed equations comprise a regular perturbation problem. In solving this problem it will now be possible to satisfy the boundary conditions. It is also important to notice that the transformed equations can be solved. If they cannot be solved, or if the untransformed equations cannot be solved, then the boundary layer transformation is of no value, since solutions of both problems are required.

The zeroth order problem is solved directly as

$$x_0 = k_0 \tag{2.45}$$

$$y_0 = k_1 e^{-\tau} - k_0 \tag{2.46}$$

The first order problem is solved as

$$\frac{d}{d\tau} x_1 = k_1 e^{-\tau} - k_0 \tag{2.47}$$

$$x_1 = -k_1 e^{-\tau} - k_0 \tau + k_2 \tag{2.48}$$

$$\frac{d}{d\tau} y_1 = -y_1 + k_1 e^{-\tau} + k_0 \tau - k_2 \tag{2.49}$$

$$y_1 = -(k_0 + k_2) + k_0 \tau + k_1 \tau e^{-\tau} + k_3 e^{-\tau} \tag{2.50}$$

Now it is possible to satisfy initial conditions to zeroth order in ϵ. From the zeroth order solution we have

$$k_0 = a \tag{2.51}$$

$$k_1 = a + b \tag{2.52}$$

The first order solution yields

$$k_2 = a + b \tag{2.53}$$

$$k_3 = 2a + b \tag{2.54}$$

Now we have a solution to first order in ϵ, or to two terms of the expansion, that is uniformly valid in the boundary layer.

$$x = a + \epsilon\left[a(1 - \tau - e^{-\tau}) + b(1 - e^{-\tau})\right] \tag{2.55}$$

$$y = -a(1 - e^{-\tau}) + be^{-\tau} + \epsilon\left[a(-2 + \tau + 2e^{-\tau} + \tau e^{-\tau}) + b(-1 + \tau e^{-\tau} + e^{-\tau})\right] \tag{2.56}$$

The problem now is to reconcile this solution, valid in the boundary layer, with the previous solution, valid everywhere in the region except the boundary layer. It is common to call the boundary layer solution the inner solution. The other solution, which is valid everywhere in the region of interest except the boundary layer, is called the outer solution. In the problem of maximizing range in cruise there will be one outer solution (cruise) and two inner solutions. The first inner solution will describe the problem in the neighborhood of the initial conditions. The second inner solution will describe the problem in the neighborhood of the final conditions.

In our demonstration problem the inner and outer solutions are combined t give a single uniformly valid solution by the technique of matched asymptotic expansions. The solutions are not matched at a point as one would match boundary conditions. Matching is based on the notion that the inner solution, valid in the boundary layer, and the outer solution, valid outside of the boundary layer, must both be valid in some overlap region.

The inner solution is now extended to a form that it approaches beyond the boundary layer. First the independent variable is transformed to that of the outer solution

$$\tau = \frac{t}{\epsilon} \tag{2.57}$$

The solution is then expanded in powers of ϵ. The resulting expansion is called the outer expansion of the inner solution.

The outer solution is now extended to a form that is approaches as it approaches the boundary layer from some large value of t. First the independent variable is transformed to that of the inner solution

$$t = \epsilon\tau \tag{2.58}$$

The solution is then expanded in powers of ϵ. The resulting expansion is called the inner expansion of the outer solution. By suitable choice of the undetermined constants of the inner solution it will be possible to make the inner expansion of the outer solution identical, up to a certain order of ϵ, to the outer expansion of the inner solution.

First we evaluate the outer expansion of the inner solution. Using the transformation (2.57) we have

$$x^{i2} = a(1 - t) + \epsilon[a + b](1 - e^{-t/\epsilon}) \tag{2.59}$$

$$y^{i2} = -a(1 - t) + [a + b]e^{-t/\epsilon}(1 + t)$$

$$-\epsilon[2a + b](1 - e^{-t/\epsilon}) \tag{2.60}$$

These equations are expanded in powers of ϵ, and in the limit of small ϵ the exponential terms are vanishingly small. Using notation similar to that of O'Malley we write the outer expansion to two terms (order zero and one in ϵ) of the inner solution to two terms as a function of t and ϵ as

$$[x(t, \epsilon)^{i2}]^{o2} = a(1 - t) + \epsilon(a + b) \tag{2.61}$$

$$[y(t, \epsilon)^{i2}]^{o2} = -a(1 - t) - \epsilon(2a + b) \tag{2.62}$$

Now we evaluate the inner expansion of the outer solution. Using the transformation (2.58) we have

$$x^{o2} = c_3e^{-\epsilon\tau} - \epsilon c_4e^{-\tau} - \epsilon c_3(1 + \epsilon\tau)e^{-\epsilon\tau} \tag{2.63}$$

$$y^{o2} = -c_3e^{-\epsilon\tau} + \epsilon c_4e^{-\epsilon\tau} + \epsilon^2c_3\tau e^{-\epsilon\tau} \tag{2.64}$$

Expanding in powers of ϵ we can write the inner expansion to first order in ϵ of the outer solution to first order in ϵ as a function of τ and ϵ

$$[x(\tau, \epsilon)^{o2}]^{i2} = c_3 - \epsilon(c_3\tau + c_4e^{-\tau} + c_3) \tag{2.65}$$

$$[y(\tau, \epsilon)^{o2}]^{i2} = -c_3 + \epsilon(c_3\tau + c_4) \tag{2.66}$$

Transforming back to functions of t we have

$$[x(t, \epsilon)^{o2}]^{i2} = c_3(1 - t) - \epsilon(c_3 + c_4) \tag{2.67}$$

$$[y(t, \epsilon)^{o2}]^{i2} = -c_3(1 - t) + \epsilon c_4 \tag{2.68}$$

Now, the condition for matching is that

$$[x(t, \epsilon)^{o2}]^{i2} = [x(t, \epsilon)^{i2}]^{o2} \tag{2.69}$$

$$[y(t, \epsilon)^{o2}]^{i2} = [y(t, \epsilon)^{i2}]^{o2} \tag{2.70}$$

This is accomplished if we select the constants c_3 and c_4 as

$$c_3 = a \tag{2.71}$$

$$c_4 = -2a - b \tag{2.72}$$

Now we can proceed to write a composite solution for x and y in terms of t valid to first order in ϵ throughout the region of interest. To begin with, this solution will be the sum of the inner and outer solutions. However, that implies doubly describing the variables in the overlap region where matching takes place. To remove this effect we subtract out the inner (or outer) expansion of the outer (or inner) solution. Finally, for our composite soultion we have

$$x(t, \epsilon)^{c2} = x(t, \epsilon)^{i2} + x(t, \epsilon)^{o2} - [x(t, \epsilon)^{i2}]^{o2} \tag{2.73}$$

$$\begin{aligned} x(t)^{c2} = {} & a(1 - t) + \epsilon[a + b](1 - e^{-t/\epsilon}) + ae^{-t} \\ & - \epsilon a(1 + t)e^{-t} + \epsilon[2a + b]e^{-t/\epsilon} - a(1 - t) \\ & - \epsilon(a + b) \end{aligned} \tag{2.74}$$

$$y(t)^{c2} = -a(1 - t) + [a + b]e^{-t/\epsilon}(1 + t)$$

$$-\epsilon[2a + b](1 - e^{-t/\epsilon}) - ae^{-t} - \epsilon[2a + b]e^{-t}$$

$$+ \epsilon ate^{-t} + a(1 - t) + \epsilon(2a + b) \qquad (2.75)$$

These solutions simplify to

$$x(t)^{c2} = ae^{-t} + \epsilon[ae^{-t/\epsilon} - a(1 + t)e^{-t}] \qquad (2.76)$$

$$y(t)^{c2} = (a + b)(1 + t)e^{-t/\epsilon} - ae^{-t}(1 - \epsilon t)$$

$$- \epsilon[2a + b](e^{-t} - e^{-t/\epsilon}) \qquad (2.77)$$

It is evident that these composite solutions satisfy the initial conditions exactly. For other values of t the error between these solutions and the exact solutions given by (2.5) and (2.6) will be $O(\epsilon^2)$.

CHAPTER III

THE MAXIMUM RANGE PROBLEM AS A SINGULAR PERTURBATION PROBLEM

In this Chapter it will be demonstrated that the state differential equations of the maximum range problem can be formulated in terms of a small dimensionless parameter, ϵ, and that in the limit as ϵ approaches zero, the order of the problem is reduced. Some assumptions that are used to simplify the equations are discussed and symbols are defined.

The state differential equations for altitude (h), range (r), mass (m) and flight path angle (γ) are, respectively

$$\dot{h} = v \sin\gamma \tag{3.1}$$

$$\dot{r} = v \cos\gamma \tag{3.2}$$

$$\dot{m} = -\frac{T}{g}(SFC) \tag{3.3}$$

$$\dot{\gamma} = \frac{g}{v}\left(\frac{L}{mg} - \cos\gamma\right) \tag{3.4}$$

Equation (3.4) incorporates the conventional assumption that the component of thrust (T) in the direction of lift (L) is degligible [1].

True air speed (v) is assumed to be a constant. The constant speed cruise condition is predicted by range-optimal energy state solutions [1]. It will occur either at the transonic drag rise or at the maximum supersonic Mach number. This assumption means that the problem is restricted to cruise and to constant speed transitions to and from cruise. It also means

that thrust is no longer a control variable. Instead its value is constrained to be such that v remains constant

$$T = W \sin\gamma + D \tag{3.5}$$

Aerodynamic drag, D, is described as follows

$$D = C_D S q \tag{3.6}$$

The wing area is S. The drag coefficient, C_D, is assumed to be the sum of a profile drag coefficient, or drag coefficient for zero lift, C_{D_o}, and a term proportional to the square of the lift coefficient. The proportionality factor, K, is the coefficient of induced drag. Both C_{D_o} and K are functions of Mach number.

$$C_D = C_{D_o} + KC_L^2 \tag{3.7}$$

$$D = C_{D_o} Sq + \frac{KL^2}{Sq} \tag{3.8}$$

$$D = C_{D_o} Sq + \frac{KW^2(1 + u)^2}{Sq} \tag{3.9}$$

The dynamic pressure, q, is

$$q = \frac{1}{2}\rho v^2 \tag{3.10}$$

and if we restrict our problem to the stratosphere we have an isothermal atmosphere and two simplifications result: atmospheric density, ρ, becomes an exponential function of altitude

$$\rho = \rho_i e^{-\beta(h - h_i)} \tag{3.11}$$

and the speed of sound becomes a constant. Mach number is therefore a constant in view of our assumption of constant v, and C_{D_o} and K are also constants.

The control variable, u, is defined as

$$u = \frac{L}{W} - 1 \tag{3.12}$$

Specific Fuel Consumption, SFC, is assumed to be describable as a function of thrust coefficient, C_F, where

$$C_F = \frac{T}{Sq} \tag{3.13}$$

It has been shown in [4] that SFC is a function of Mach number, power setting, and atmospheric temperature. Since our problem is restricted to constant Mach number flight in an isothermal atmosphere, SFC depends only on power setting, or C_F. The nature of this dependence is shown for typical transonic and supersonic cruising aircraft in Appendix A. In this problem we assume that in the vicinity of the cruise value of C_F we can express SFC as a linear function of C_F

$$SFC = SFC_c + \left(\frac{dSFC}{dC_F}\right)(C_F - C_{Fc}) \tag{3.14}$$

We can also write SFC as

$$SFC = c_0 + c_1 C_F \tag{3.15}$$

and, defining a constant, y, as

$$y = \frac{c_1}{c_0} C_{D_o} \tag{3.16}$$

we have

$$SFC = c_0 \left[1 + y \frac{C_F}{C_{D_o}}\right] \tag{3.17}$$

The constant, y, is a measure of the slope of the curve of SFC as a function of C_F. It recurs throughout the rest of this thesis in connection with the description and derivation of the optimal solution.

We now define the following dimensionless variables

$$\Delta H = \beta(h - h^*) \tag{3.18}$$

$$Q = \frac{q}{\frac{W^*}{S}\sqrt{\frac{K}{C_{D_o}}}} \tag{3.19}$$

$$M = \frac{m^* - m}{m^*} \tag{3.20}$$

$$R = \frac{r}{r^*} \tag{3.21}$$

where β is the scale height of the atmosphere and the asterisks denote reference values. For mass the reference value will be the initial value, m_i. It is now possible to express weight as

$$W = W_i(1 - M) \tag{3.22}$$

and drag as

$$D = W_i\sqrt{C_{D_o}K}\,Q\left[1 + \frac{(1 - M)^2(1 + u)^2}{Q^2}\right] \tag{3.23}$$

The reference value of q is the value that minimizes the expression for drag in equilibrium flight, that is, when T = D and L = W (u = 0). Differentiating (3.8) with respect to q and solving for q we have

$$q = \frac{W}{S}\sqrt{\frac{K}{C_{D_o}}} \tag{3.24}$$

Since L = W and v is constant in equilibrium flight, this equation defines the altitude for maximum lift-drag ratio. Also, since Q is a ratio of two dynamic pressures, we can express it as

$$Q = \frac{\frac{1}{2}\rho_0 e^{-\beta h} v^2}{\frac{1}{2}\rho_0 e^{-\beta h^*} v^2} \tag{3.25}$$

$$Q = e^{-\beta(h - h^*)} \tag{3.26}$$

$$Q = e^{-\Delta H} \tag{3.27}$$

Thus, from (3.19) and (3.26) we have tied the reference altitude to the reference weight. The reference value of altitude is the altitude for the maximum lift-drag ratio attainable at the initial value of mass. At that altitude we have

$$\Delta H = 0 \tag{3.28}$$

$$Q = 1 \tag{3.29}$$

Now if we express the state equations in terms of the dimensionless variables ΔH, R, M and γ, and convert from time to R as the independent variable by dividing by the equation

$$\frac{dR}{dt} = \frac{u}{r^*} \cos\gamma \tag{3.30}$$

we have

$$\frac{d\Delta H}{dR} = \beta r^* \tan\gamma \tag{3.31}$$

$$\frac{d\gamma}{dR} = \beta r^*\left(\frac{g}{\beta v^2}\right)\left(\frac{1+u}{\cos\gamma} - 1\right) \tag{3.32}$$

$$\frac{dM}{dR} = \frac{c_0 r^*}{v}\left(\frac{W\sin\gamma + D}{W_i}\right)\left(1 + \frac{c_1}{c_0}\frac{T}{Sq}\right)\sec\gamma \tag{3.33}$$

Since c_0 has the dimensions of SFC, namely inverse seconds, the quantity (v/c_0) is a distance. We can therefore define r^* as

$$r^* = \frac{v}{c_0} \tag{3.34}$$

Velocity will be on the order of 1,000 ft/sec, and typical values of c_0 are about 0.0005/sec [1, 10, 20], so r^* will be on the order of 2×10^6 ft.

In the stratosphere, β^{-1} can be taken as 20,800 ft [21]. Therefore, βr^*, or $\frac{\beta v}{c_0}$, is dimensionless and its value is on the order of 100. We therefore choose its inverse as our perturbation parameter.

$$\epsilon = \frac{c_0}{\beta v} \tag{3.35}$$

It will be shown in Chapter IV that this parameter can be related to the cruise flight path angle. In fact, for $y = 0$ we have

$$\epsilon = \left[\gamma\left(\frac{L}{D}\right)\right]_{cruise} \tag{3.36}$$

Finally, in the equation for flight path angle we shall use the following definition

$$a = \frac{g}{\beta v^2} \tag{3.37}$$

This parameter is much larger than ϵ. In fact it can be related to Mach number, $\mathcal{M}$, and hence is on the order of one.

$$\mathcal{M} = (k_{air}\, \beta v^2/g)^{1/2} \tag{3.38}$$

where k_{air} is the ratio of the specific heat of air at constant pressure to that at constant volume and has a value of approximately 1.4.

The state equations can now be written as follows

$$\frac{d\Delta H}{dR} = \frac{1}{\epsilon} \tan\gamma \tag{3.39}$$

$$\frac{d\gamma}{dR} = \frac{a}{\epsilon}\left(\frac{1+u}{\cos\gamma} - 1\right) \tag{3.40}$$

$$\frac{dM}{dR} = \left(\frac{W\sin\gamma + D}{W_i}\right)\left(1 + \frac{c_1}{c_0}\,\frac{T}{Sq}\right)\sec\gamma \tag{3.41}$$

It is seen that in the limit as ϵ approaches zero the differential equations for γ and ΔH become algebraic equations defining γ and u as zero, and arbitrary initial conditions on γ and ΔH could not be met. That is, for some non-zero initial value, γ would have to go to zero in a zero interval of range. It will be shown later that optimality considerations fix the constant value of ΔH which would also have to be achieved in a zero interval of range. Thus by demonstrating a dependence on ϵ, a reduction in order in the limit as ϵ approaches zero, and an inability to match given initial conditions in the limit as ϵ approaches zero, we have demonstrated that the state differential equations of the maximum range problem can be formulated as a singular perturbation problem.

CHAPTER IV

MAXIMUM RANGE CRUISE AND THE BREGUET RANGE EQUATION

We now develop the outer, or cruise, solution. If in the state differential equations we assume that we can use a series approximation to two terms for trigonometric functions of γ, we have

$$\tan \gamma = \gamma \tag{4.1}$$

$$\cos \gamma = 1 - \frac{\gamma^2}{2} \tag{4.2}$$

Using these approximations and substituting for T, W and D from (3.5), (3.22) and (3.23), the state differential equations become

$$\frac{d\Delta H}{dR} = \frac{1}{\epsilon}\gamma \tag{4.3}$$

$$\frac{d\gamma}{dR} = \frac{a}{\epsilon}\left(u + \frac{\gamma^2}{2}\right) \tag{4.4}$$

$$\frac{dM}{dR} = \left\{\left[(1-M)\gamma + \sqrt{C_{D_o}K}\,Q\left(1 + \frac{(1-M)^2(1+u)^2}{Q^2}\right)\right]\left[1 + \frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q}\right.\right.$$

$$\left.\left. + \; y\left(1 + \frac{(1-M)^2}{Q^2}(1+u)^2\right)\right]\left(1 + \frac{\gamma^2}{2}\right)\right\} \tag{4.5}$$

The optimal control problem is to find the control, u, that transfers the state (ΔH, γ, M), which is defined by the above three equations, from a given initial value, (ΔH_i, γ_i, M_i) to a fixed final value (ΔH_f, γ_f, M_f) while maximizing the final value of the independent variable, R_f.

We can write the variational Hamiltonian as

$$\mathcal{H} = -1 + \frac{a}{\epsilon}\lambda_\gamma\left(u + \frac{\gamma^2}{2}\right) + \frac{1}{\epsilon}\lambda_H\gamma + \lambda_M\left\{\left[(1-M)\gamma + \sqrt{C_{D_o}K}\left(Q\right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{Q}(1-M)^2(1+u)^2\right)\right]\left[1 + \frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q} + y\left(1 + \frac{(1-M)^2}{Q^2}(1+u)^2\right)\right]\left(1 + \frac{\gamma^2}{2}\right)\right\} \tag{4.6}$$

Expanding the state and control variables in powers of ϵ we have

$$\frac{d}{dR}(\gamma_0 + \epsilon\gamma_1) = \frac{a}{\epsilon}\left[\left(u_0 + \frac{1}{2}\gamma_0^2\right) + \epsilon(u_1 + \gamma_0\gamma_1) + \epsilon^2\left(u_2 + \frac{1}{2}\gamma_1^2 + \gamma_0\gamma_2\right)\right] \tag{4.7}$$

$$\frac{d}{dR}(\Delta H_0 + \epsilon\Delta H_1) = \frac{1}{\epsilon}\left(\gamma_0 + \epsilon\gamma_1 + \epsilon^2\gamma_2\right) \tag{4.8}$$

It is already apparent that

$$\gamma_0 = 0 \quad \text{and} \tag{4.9}$$

$$u_0 = \frac{1}{2}\gamma_0^2 = 0 \quad \text{and} \tag{4.10}$$

$$\frac{d}{dR}\gamma_0 = 0 \tag{4.11}$$

which implies that

$$u_1 = 0 \tag{4.12}$$

also. Therefore

$$\frac{d}{dR}(M_0+\epsilon M_1) = \left\{\sqrt{C_{D_o}K}\left[Q_0+\frac{(1-M_0)^2}{Q_0}\right]\left[1+y\left(1+\frac{(1-M_0)^2}{Q_0^2}\right)\right]\right\}$$

$$+\epsilon\left\{\gamma_1(1-M_0)\left[1+2y\left(1+\frac{(1-M_0)^2}{Q_0^2}\right)\right]+\Delta H_1 Q_0\sqrt{C_{D_o}K}\left[\left(-1\right.\right.\right.$$

$$\left.+\frac{(1-M_0)^2}{Q_0^2}\right)\left[1+y\left(1+\frac{(1-M_0)^2}{Q_0^2}\right)\right]+2y\frac{(1-M_0)^2}{Q_0^2}\left(1+\frac{(1-M_0)^2}{Q_0^2}\right)\Bigg]$$

$$\left.+M_1\frac{Q_0\sqrt{C_{D_o}K}}{1-M_0}\left[-2\frac{(1-M_0)^2}{Q_0^2}\left[1+y\left(1+\frac{(1-M_0)^2}{Q_0^2}\right)\right]-2y\frac{(1-M_0)^2}{Q_0^2}\left(1+\frac{(1-M_0)^2}{Q_0^2}\right)\right]\right\} \tag{4.13}$$

The costate differential equations are

$$\frac{d}{dR}\lambda_\gamma = -\left\{\frac{a}{\epsilon}\lambda_\gamma\gamma+\frac{1}{\epsilon}\lambda_H+\lambda_M(1-M)\left(1+\frac{\gamma^2}{2}\right)\left[1+\frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q}\right.\right.$$

$$\left.+y\left(1+\frac{(1-M)^2}{Q^2}(1+u)^2\right)\right]+\lambda_M\gamma\left[(1-M)\gamma+\sqrt{C_{D_o}K}\left(Q\right.\right.$$

$$\left.\left.+\frac{1}{Q}(1-M)^2(1+u)^2\right)\right]\left[1+\frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q}+y\left(1+\frac{(1-M)^2}{Q^2}(1+u)^2\right)\right]$$

$$\left.+\lambda_M\left[(1-M)\gamma+\sqrt{C_{D_o}K}\left(Q+\frac{1}{Q}(1-M)^2(1+u)^2\right)\right]\left(\frac{y}{\sqrt{C_{D_o}K}}\right)\left(\frac{1-M}{Q}\right)\left(1+\frac{\gamma^2}{2}\right)\right\} \tag{4.14}$$

$$\frac{d}{dR}\lambda_H = +\left\{\lambda_M\sqrt{C_{D_o}K}\left(1-\frac{(1-M)^2}{Q^2}(1+u)^2\right)\left[1+\frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q}\right.\right.$$

$$\left.+y\left(1+\frac{(1-M)^2}{Q^2}(1+u)^2\right)\right]+\lambda_M\left[(1-M)\gamma+\sqrt{C_{D_o}K}\left(Q\right.\right.$$

$$\left.\left.+\frac{(1-M)^2}{Q}(1+u)^2\right)\right]\left[-\frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q^2}-2y\frac{(1-M)^2}{Q^3}\right]\Bigg\}(Q)\left(1+\frac{\gamma^2}{2}\right) \tag{4.15}$$

$$\frac{d}{dR}\lambda_M = -\Big\{\lambda_M\Big[-\gamma - 2\sqrt{C_{D_o}K}\left(\frac{(1-M)}{Q}\right)(1+u)^2\Big]\Big[1+\frac{y}{\sqrt{C_{D_o}K}}\left(\frac{1-M}{Q}\right)\gamma$$

$$+ y\left(1+\left(\frac{1-M}{Q}\right)^2(1+u)^2\right)\Big] + \lambda_M\Big[(1-M)\gamma + \sqrt{C_{D_o}K}\,\Big(Q$$

$$+\frac{1}{Q}(1-M)^2(1+u)^2\Big)\Big]\Big[-\frac{y}{\sqrt{C_{D_o}K}}\frac{\gamma}{Q} - 2y\frac{(1-M)}{Q^2}(1+u)^2\Big]\Big\}\left(1+\frac{\gamma^2}{2}\right) \tag{4.16}$$

We wish to find a control, u, such that

$$\frac{\partial\mathscr{H}}{\partial u} = 0 \tag{4.17}$$

Performing the indicated partial differentiation of (4.6) we have

$$\mathscr{H}_u = \frac{a\lambda_\gamma}{\epsilon} + \lambda_M\Big\{2\sqrt{C_{D_o}K}\,\frac{(1-M)^2}{Q}(1+u)\left(1+\frac{\gamma^2}{2}\right)\Big[1+\frac{y(1-M)\gamma}{Q\sqrt{C_{D_o}K}}$$

$$+ y\left(1+\frac{(1-M)^2}{Q^2}(1+u)^2\right)\Big] + 2y\Big[(1-M)\gamma + \sqrt{C_{D_o}K}\,\Big(Q$$

$$+\frac{1}{Q}(1-M)^2(1+u)^2\Big)\Big](1+u)\left(1+\frac{\gamma^2}{2}\right)\frac{(1-M)^2}{Q^2}\Big\} \tag{4.18}$$

Now expand the state, costate and control variables in these equations in powers of ϵ. In doing so the following relations are used

$$\gamma_0 = u_0 = u_1 = 0 \tag{4.19}$$

$$Q = Q_0 + \epsilon Q_1 = e^{-\Delta H} = e^{-\Delta H_0 - \epsilon\Delta H_1} = Q_0(1-\epsilon\Delta H_1) \tag{4.20}$$

and, as a notational convenience

$$\frac{1-M_0}{Q_0} = x \tag{4.21}$$

The following expanded equations result

$$\frac{d}{dR}(\lambda_{\gamma 0}+\epsilon\lambda_{\gamma 1}) = -\frac{1}{\epsilon}\lambda_{H0} + \epsilon^0\Big\{-\lambda_{H1} - a\lambda_{\gamma 0}\gamma_1 - \lambda_{M0}(1-M_0)[1$$

$$+ 2y(1+x^2)]\Big\} + \epsilon\Big\{-\lambda_{H2} - a(\lambda_{\gamma 1}\gamma_1 + \lambda_{\gamma 0}\gamma_2) - \lambda_{M1}(1-M_0)[1$$

$$+ 2y(1+x^2)] - \lambda_{M0}\gamma_1\Big[\sqrt{C_{D_o}K}\,Q_0(1+x^2)(1+y(1+x^2))\Big]$$

$$+ 2\frac{x^2y}{\sqrt{C_{D_o}K}}Q_0 + \lambda_{M0}M_1[1+2y(1+3x^2)] - \lambda_{M0}\Delta H_1(1-M_0)[4yx^2]\Big\} \tag{4.22}$$

$$\frac{d}{dR}(\lambda_{H0}+\epsilon\lambda_{H1}) = \lambda_{M0}\sqrt{C_{D_o}K}\,Q_0\Big[(1-x^2)(1+y(1+x^2))$$

$$- 2yx^2(1+x^2)\Big] + \epsilon\Big\{\lambda_{M1}\sqrt{C_{D_o}K}\,Q_0\Big[(1-x^2)(1+y(1+x^2))$$

$$- 2yx^2(1+x^2)\Big] - \lambda_{M0}\gamma_1 Q_0 4yx^3 + \lambda_{M0}\sqrt{C_{D_o}K}\,Q_0\,\Delta H_1\Big[$$

$$- (1+x^2)(1+y(1+x^2)) - 8yx^4\Big] + \lambda_{M0}\sqrt{C_{D_o}K}\,M_1(2x)[1+y(2+6x^2)]\Big\} \tag{4.23}$$

$$\frac{d}{dR}(\lambda_{M0}+\epsilon\lambda_{M1}) = \lambda_{M0}2x\sqrt{C_{D_o}K}\,[1+2y(1+x^2)]$$

$$+ \epsilon\Big\{\lambda_{M1}2x\sqrt{C_{D_o}K}\,[1+2y(1+x^2)] + \lambda_{M0}\gamma_1[1+2y+6x^2y]$$

$$+ \lambda_{M0}\sqrt{C_{D_o}K}\,\Delta H_1(2x)[1+2y+6x^2y]$$

$$- \lambda_{M0}\sqrt{C_{D_o}K}\left(\frac{M_1}{1-M_0}\right)(2x)[1+2y+6x^2y]\Big\} \tag{4.24}$$

$$\epsilon^{-1}(\mathcal{H}_u)_{-1} + \epsilon^0(\mathcal{H}_u)_0 = \epsilon^{-1}a\lambda_{\gamma 0} + \epsilon^0\Big\{a\lambda_{\gamma 1}$$

$$+ \lambda_{M0}\Big[2x\sqrt{C_{D_o}K}\,(1-M_0)(1+2y[1+x^2])\Big]\Big\} \tag{4.25}$$

Similarly we can express the expanded Hamiltonian to first order in ϵ as

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1 = \left\{-1 + \lambda_{H0}\gamma_1 + \lambda_{M0}\sqrt{C_{D_o}K}\,\frac{(1-M_0)}{x}\,(1+x^2)\,[1+y(1+x^2)]\right\}$$

$$+ \epsilon \left\{\lambda_{H0}\gamma_2 + \lambda_{H1}\gamma_1 + a\lambda_{\gamma 0}\left(u_2 + \frac{1}{2}\gamma_1^2\right) + \lambda_{M1}\sqrt{C_{D_o}K}\,Q_0\,(1+x^2)(1+y(1+x^2))\right.$$

$$+ \lambda_{M0}\gamma_1(1-M_0)[1+2y(1+x^2)] + \lambda_{M0}\sqrt{C_{D_o}K}\,Q_0\Delta H_1\,[(x^2-1)(1+y)(1+x^2))$$

$$\left. + 2yx^2(1+x^2)] - \lambda_{M0}M_1\,(2x\sqrt{C_{D_o}K})\,[1+2y\,(1+x^2)]\right\} \tag{4.26}$$

Now from (4.25) we must have

$$\lambda_{\gamma 0} = 0 \tag{4.27}$$

Also, in order to satisfy the expanded canonical equation for λ_γ we must have

$$\lambda_{H0} = 0 \tag{4.28}$$

For these costates to remain zero over a non-zero interval of range, their derivatives must also be zero. Consider first λ_{H0}

$$\frac{d}{dR}\lambda_{H0} = 0 \tag{4.29}$$

$$\frac{d}{dR}\lambda_{H0} = \lambda_{M0}\sqrt{C_{D_o}K}\,Q_0\,[(1-x^2)(1+y(1+x^2)) - 2yx^2(1+x^2)] \tag{4.30}$$

$$0 = 3yx^4 + (1+2y)\,x^2 - (1+y) \tag{4.31}$$

The positive real solution to this equation is

$$x = +\sqrt{\frac{-1-2y+\sqrt{1+16y+16y^2}}{6y}} \tag{4.32}$$

It also will be useful later to express y as a function of x

$$y = \frac{(1 - x^2)}{(3x^2 - 1)(x^2 + 1)} \tag{4.33}$$

Figure 4.1 is a plot of x vs y for physically realizeable values of y. Since y is the product of C_{D_o} and the slope of SFC vs C_F, y can never be negative. Negative C_{D_o} is impossible in any case and a negative $d(SFC)/dC_F$ would result in a "chattering" solution: the engine could be cycled on and off in such a way that its duty cycle would achieve minimum SFC. Therefore, x remains less than or equal to one, and other branches of this curve have no meaning for this problem.

The fact that x remains less than or equal to one means that the optimal initial cruise altitude is below or at most equal to the optimal glide altitude, which is the altitude for maximum lift-drag ratio. This is seen from the definition of x when R is zero

$$x = (1 - M_{0i})/Q_{0i} = 1/Q_{0i} = e^{+\Delta H_{0i}} \tag{4.34}$$

or

$$\Delta H_{0i} = \ln x \qquad \frac{1}{\sqrt{3}} < x \leq 1 \tag{4.35}$$

and from the definition of ΔH as the normalized altitude difference measured from the altitude for maximum lift-drag ratio. Therefore, the value of x given by Eq. (4.32) specifies an altitude at which the product of $1/(SFC)_0$ and lift-drag ratio has been maximized (to zero order in ϵ) and this is lower than (possibly equal to) the altitude that corresponds to maximizing only the lift-drag ratio.

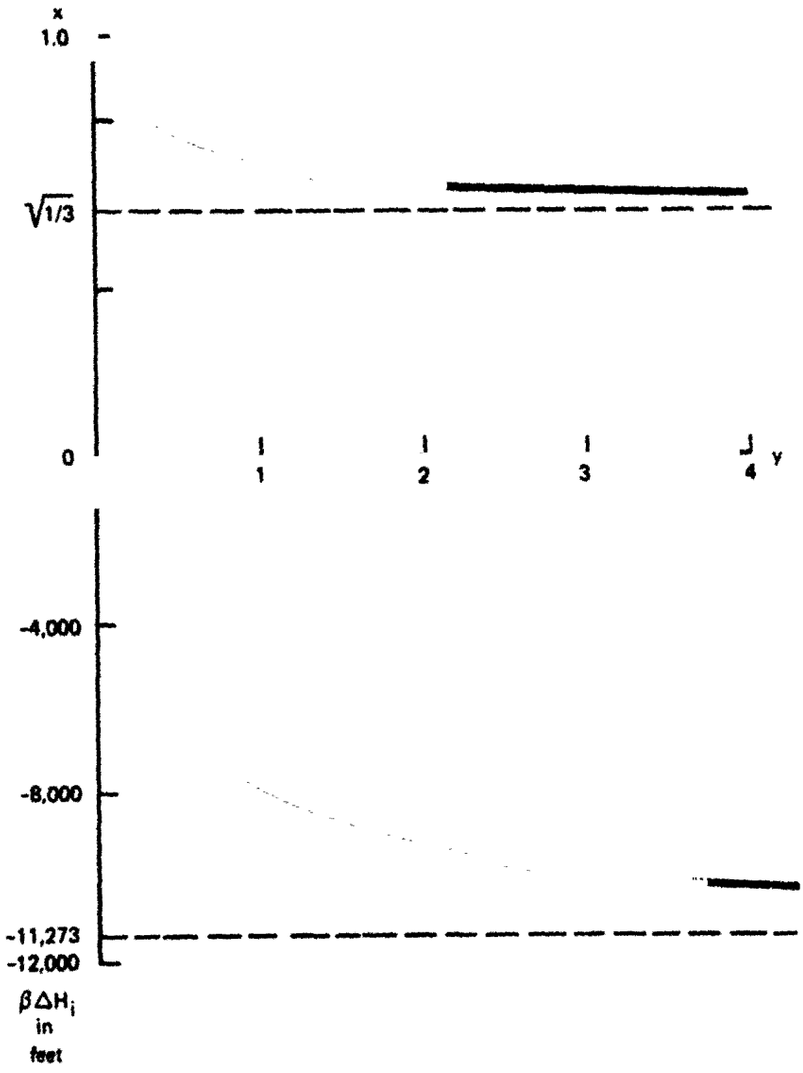

Figure 4.1. x and $\beta \Delta H_i$ as Functions of y

Using the optimal value of x, a constant, it is possible to solve the differential equation for M_0

$$\frac{d}{dR} M_0 = \sqrt{C_{D_o} K}\,(1-M_0)\left(x + \frac{1}{x}\right)[1+y\,(1+x^2)] \tag{4.36}$$

$$= (1 - M_0)\,2x\sqrt{C_{D_o} K}\,\frac{(1+x^2)}{(3x^2-1)} \tag{4.37}$$

$$1 - M_0 = c e^{-2x\sqrt{C_{D_o} K}(1+x^2)\,R/(3x^2-1)} \tag{4.38}$$

and the constant, c, is 1 so that M_0 is zero when R is zero. And now, since

$$1 - M_0 = xQ_0 = xe^{-\Delta H_0} \tag{4.39}$$

we have

$$\Delta H_0 = 2x\sqrt{C_{D_o} K}\,\frac{(1+x^2)}{(3x^2-1)}\,R + \ln x \tag{4.40}$$

which implies that

$$\gamma_1 = 2x\sqrt{C_{D_o} K}\,\frac{(1+x^2)}{(3x^2-1)} \tag{4.41}$$

which is a constant, implying that

$$u_2 = -\frac{1}{2}\gamma_1^2 \tag{4.42}$$

Notice that initial values of γ_0 and ΔH_0 are fixed, and cannot be matched to arbitrary initial conditions. Alternatively it could be said that from arbitrary initial conditions γ_0 and ΔH_0 must move to their optimal values in a zero interval of range, thus demonstrating the singular nature of the problem in the vicinity of initial conditions.

Finally, the equation for M_0 can be written as

$$1 - M_0 = e^{-\gamma_1 R} \tag{4.43}$$

Another way of defining γ_1 is as follows

$$\gamma_1 = \sqrt{C_{D_o} K}\left(x + \frac{1}{x}\right)[1 + y\,(1 + x^2)] \tag{4.44}$$

$$= \left[\frac{DRAG_0}{LIFT_0}\right]\left[\frac{SFC_0}{c_0}\right] \tag{4.45}$$

This will be useful later in developing the Breguet equation. In this form $\left(\frac{L}{D}\right)_0$ and $(SFC)_0$ refer to the zero order problem, in which lift equals weight and thrust equals drag. Since the cruise flight path angle is $\epsilon\gamma_1$ to first order in ϵ we see that ϵ can be expressed as

$$\epsilon = \left(\frac{c_0}{SFC_0}\right)\left[\gamma\left(\frac{L}{D}\right)_0\right]_{BREGUET} \tag{4.46}$$

If y=0, this expression is further simplified by the fact that $c_0 = SFC_0$.

Concluding now with the zero-order problem we have from the condition that $\mathscr{H}_0 = 0$ that

$$-1 + \lambda_{M0}\sqrt{C_{D_o} K}\,(1-M_0)\left(x + \frac{1}{x}\right)[1 + y\,(1 + x^2)] = 0 \tag{4.47}$$

$$-1 + \lambda_{M0}\,\gamma_1\,(1-M_0) = 0 \tag{4.48}$$

$$\lambda_{M0} = \left(\frac{1}{\gamma_1}\right) e^{+\gamma_1 R} \tag{4.49}$$

This result is consistent with the canonical equation for λ_{M0} which states that

$$\frac{d}{dR}\lambda_{M0} = \lambda_{M0}\,2x\sqrt{C_{D_o} K}\,[1+2y\,(1+x^2)] = \gamma_1\,\lambda_{M0} \tag{4.50}$$

Now using the optimal constant value of x and resulting expression for M_0 it is possible to solve the differential equation for M_1

$$\frac{d}{dR} M_1 = \gamma_1 (1-M_0) [1+2y (1+x^2)] + \Delta H_1 Q_0 \sqrt{C_{D_o} K} [(x^2-1) (1+y (1+x^2))$$
$$+ 2yx^2 (1+x^2)] + M_1 \frac{\sqrt{C_{D_o} K}}{x} [-2x^2 (1+2y (1+x^2))] \tag{4.51}$$

From the optimality condition on x the coefficient of ΔH_1 is identically zero. Using the definitions of γ_1 and $1-M_0$ the equation becomes

$$\frac{d}{dR} M_1 = \frac{\gamma_1^2}{2x\sqrt{C_{D_o} K}} e^{-\gamma_1 R} - \gamma_1 M_1 \tag{4.52}$$

An integrating factor is $e^{+\gamma_1 R}$ and we have

$$M_1 = e^{-\gamma_1 R} \left[\int \frac{\gamma_1^2}{2x\sqrt{C_{D_o} K}} dR + c \right] \tag{4.53}$$

$$M_1 = e^{-\gamma_1 R} \left[\left(\frac{\gamma_1^2 R}{2x\sqrt{C_{D_o} K}} \right) + M_{1b}(0) \right] \tag{4.54}$$

The constant of integration is not necessarily zero. It represents a first order initial value of mass in cruise.

At this point is is possible to develop the Breguet range equation. However, first we shall investigate the first order necessary conditions for optimality and determine ΔH_1, γ_2 and u_3. The condition that the derivative of $\lambda_{\gamma 0}$ must be zero requires that

$$- \lambda_{H1} - \lambda_{M0} (1-M_0) [1+2y (1+x^2)] = 0 \tag{4.55}$$

$$\lambda_{H1} = - \frac{1}{2x\sqrt{C_{D_o} K}} \tag{4.56}$$

This is a constant and hence its derivative must be zero

$$0 = \lambda_{M1}\sqrt{C_{D_o}K}\,Q_0\,[(1-x^2)\,(1+y\,(1+x^2)) - 2yx^2\,(1+x^2)]$$

$$- \lambda_{M0}\gamma_1\,Q_0\,4yx^3 + \lambda_{M0}\sqrt{C_{D_o}K}\,Q_0\,\Delta H_1\,[-(1+x^2)\,(1+y\,(1+x^2))-8yx^4]$$

$$+ \lambda_{M0}\,2x\sqrt{C_{D_o}K}\,M_1\,[1+2y\,(1+3x^2)] \tag{4.57}$$

Here the coefficient of λ_{M1} is equal to zero at the optimal value of x.

$$0 = -\frac{4yx^3\,\gamma_1}{\sqrt{C_{D_o}K}} + \Delta H_1\,[-(1+x^2) - y(1+2x^2+9x^4)]$$

$$+ 2x^2\,\frac{M_1}{1-M_0}\,(1+2y\,(1+3x^2)) \tag{4.58}$$

$$0 = -4\gamma_1 x^3\,(1-x^2) - 2x^2\sqrt{C_{D_o}K}\,\Delta H_1\,(-3x^4 + 6x^2 + 1)$$

$$+ \left(2x^2\sqrt{C_{D_o}K}\,M_{1b}(0) + x\,\gamma_1^{\,2}R\right)(-3x^4 + 6x^2 + 1) \tag{4.59}$$

$$\Delta H_1 = \frac{\gamma_1^{\,2}\,R}{2x\sqrt{C_{D_o}K}} - \frac{2x\,\gamma_1(1-x^2)}{\sqrt{C_{D_o}K}\,(1+6x^2-3x^4)} + M_{1b}(0) \tag{4.60}$$

This implies that

$$\gamma_2 = \frac{\gamma_1^{\,2}}{2x\sqrt{C_{D_o}K}} \tag{4.61}$$

Again this is a constant. From comparing second power terms in the expanded differential equation for γ, we must have

$$\frac{d}{dR}\gamma_2 = u_3 + \gamma_1\gamma_2 = 0 \tag{4.62}$$

$$u_3 = -\frac{\gamma_1^3}{2x\sqrt{C_{D_o}K}} \tag{4.63}$$

It is now possible to determine λ_{M1} from its equation

$$\frac{d}{dR}\lambda_{M1} = \gamma_1\lambda_{M1} + \lambda_{M0}(1+2y+6x^2y)\left[\gamma_1 + 2x\sqrt{C_{D_o}K}\left(\Delta H_1 - \frac{M_1}{1-M_0}\right)\right] \tag{4.64}$$

An integrating factor is $e^{-\gamma_1 R}$, and we have

$$\lambda_{M1} = e^{+\gamma_1 R}\int\left[c + (1+2y+6x^2y)\left[1 + \frac{2x\sqrt{C_{D_o}K}}{\gamma_1}\left(-\frac{2x\gamma_1}{\sqrt{C_{D_o}K}}\right)\frac{(1-x^2)}{(1+6x^2-3x^4)}\right]dR\right] \tag{4.65}$$

$$\lambda_{M1} = e^{+\gamma_1 R}\left[c + \frac{1+x^2}{(3x^2-1)}R\right] \tag{4.66}$$

Now the first order term in the Hamiltonian becomes

$$\begin{aligned}\mathscr{H}_1 = {} & \lambda_{H1}\gamma_1 + \lambda_{M1}(1-M_0)\sqrt{C_{D_o}K}\left(x+\frac{1}{x}\right)[1+y(1+x^2)] \\ & + \lambda_{M0}\gamma_1(1-M_0)[1+2y(1+x^2)] + \lambda_{M0}(1-M_0)\sqrt{C_{D_o}K}\,\Delta H_1\left[\left(x-\frac{1}{x}\right)(1\right. \\ & \left. + y(1+x^2)) + 2xy(1+x^2)\right] - \lambda_{M0}M_1\,2x\sqrt{C_{D_o}K}\,[1+2y(1+x^2)]\end{aligned} \tag{4.67}$$

The coefficient of ΔH_1 is zero at the optimal value of x, and the remaining coefficients are more conveniently expressed in terms of γ_1

$$\mathscr{H}_1 = \lambda_{H1}\gamma_1 + \lambda_{M1}(1-M_0)\gamma_1 + \lambda_{M0}(1-M_0)\frac{\gamma_1^2}{2x\sqrt{C_{D_o}K}} - \lambda_{M0}M_1\gamma_1 \tag{4.68}$$

Substituting for λ_{H1}, λ_{M1}, $1-M_0$, λ_{M0}, and M_1 from (4.56), (4.66),

(4.43), (4.49), and (4.54) we have

$$\mathscr{H}_1 = -\frac{\gamma_1}{2x\sqrt{C_{D_o}K}} + \gamma_1\left[c + \frac{\gamma_1 R}{2x\sqrt{C_{D_o}K}}\right] + \frac{\gamma_1}{2x\sqrt{C_{D_o}K}} - \frac{\gamma_1^2 R}{2x\sqrt{C_{D_o}K}} - M_{1b}(0) \quad (4.69)$$

$$\mathscr{H}_1 = \gamma_1 c - M_{1b}(0) \quad (4.70)$$

and for $\mathscr{H}_1 = 0$ we have

$$c = \frac{M_{1b}(0)}{\gamma_{1b}} \quad (4.71)$$

This constant originated in (4.66) so we now have for λ_{M1}

$$\lambda_{M1} = \left[\frac{\gamma_{1b} R}{2x\sqrt{C_{D_o}K}} + \frac{M_{1b}(0)}{\gamma_{1b}}\right] e^{+\gamma_{1b} R} \quad (4.72)$$

Finally, $\lambda_{\gamma 1}$ can be determined from the condition that $(\mathscr{H}_u)_0 = 0$ as follows

$$a\lambda_{\gamma 1} + \lambda_{M0}(1-M_0)\, 2x\sqrt{C_{D_o}K}\,[1+2y(1+x^2)] = 0 \quad (4.73)$$

and, substituting for λ_{M0}, M_0, and γ_1 we have

$$\lambda_{\gamma 1} = -\frac{1}{a} \quad (4.74)$$

We have now satisfied the necessary conditions for optimality to first order in ϵ. Figure 4.2 is a diagram of the sequences that led to the zero order optimal solution. Figure 4.3 is the same for the first order optimal solution. An arrowhead from one box to another indicates that the information in the first box leads to the conclusion in the second. A summation of two arrowheads indicates that two information sources are necessary to draw the indicated conclusion. It does not indicate a summation of equations.

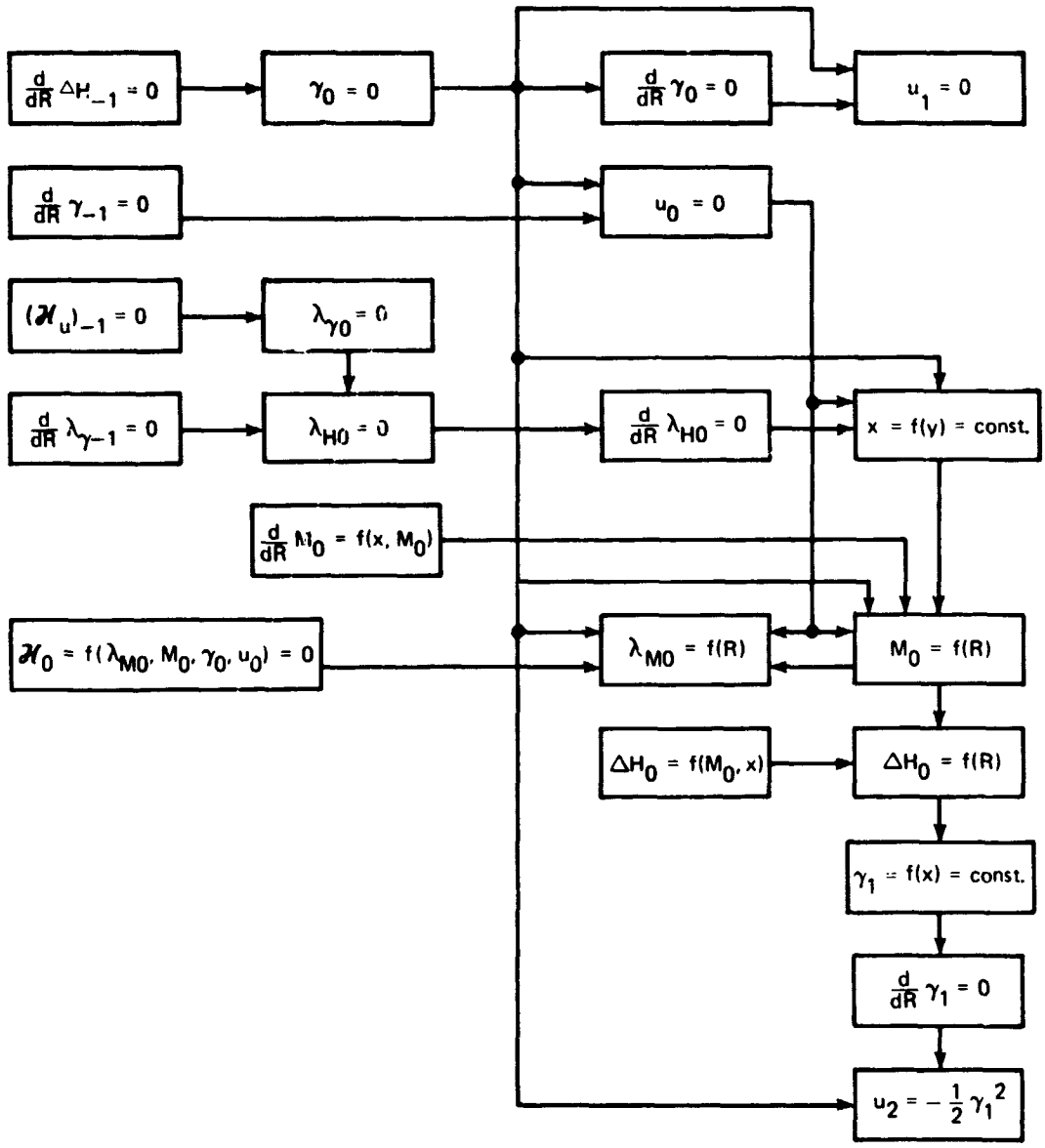

Figure 4.2. Zero Order Optimal Outer Solution Sequence

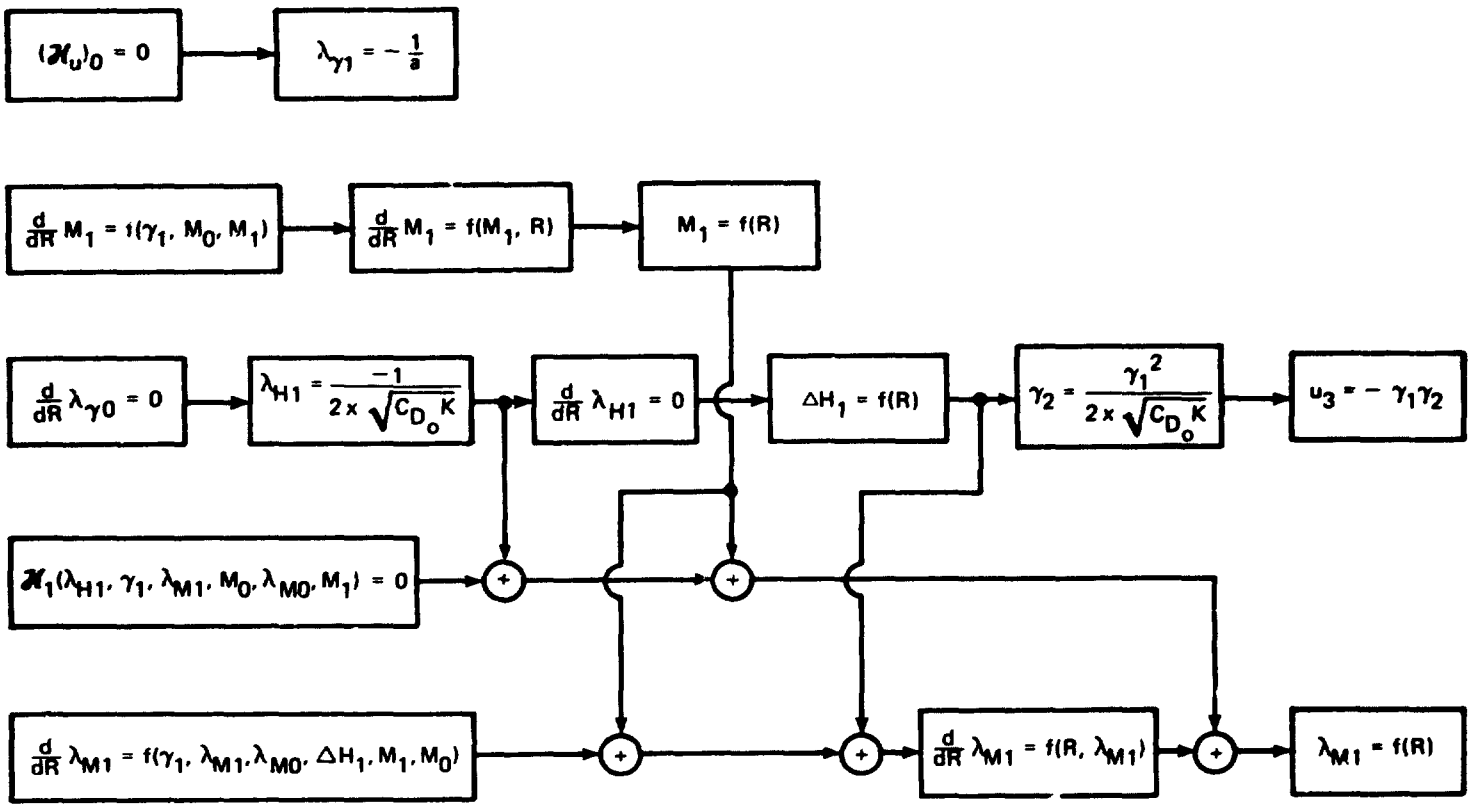

Figure 4.3. First Order Optimal Outer Solution Sequence

We return now to the expression for M and develop the Breguet range equation.

$$M_f = (M_0 + \epsilon M_1)_f = 1 - e^{-\gamma_1 R_f} + \epsilon e^{-\gamma_1 R_f} \left[\frac{\gamma_1^2 R_f}{2x\sqrt{C_{D_o} K}}\right] \tag{4.75}$$

$$1 - M_f = e^{-\gamma_1 R_f} \left[1 - \epsilon \frac{\gamma_1^2 R_f}{2x\sqrt{C_{D_o} K}}\right] \tag{4.76}$$

$$\ln(1-M_f) = -\gamma_1 R_f + \ln \left[1 - \epsilon \frac{\gamma_1^2 R_f}{2x\sqrt{C_{D_o} K}}\right] \tag{4.77}$$

$$\ln(1-M_f) = -\gamma_1 R_f - \epsilon \frac{\gamma_1^2 R_f}{2x\sqrt{C_{D_o} K}} \tag{4.78}$$

$$R_f = -\ln(1-M_f) \left[\frac{1}{\gamma_1\left(1 + \epsilon \frac{\gamma_1}{2x\sqrt{C_{D_o} K}}\right)}\right] \tag{4.79}$$

$$R_f = -\ln(1-M_f) \left(\frac{1}{\gamma_1}\right) \left(1 - \epsilon \frac{\gamma_1}{2x\sqrt{C_{D_o} K}}\right) \tag{4.80}$$

Now using the following relations

$$-\ln(1-M_f) = -\ln\left(1 - \frac{m_i - m_f}{m_i}\right) = +\ln\left(\frac{m_i}{m_f}\right) \tag{4.81}$$

and

$$\frac{1}{\gamma_1} = \frac{LIFT_0\, c_0}{DRAG_0\, SFC_0} \tag{4.82}$$

and

$$r_f = R_f \left(\frac{v}{c_0}\right) \tag{4.83}$$

we have

$$(r_f)_{max} = \left[\left(\frac{v}{SFC_0}\left(\frac{L}{D}\right)_0\right)_{max} \ln\left(\frac{m_i}{m_f}\right)\right]\left[1 - \epsilon\frac{\gamma_1}{2x\sqrt{C_{D_o}K}}\right] \qquad (4.84)$$

The first brackets contain the usual Breguet range expression. The optimal value of x maximizes the product of $v/(SFC)_0$ and $(L/D)_0$ and hence maximizes range at constant velocity and fixed initial and final values of mass.

The second brackets contain a first order correction factor resulting from the fact that flight path angle is not zero but a small position quantity and therefore thrust equals not only drag but also a component of weight in the thrust direction. This extra thrust required for climbing results in a smaller final range, but the difference is of order ϵ.

Rutowski [2] derived an expression for this range correction factor from consideration of the increase in potential energy due to climbing during Breguet cruise. He considered SFC to be constant in his development and did not attempt a mathematical maximization of the resulting range expression. If the above expression for maximum range were derived based on a constant value of SFC, then the resulting maximizing value of x would be unity and γ_1 would be $2\sqrt{C_{D_o}K}$. As a result, the correction factor appearing above would become $(1-\epsilon)$ which agrees exactly with Rutowski's factor of $\left(1 - \frac{SFC_0}{\beta v}\right)$.

Teren and Daniele [10] have analyzed the maximum cruise range problem using only mass and altitude as state variables and using thrust coefficient as the control variable. They derive the following equation to define the optimum value of C_F

$$C_F = C_{D_o} + \frac{(SFC)(C_F)}{2\frac{d}{dC_F}[(SFC)(C_F)]} + \frac{(SFC)(C_F)}{2\beta v} \tag{4.85}$$

If SFC is taken as varying linearly with C_F in the neighborhood of a constant operating value, this equation is a cubic in $\left(C_F/C_{D_o}\right)$ and ϵ:

$$2\left[\frac{C_F}{C_{D_o}} - 1\right]\left[1 + \frac{2yC_F}{C_{D_o}}\right] = \left(\frac{C_F}{C_{D_o}}\right)\left(1 + y\frac{C_F}{C_{D_o}}\right)\left[1 + \epsilon\left(1 + 2y\frac{C_F}{C_{D_o}}\right)\right] \tag{4.86}$$

For $\epsilon = 0$ the order is reduced by one and the positive real root is easily found as

$$(C_F)_0 = C_{D_o}(1+x^2) \tag{4.87}$$

For $\epsilon \neq 0$ a first improvement to this root is found by assuming it to be of the form

$$C_F = (C_F)_0(1 + \delta) \tag{4.88}$$

This is substituted into the cubic and solved for δ, retaining terms of $0(\epsilon)$ and eliminating terms that comprise the second order equation for $(C_F)_0$. The result is

$$C_F = \left(C_{D_o}\right)(1+x^2)\left[1 + \epsilon\frac{2x^2}{(3x^2-1)}\frac{(x^2+1)^2}{(1+6x^2-3x^4)}\right] \tag{4.89}$$

The identical result is achieved if (T/Sq) is expanded using the optimal values of M_0, M_1, ΔH_1, and γ_1 developed in this chapter.

Teren and Daniele do not develop an expression for maximum range in cruise. However this is easily done using their differential equations for mass and range together with the assumptions of constant lift coefficient and lift equals weight. The result is

$$(r_f)_{max} = \left[\left(\frac{v}{SFC}\right)\left(\frac{C_L}{C_F}\right)\right]_{max} \ln\left(\frac{m_i}{m_f}\right) \tag{4.90}$$

where SFC and C_L are determined by the optimal value of C_F.

Now if SFC is a linear function of C_F and if the above expanded form of the optimal value of C_F is used, together with the resulting expanded forms of the optimal C_L and SFC, this expression for maximum range will be identical to the one derived in this chapter.

CHAPTER V

MAXIMUM RANGE TRANSITIONS IN THE BOUNDARY LAYERS

5.1 Introduction

In this Chapter the two inner, or boundary layer, solutions are derived, and the results are matched with the outer or cruise solution. The optimal control is examined and shown to produce a damped oscillatory transition to and from cruise. The over-damped, or pure exponential, case is examined separately. The short-range problem, for which no cruise segment is required, is also examined. Finally, a modified Breguet range equation is derived which includes changes in range due to transitions between cruise and initial and final conditions that are not on a Breguet cruise.

5.2 The Problem in the Boundary Layer

The boundary layer problem is described by stretching the independent variable in the state equations by the transformations

$$R_L = \frac{R}{\epsilon} \tag{5.1}$$

in the left side boundary (vicinity of $R = 0$) and

$$R_R = \frac{R_f - R}{\epsilon} \tag{5.2}$$

in the right side boundary (vicinity of $R = R_f$). In the left side boundary layer $\frac{d}{dR} = \epsilon \frac{d}{dR_L}$ and the equations become

$$\frac{d\gamma}{dR_L} = a\left(u + \frac{\gamma^2}{2}\right) \tag{5.3}$$

$$\frac{d\Delta H}{dR_L} = \gamma \tag{5.4}$$

$$\frac{dM}{dR_L} = \epsilon \left\{\left[(1-M)\gamma + \sqrt{C_{D_o}K}\left[Q + \frac{(1-M)^2(1+u)^2}{Q}\right]\right]\left[1 + \frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q} + y\left[1 + \frac{(1-M)^2}{Q^2}(1+u)^2\right]\right]\left(1 + \frac{\gamma^2}{2}\right)\right\} \tag{5.5}$$

The cost to be minimized becomes

$$J = -\int_0^{R_f/\epsilon} \epsilon \, dR_L \tag{5.6}$$

and the variational Hamiltonian becomes

$$\mathcal{H} = -\epsilon + \lambda_\gamma a\left(u + \frac{\gamma^2}{2}\right) + \lambda_H \gamma + \epsilon\lambda_M \left\{\left[(1-M)\gamma + \sqrt{C_{D_o}K}\left[Q + \frac{(1-M)^2}{Q}(1+u)^2\right]\right]\left[1 + \frac{y}{\sqrt{C_{D_o}K}}\frac{(1-M)\gamma}{Q} + y\left[1 + \frac{(1-M)^2}{Q^2}(1+u)^2\right]\right]\left(1 + \frac{\gamma^2}{2}\right)\right\} \tag{5.7}$$

Unlike the situation in cruise, a zero order analytic solution to the boundary layer problem can not be found unless some further simplifying assumptions are made about the state and control variables. Typically in such cases one might consider a linear expansion of the boundary layer state equations in the vicinity of cruise. There u, γ and M will be small. Let $\Delta\hat{H}$ be defined relative to the initial optimal cruise altitude so that $\Delta\hat{H}$ also remains small.

$$\hat{Q} = \left(\frac{q}{\frac{W_i}{S} \sqrt{\frac{K}{C_{D_o}}}} \right) \left(\frac{\frac{W_i}{S} \sqrt{\frac{K}{C_{D_o}}}}{q_{0i}} \right) = Q/Q_{0i} = Qx \tag{5.8}$$

$$\hat{Q} = xe^{-\Delta H} = e^{-\Delta H + \ln x} = e^{-\Delta \hat{H}} \tag{5.9}$$

$$\Delta \hat{H} = \Delta H - \ln x \tag{5.10}$$

The linear differential equations are

$$\frac{d\gamma}{dR_L} = a\,u \tag{5.11}$$

$$\frac{d\Delta H}{dR_L} = \gamma \tag{5.12}$$

$$\begin{aligned} \frac{dM}{dR_L} = \epsilon \Big\{ & \sqrt{C_{D_o} K}\, (x + \tfrac{1}{x})\, [1 + y\,(1 + x^2)] + \gamma\, [1 + 2y\,(1 + x^2)] \\ & + \Delta \hat{H} \sqrt{C_{D_o} K}\, [2x^2 y\, (\tfrac{1}{x} + x) + (x - \tfrac{1}{x})(1 + y\,(1 + x^2))] \\ & + 2\sqrt{C_{D_o} K}\, x\,(u - M)\, [1 + 2y\,(1 + x^2)] \Big\} \end{aligned} \tag{5.13}$$

The equation for M is derived by expressing the exponential form of $\hat{Q}$ as a Taylor series in $\Delta \hat{H}$ to first order, expanding, and retaining only linear terms. It is then greatly simplified by making use of two algebraic identities. First, from the definition of the optimal value of x (Eq. 4.31, 4.32)

$$2x^2 y\, (x + \tfrac{1}{x}) + (x - \tfrac{1}{x})\, [1 + y\,(1 + x^2)] = 0 \tag{5.14}$$

Second, the first-order value of the cruise flight path angle is defined by the two equivalent expressions (Eq. 4.44, 4.50)

$$\gamma_{1b} = \sqrt{C_{D_o} K}\,(x + \frac{1}{x})[1 + y(1 + x^2)] = 2\sqrt{C_{D_o} K}\,x\,[1 + 2y(1 + x^2)] \qquad (5.15)$$

The differential equation for M then reduces to

$$\frac{dM}{dR_L} = \epsilon\,\gamma_{1b}\left(1 + u - M + \frac{\gamma}{2x\sqrt{C_{D_o} K}}\right) \qquad (5.16)$$

Now the Hamiltonian can be written as

$$\mathcal{H} = -\epsilon + a\lambda_\gamma u + \lambda_H \gamma + \epsilon\lambda_M \gamma_{1b}\left(1 + u - M + \frac{\gamma}{2x\sqrt{C_{D_o} K}}\right) \qquad (5.17)$$

It can be seen at this point that $\Delta\hat{H}$ does not appear at all in the state equations or the Hamiltonian. An optimal control, if one could be found, would be independent of $\Delta\hat{H}$. Thus linearization of the state equations fails to yield a meaningful solution because it fails to yield a meaningful mass equation. The reason for the failure is that the drag term, which appears in the mass equation, has no linear dependence on altitude in the vicinity of cruise altitude.

As a second attempt to simplify the boundary layer state equations, and thereby to be able to develop an analytic solution, consider a quadratic expansion of the equations in the vicinity of cruise. The differential equation for $\Delta\hat{H}$ remains linear (5.4). The equation for γ now includes a term in γ^2 (5.3). The equation for M becomes

$$\frac{dM}{dR_L} = \epsilon \Big\{ \Big[(1-M)\gamma + \sqrt{C_{D_o} K}\, [\tfrac{1}{x}(1 - \Delta\hat{H} + \tfrac{1}{2}\Delta\hat{H}^2) + x(1 + \Delta\hat{H} + \tfrac{1}{2}\Delta\hat{H}^2)(1$$

$$- M)^2(1+u)^2] \Big] \Big[1 + \frac{xy}{\sqrt{C_{D_o} K}} (1 + \Delta\hat{H} + \tfrac{1}{2}\Delta\hat{H}^2) \Big[(1-M)\gamma + \sqrt{C_{D_o} K}\, [\tfrac{1}{x}(1 - \Delta\hat{H}$$

$$+ \tfrac{1}{2}\Delta\hat{H}^2) + x(1 + \Delta\hat{H} + \tfrac{1}{2}\Delta\hat{H}^2)(1-M)^2(1+u)^2] \Big] \Big] (1 + \frac{\gamma^2}{2}) \Big\} \qquad (5.18)$$

with $\hat{Q}$ expressed as a Taylor series in $\Delta\hat{H}$ to second order. Performing the indicated multiplications and neglecting terms of higher order than second results in the following equation

$$\frac{dM}{dR_L} = \epsilon \Big\{ \sqrt{C_{D_o} K}\,(x + \tfrac{1}{x})[1 + y(1 + x^2)] + \gamma[1 + 2y(1 + x^2)]$$

$$+ \Delta\hat{H}\sqrt{C_{D_o} K}\Big(2x^2 y(x + \tfrac{1}{x}) + (x - \tfrac{1}{x})[1 + y(1 + x^2)]\Big) + (u - M)\, 2x\sqrt{C_{D_o} K}[1 + 2y(1 + x^2)]$$

$$+ \gamma^2\Big(\frac{xy}{\sqrt{C_{D_o} K}} + \frac{\sqrt{C_{D_o} K}}{2}(x + \tfrac{1}{x})[1 + y(1 + x^2)]\Big) + \gamma\Delta\hat{H}4x^2 y$$

$$+ \Delta\hat{H}^2\sqrt{C_{D_o} K}\Big(4x^3 y + \tfrac{1}{2}(x + \tfrac{1}{x})[1 + y(1 + x^2)]\Big) + \gamma u 4x^2 y$$

$$+ (M^2 + u^2)\, x\sqrt{C_{D_o} K}\,[1 + 2y(1 + x^2) + 4yx^2] - \gamma M[1 + 4yx^2 + 2y(1 + x^2)]$$

$$+ (\Delta\hat{H} u - \Delta\hat{H} M)\, 2x\sqrt{C_{D_o} K}\,[1 + 2y(1 + x^2) + 4x^2 y]$$

$$- Mu4x\sqrt{C_{D_o} K}\,[1 + 2y(1 + x^2) + 2x^2 y] \Big\} \qquad (5.19)$$

These equations (5.3, 5.4, 5.19) are solvable as the state equations of a linear-quadratic singular perturbation optimal control problem if we can neglect the term $\frac{\gamma^2}{2}$ in comparison to u in Eq. (5.3). In the cruise problem this term had no effect on the solution through first order in ϵ.

To show qualitatively that this term is negligible in the boundary layer, consider the effect of the transformation

$$U = u + \frac{\gamma^2}{2} \tag{5.20}$$

on the state equations. The equation for γ (5.3) is linearized and the equation for $\Delta\hat{H}$ (5.4) is unchanged. The effect on the equation for M (5.19) can be shown to be merely the replacement of u by U and the inclusion of an additive term in the coefficient of γ^2. This additive term can be shown to be so small in comparison to the principal term in that coefficient that it is safely neglected. But if that term is neglected, the transformed equation set can not be distinguished from the set that results from merely neglecting the term $\frac{\gamma^2}{2}$ in comparison to u in the original equation for γ.

Of course, the entire solution of this problem could be based on this transformation and no appreciable difference would occur in the result. For our present purposes we merely cite the potential of this transformation and proceed to linearize the equation for γ by neglecting $\frac{\gamma^2}{2}$ in comparison to u. The equation for M is simplified by the use of (5.14) and (5.15). Finally, the state equations of the linear-quadratic singular perturbation optimal control problem and the variational Hamiltonian are

$$\frac{d\gamma}{dR_L} = a u \tag{5.21}$$

$$\frac{d\Delta H}{dR_L} = \gamma \tag{5.22}$$

$$\frac{dM}{dR_L} = \epsilon\Big\{\gamma_{1b}\Big(1+u-M+\frac{\gamma}{2x\sqrt{C_{D_o}K}}\Big)+\gamma^2\Big(\frac{\gamma_{1b}}{2}+\frac{xy}{\sqrt{C_{D_o}K}}\Big)$$

$$+\Big(4yx^2+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\Big)\Big[-\gamma M+2x\sqrt{C_{D_o}K}\Big(\Delta\hat{H}[u-M]+\frac{u^2+M^2+\Delta H^2}{2}\Big)\Big]$$

$$+\gamma(u+\Delta\hat{H})4x^2y-4x\sqrt{C_{D_o}K}\,Mu\Big(\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}+2x^2y\Big)\Big\} \tag{5.23}$$

$$\mathcal{H} = -\epsilon+a\lambda_\gamma u+\lambda_H\gamma+\epsilon\lambda_M\Big\{\gamma_{1b}\Big(1+u-M+\frac{\gamma}{2x\sqrt{C_{D_o}K}}\Big)$$

$$+\gamma^2\Big(\frac{\gamma_{1b}}{2}+\frac{xy}{\sqrt{C_{D_o}K}}\Big)+\gamma(u+\Delta\hat{H})4x^2y-4x\sqrt{C_{D_o}K}\,Mu\Big(2x^2y+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\Big)$$

$$+\Big(4yx^2+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\Big)\Big[-\gamma M+2x\sqrt{C_{D_o}K}\Big(\Delta\hat{H}[u-M]+\frac{u^2+M^2+\Delta\hat{H}^2}{2}\Big)\Big]\Big\} \tag{5.24}$$

The factor of ϵ in the equation for M makes this a solvable problem. The ϵ means that the equation for M is not in the Hamiltonian to zero order in ϵ and, as a result, the zero order value of the costate for mass, λ_{M0}, is constant. Thus the zero order problem, for which mass is constant, has linear differential equations for γ and $\Delta\hat{H}$ and a quadratic cost functional which represents an "out of the loop" equation for M weighted by a constant, λ_{M0}. The first order problem is of course linear with coefficients depending on the zero order problem.

We now proceed to develop the costate differential equation

$$\frac{d\lambda_\gamma}{dR_L} = -\lambda_H - \epsilon\lambda_M \left\{ \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}} + \gamma\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) + (u + \Delta\hat{H})\, 4x^2y \right.$$

$$\left. - M\left(4x^2y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\right\} \tag{5.25}$$

$$\frac{d\lambda_H}{dR_L} = -\epsilon\lambda_M \left\{\gamma 4x^2y + \left(\gamma_{1b} + 8x^3y\sqrt{C_{D_o}K}\right)\left(u - M + \Delta\hat{H}\right)\right\} \tag{5.26}$$

$$\frac{d\lambda_M}{dR_L} = -\epsilon\lambda_M \left\{-\gamma_{1b} - u\left(2\gamma_{1b} + 8\sqrt{C_{D_o}K}\, x^3y\right)\right.$$

$$\left. + \left(4x^2y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\left[-\gamma + 2x\sqrt{C_{D_o}K}\,(M - \Delta\hat{H})\right]\right\} \tag{5.27}$$

The condition that the optimal control, u, minimizes the Hamiltonian is expressed as follows

$$\mathscr{H}_u = a\lambda_\gamma + \epsilon\lambda_M \left[\gamma_{1b} + 4x^2y\,\gamma_0 - \left(2\gamma_{1b} + 8x^3y\sqrt{C_{D_o}K}\right) M\right.$$

$$\left. + \left(\gamma_{1b} + 8x^3y\sqrt{C_{D_o}K}\right)(\Delta\hat{H} + u)\right] = 0 \tag{5.28}$$

Now if we expand the state, costate and control variables in power series in ϵ, the differential equations of the state and costate variables become

$$\frac{d}{dR_L}(\gamma_0 + \epsilon\,\gamma_1) = a\,(u_0 + \epsilon\,u_1) \tag{5.29}$$

$$\frac{d}{dR_L}(\Delta\hat{H}_0 + \epsilon\,\Delta\hat{H}_1) = \gamma_0 + \epsilon\,\gamma_1 \tag{5.30}$$

$$\frac{d}{dR_L}(M_0+\epsilon M_1) = \epsilon\left\{\gamma_{1b}\left(1+u_0+\frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\right)+\gamma_0^{\,2}\left(\frac{\gamma_{1b}}{2}+\frac{xy}{\sqrt{C_{D_o}K}}\right)\right.$$

$$\left.+\left(4x^3y\sqrt{C_{D_o}K}+\frac{\gamma_{1b}}{2}\right)\left[u_0+\Delta\hat{H}_0\right]^2+4x^2y\,\gamma_0\left(u_0+\Delta\hat{H}_0\right)\right\} \quad (5.31)$$

$$\frac{d}{dR_L}(\lambda_{\gamma 0}+\epsilon\lambda_{\gamma 1}+\epsilon^2\lambda_{\gamma 2}) = -\lambda_{H0}-\epsilon\left\{\lambda_{H1}+\lambda_{M0}\left[\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}+\gamma_0\left(\gamma_{1b}+\frac{2xy}{\sqrt{C_{D_o}K}}\right)\right.\right.$$

$$\left.\left.+4x^2y\,(u_0+\Delta\hat{H}_0)\right]\right\}+\epsilon^2\left\{-\lambda_{H2}-\lambda_{M1}\left[\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right.\right.$$

$$\left.+\gamma_0\left(\gamma_{1b}+\frac{2xy}{\sqrt{C_{D_o}K}}\right)+4x^2y\,(u_0+\Delta\hat{H}_0)\right]-\lambda_{M0}\left[\gamma_1\left(\gamma_{1b}\right.\right.$$

$$\left.\left.\left.+\frac{2xy}{\sqrt{C_{D_o}K}}\right)+4x^2y\,(u_1+\Delta\hat{H}_1)-M_1\left(4x^2y+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\right]\right\}$$

(5.32)

$$\frac{d}{dR_L}(\lambda_{H0}+\epsilon\lambda_{H1}+\epsilon^2\lambda_{H2}) = -\epsilon\left\{\lambda_{M0}\left[4x^2y\,\gamma_0+\left(\gamma_{1b}+8x^3y\sqrt{C_{D_o}K}\right)\left(u_0+\Delta\hat{H}_0\right)\right]\right\}$$

$$+\epsilon^2\left\{-\lambda_{M1}\left[4x^2y\,\gamma_0+\left(\gamma_{1b}+8x^3y\sqrt{C_{D_o}K}\right)\left(u_0+\Delta\hat{H}_0\right)\right]\right.$$

$$\left.-\lambda_{M0}\left[4x^2y\,\gamma_1+\left(\gamma_{1b}+8x^3y\sqrt{C_{D_o}K}\right)\left(u_1-M_1+\Delta\hat{H}_1\right)\right]\right\}$$

(5.33)

$$\frac{d}{dR_L}(\lambda_{M0}+\epsilon\lambda_{M1}) = -\epsilon\lambda_{M0}\left\{-\gamma_{1b}-\gamma_0\left(4x^3y\sqrt{C_{D_o}K}+\gamma_{1b}\right)\right.$$

$$\left.-\left(4x^2y+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\left(\gamma_0+2x\sqrt{C_{D_o}K}\;\Delta\hat{H}_0\right)\right\} \quad (5.34)$$

Expanding the equation $\mathscr{H}_u = 0$ gives

$$
\begin{aligned}
(\mathscr{H}_u)_0 + \epsilon(\mathscr{H}_u)_1 + \epsilon^2(\mathscr{H}_u)_2 &= a\lambda_{\gamma 0} \\
&+ \epsilon\left\{a\lambda_{\gamma 1} + \lambda_{M0}\left[\gamma_{1b} + 4x^2 y\,\gamma_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\Delta\hat{H}_0 + u_o\right)\right]\right\} \\
&+ \epsilon^2\left\{a\lambda_{\gamma 2} + \lambda_{M1}\left[\gamma_{1b} + 4x^2 y\,\gamma_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\Delta\hat{H}_0 + u_0\right)\right]\right. \\
&\left. + \lambda_{M0}\left[4x^2 y\,\gamma_1 - \left(2\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)M_1 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\Delta\hat{H}_1 + u_1\right)\right]\right\}
\end{aligned}
\tag{5.35}
$$

Similarly, the expanded Hamiltonian is

$$
\begin{aligned}
\mathscr{H} = &\left\{a\lambda_{\gamma 0}u_0 + \lambda_{H0}\gamma_0\right\} + \epsilon\left\{-1 + a(\lambda_{\gamma 0}u_1 + \lambda_{\gamma 1}u_1) + (\lambda_{H0}\gamma_1 + \lambda_{H1}\gamma_0)\right. \\
&+ \lambda_{M0}\left[\gamma_{1b}\left(1 + u_0 + \frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\right) + \gamma_0{}^2\left(\frac{\gamma_{1b}}{2} + \frac{xy}{\sqrt{C_{D_o}K}}\right) + \gamma_0(u_0 + \Delta\hat{H}_0)\,4x^2 y\right. \\
&\left.\left. + \left(4x^2 y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\sqrt{C_{D_o}K}\,x\,(u_0 + \Delta\hat{H}_0)^2\right]\right\} + \epsilon^2\left\{a(\lambda_{\gamma 0}u_2 + \lambda_{\gamma 1}u_1 + \lambda_{\gamma 2}u_0)\right. \\
&+ (\lambda_{H0}\gamma_0 + \lambda_{H1}\gamma_1 + \lambda_{H2}\gamma_0) + \lambda_{M1}\left[\gamma_{1b}\left(1 + u_0 + \frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\right) + \gamma_0{}^2\left(\frac{\gamma_{1b}}{2} + \frac{xy}{\sqrt{C_{D_o}K}}\right)\right. \\
&\left. + \gamma_0(u_0 + \Delta\hat{H}_0)\,4x^2 y + \left(4x^2 y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\sqrt{C_{D_o}K}\,x\,(u_0 + \Delta\hat{H}_0)^2\right] \\
&+ \lambda_{M0}\left[\gamma_{1b}\left(u_1 - M_1 + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right) + \gamma_0\gamma_1\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) + 4x^2 y\,(\gamma_1 u_0 + \gamma_1\Delta\hat{H}_0\right. \\
&+ \gamma_0 u_1 + \gamma_0\Delta\hat{H}_1) - M_1 u_0\left(8x^3 y\sqrt{C_{D_o}K} + 2\gamma_{1b}\right) \\
&+ \left(4x^2 y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\left[-\gamma_0 M_1 + 2x\sqrt{C_{D_o}K}\,\left(\Delta\hat{H}_0 u_1\right.\right. \\
&\left.\left.\left. + \Delta\hat{H}_1 u_0 - \Delta\hat{H}_0 M_1 + u_0 u_1 + \Delta\hat{H}_0\Delta\hat{H}_1\right)\right]\right\}
\end{aligned}
\tag{5.36}
$$

Because λ_{M0} is a constant we can now see that the costate equations are linear and the Hamiltonian is quadratic and we should now be able to develop and analytic solution.

5.3 The Optimal Control

The required optimal control, u, must minimize the Hamiltonian, that is

$$\frac{\partial}{\partial u} \mathscr{H} = 0 \tag{5.37}$$

Considering first the zero-order part of $\mathscr{H}_u$ we must have

$$\lambda_{\gamma 0} = 0 \tag{5.38}$$

and since the Hamiltonian must itself be zero along the optimal trajectory we must also have, from $\mathscr{H}_0 = 0$

$$\lambda_{H0} = 0 \tag{5.39}$$

Now the optimal u_0 can be determined from the first order part of $\mathscr{H}_u = 0$

$$(\mathscr{H}_u)_1 = 0 \tag{5.40}$$

$$a\lambda_{\gamma 1} + \lambda_{M0}\left[\gamma_{1b} + 4x^2 y\, \gamma_0 + \left(\gamma_{1b} + 8\sqrt{C_{D_o} K}\, x^3 y\right)\left(u_0 + \Delta\hat{H}_0\right)\right] = 0 \tag{5.41}$$

An expression for the optimal value of u_1 results from considering the second order part of $\mathscr{H}_u = 0$

$$(\mathscr{H}_u)_2 = 0 \tag{5.42}$$

$$(\mathscr{H}_u)_2 = a\lambda_{\gamma 2} + \lambda_{M1}\left[\gamma_{1b} + 4x^2 y\,\gamma_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u_0 + \Delta\hat{H}_0\right)\right]$$

$$+ \lambda_{M0}\left[4x^2 y\,\gamma_1 - M_1\left(2\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right) + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\Delta\hat{H}_1 + u_1\right)\right] = 0 \tag{5.43}$$

We can now solve for the optimal u_0 and u_1. It will be shown that the optimal u_0 satisfies a homogeneous fourth order linear differential equation with constant coefficients. Furthermore, the coefficients of the first and third derivatives are zero, indicating that the roots are symmetrically located in the complex plane with respect to both the real and the imaginary axes. Next the optimal u_1 will be shown to satisfy a non-homogeneous equation, the homogeneous part of which is identical to the equation for the optimal u_0. The non-homogeneous part, or forcing part, will be a function of M_1 which is itself a function of the optimal trajectory of the zero-order problem.

First, since $(\partial\mathscr{H}/\partial u)_1 = 0$ over the entire optimal trajectory, it follows that the derivative of this partial derivative, with respect to R, the independent variable, will also be zero over the entire trajectory. Such derivatives will be indicated by primes. Similarly, all higher order derivatives will be zero. Proceeding, then, to tak ccessive derivatives of $(\partial\mathscr{H}/\partial u)_1$, the differential equation for the optimal u_0 is derived. At each step the derivative of a state or costate variable is replaced by its defining canonical differential equation.

$$\frac{d}{dR_L}(\mathscr{H}_u)_1 = a\lambda'_{\gamma 1} + \lambda_{M0}\left[4x^2 y\,\gamma'_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u'_0 + \Delta\hat{H}'_0\right)\right] \tag{5.44}$$

$$0 = a\left[-\lambda_{H1} - \lambda_{M0}\left[\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}} + \gamma_0\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) + 4x^2 y\Delta\hat{H}_0\right]\right]$$

$$+ \lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\gamma_0 + u'_0\right)\Big] \tag{5.45}$$

$$\frac{d^2}{dR_L^2}(\mathscr{H}_u)_1 = a\left[-\lambda'_{H1} - \lambda_{M0}\left[\gamma'_0\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) + 4x^2 y\,\Delta\hat{H}'_0\right]\right]$$

$$+ \lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\gamma'_0 + u''_0\right) \tag{5.46}$$

$$0 = a\lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u_0 + \Delta\hat{H}_0\right) - \lambda_{M0}a^2 u_0\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right)$$

$$+ \lambda_{M0}\left(8x^3 y\sqrt{C_{D_o}K} + \gamma_{1b}\right)\left(au_0 + u''_0\right) \tag{5.47}$$

No further substitution for variables is required and the next two derivatives are taken as a single operation

$$u_0^{iv} + u''_0\left[2a - a^2\frac{\gamma_{1b} + \dfrac{2xy}{\sqrt{C_{D_o}K}}}{\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}}\right] + a^2 u_0 = 0 \tag{5.48}$$

Using p as a derivative operator and defining Z as

$$Z = \frac{\sqrt{a}}{2}\sqrt{\frac{\gamma_{1b} + \dfrac{2xy}{\sqrt{C_{D_o}K}}}{\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}}} \tag{5.49}$$

the differential equation for u_0 may be written as

$$\left[p^4 + p^2\left(2a - \left[2Z\sqrt{a}\right]^2\right) + a^2\right] u_0 = 0 \tag{5.50}$$

or in the equivalent factored form

$$(p^2 + 2Z\sqrt{a}\,p + a)\,(p^2 - 2Z\sqrt{a}\,p + a)\,u_0 = 0 \tag{5.51}$$

The roots of this equation depend on a, which is a function of v, and Z, which is a function of v, x, y, C_{D_o} and K. At a given Mach number C_{D_o} and K are constants. Since x is a function of y we can say that the roots depend on velocity and y, that is, $\left(\frac{d(SFC)}{dC_F}\right) C_{D_o}$. If at a given velocity $d(SFC)/dC_F$ is allowed to vary between zero (constant SFC) and $+\infty$, the root locations for a particular airplane will vary as shown in Fig. 5.1. It is obvious that the effect of increasing the slope, $d(SFC)/dC_F$, away from zero is to increase the damping ratio. The natural frequency on the oscillatory brench remains constant.

A locus of roots as velocity varies is somewhat more difficult to obtain because of the fact that y as well as C_{D_o} and K would have to be described as a function of Mach number. However, if y is zero, as is sometimes necessary to assume, this difficulty is greatly diminished. Such a locus of roots as velocity varies and y remains zero is shown for a particular airplane in Fig. 5.2. In this figure it is seen that the effect of increasing velocity is to reduce the damping and the natural frequency of the oscillatory branch. Below a velocity of 409 ft/sec the roots are non-oscillatory.

Returning now to Fig. 5.1 and the equation for u_0 we can say that on the oscillatory branch we have for natural frequency and damping ratio, respectively,

$$\omega_n = \sqrt{a} \tag{5.52}$$

$$\zeta = Z \tag{5.53}$$

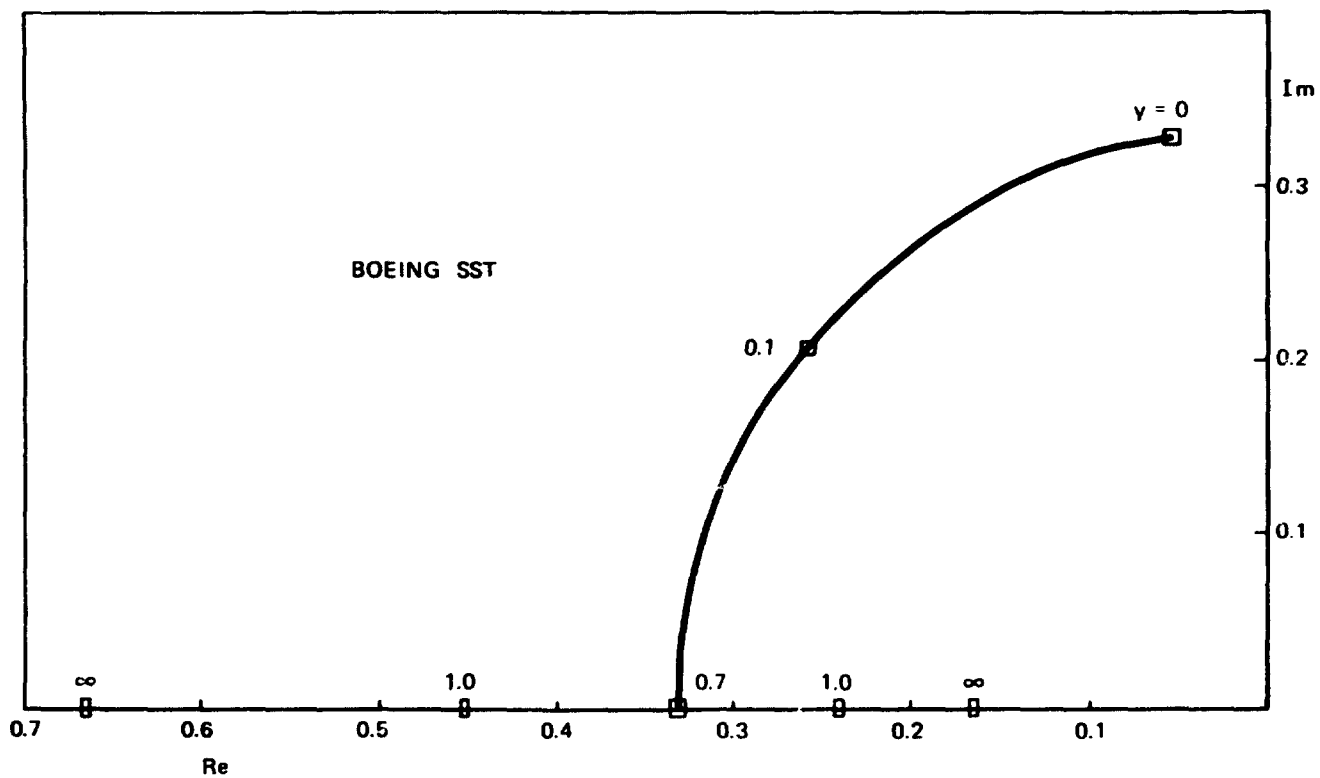

Figure 5.1. Root Locus at Constant Mach No. as y Varies

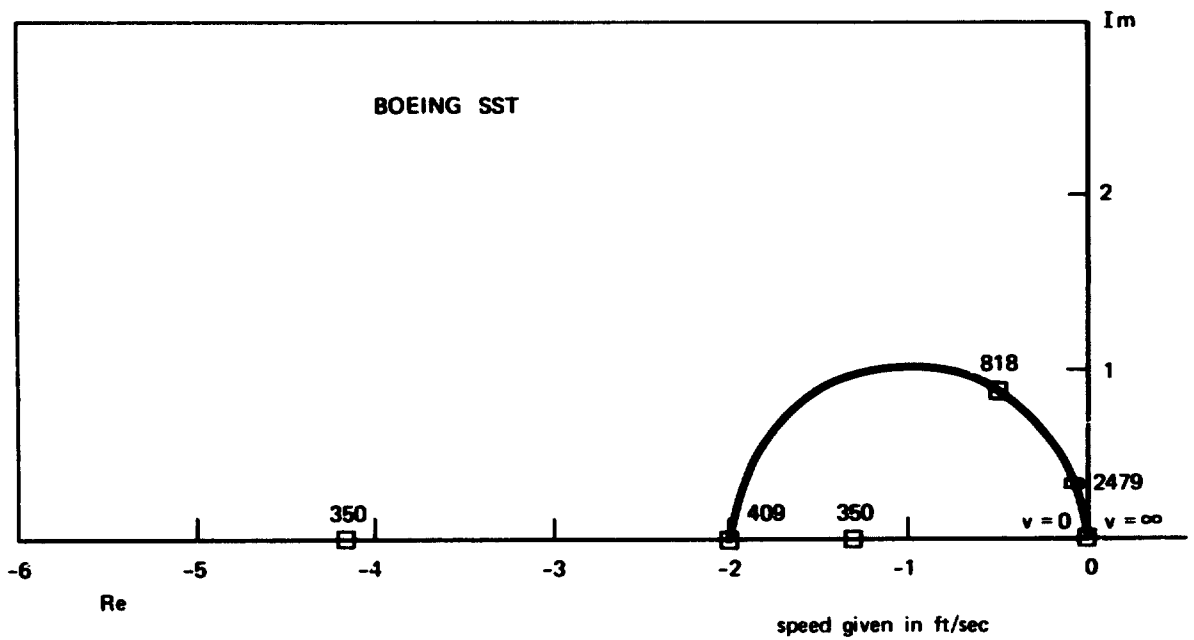

Figure 5.2 Root Locus at y = 0 as True Air Speed Varies

When the roots of the equation are all real they are

$$p = \pm\sqrt{a}\left[Z \pm \sqrt{Z^2 - 1}\right] \tag{5.54}$$

The damping is principally attributable to the slope of the curve of SFC vs C_F for the engine. From (5.51) we see that if Z were zero, the dynamic modes represented by that equation would have no damping. The damping comes from the term $a^2\left(\gamma_{1b} + 2xy/\sqrt{C_{D_o}K}\right)$. Each of these terms originates in a γ_0^2 term in the cost functional. Since γ_0 is proportional to $d\Delta\hat{H}/dR_L$, they represent a penalty on altitude rate excusions.

The first of these terms can be traced to the effect of the cosine of the flight path angle on range. The second comes from a combination of the restriction of velocity to be constant and of the modeling of SFC as a linear function of C_F. Mass rate is therefore proportional to T^2, and since thrust must have a component equal to $W\gamma$, the second γ^2 term enters the cost functional. The second contribution to damping is the larger one.

The equation for the optimal value of u_0 can be solved to express u_0 as a function of R_L. Then, in turn, the equations for γ_0 and $\Delta\hat{H}_0$ can be solved and the entire zero-order state and control are known as functions of R_L. Finally, it is seen that M_1, λ_{M1}, $\lambda_{\gamma 1}$, and λ_{H1} are functions of the zero order state and control and hence they, too, can be expressed as functions of R_L.

Returning now to the expression for the optimal value of u_1, we can say

$$a\lambda_{\gamma 2} + \lambda_{M0}\left[4x^2 y\,\gamma_1 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(\Delta\hat{H}_1 + u_1\right)\right] - f_1\left(R_L\right) = 0 \tag{5.55}$$

where

$$f_1\left(R_L\right) = -\lambda_{M1}\left[\gamma_{1b} + 4x^2 y\,\gamma_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u_0 + \Delta\hat{H}_0\right)\right]$$

$$+ \lambda_{M0}\,M_1\left[2\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right] \tag{5.56}$$

or using $(\partial\mathscr{H}/\partial u)_1 = 0$,

$$f_1\left(R_L\right) = a\lambda_{\gamma 1}\left(\frac{\lambda_{M1}}{\lambda_{M0}}\right) + \lambda_{M0}M_1\left(2\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right) \tag{5.57}$$

Proceeding to take derivatives with respect to R_L we have

$$\frac{d}{dR_L}(\mathscr{H}_u)_2 = a\left\{-\lambda_{H2} - \lambda_{M0}\left[\gamma_1\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) + 4x^2 y\Delta\hat{H}_1\right] - f_2\left(R_L\right)\right\}$$

$$+ \lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u'_1 + \gamma_1\right) - f'_1\left(R_L\right) \tag{5.58}$$

where

$$f_2\left(R_L\right) = \lambda_{M1}\left[\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}} + \gamma_0\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) + 4x^2 y\left(u_0 + \Delta\hat{H}_0\right)\right]$$

$$- \lambda_{M0}M_1\left(4x^2 y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right) \tag{5.59}$$

The next derivative is

$$\frac{d^2}{dR_L^2}(\mathscr{H}_u)_2 = a\left\{\lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u_1 + \Delta\hat{H}_1\right) - f_3\left(R_L\right)\right\}$$

$$- a^2\lambda_{M0}u_1\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) - 2f'_2\left(R_L\right) - f''_1\left(R_L\right)$$

$$+ \lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u''_1 + au_1\right) \tag{5.60}$$

where

$$f_3(R_L) = -\lambda_{M1}\left[4x^2 y\,\gamma_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u_0 + \Delta\hat{H}_0\right)\right]$$

$$+ M_1\lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right) \tag{5.61}$$

Two more derivatives yield the following expression

$$\frac{d^4}{dR_L^{\,4}}(\mathscr{H}_u)_2 = 0 = a\lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u''_1 + au_1\right) - af''_3(R_L)$$

$$- a^2\lambda_{M0}u''_1\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right) - 2f'''_2(R_L) - f^{IV}_1(R_L)$$

$$+ \lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u^{IV}_1 + au''_1\right) \tag{5.62}$$

Finally, this can be rewritten as

$$u^{IV}_1 + u''_1\left[2a - (2Z\sqrt{a})^2\right] + a^2 u_1 = \frac{f^{IV}_1 + af'''_2 + af''_3}{\lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)} \tag{5.63}$$

The homogeneous part of this equation is seen to be identical to the equation satisfied by the optimal u_0. The forcing terms are seen to be functions of the zero-order optimal state and control.

Having derived equations for the optimal u_0 and u_1 in the left side boundary layer it is easy to do the same for the right side boundary layer. Since the development is a direct parallel of that of the left side boundary layer, the details will be omitted. The results will be presented, preceeded by some comments regarding differences that appear in the right side results relative to those of the left side.

First, because of the assumption in the equation for M that $M \cong 0$ and $Q \cong 1$ in the boundary layer, we define $\tilde{M}$ and $\tilde{Q}$ as follows

$$\tilde{M} = \frac{m_f - m}{m_f} \tag{5.64}$$

$$\tilde{M}' = \frac{m_i}{m_f} M' \tag{5.65}$$

$$\tilde{Q} = e^{-\beta(h - h_f + \ln x)} = e^{-\Delta\tilde{H}} \tag{5.66}$$

That is, mass is referred to its final value and altitude to the value that would obtain on a Breguet cruise when mass reaches its final value. As a result of the definition of $\tilde{M}$, we have the costate relationship

$$\lambda_{\tilde{M}} = \frac{m_f}{m_i} \lambda_M \tag{5.67}$$

Second, because of the stretching transformation

$$R_R = \frac{R_f - R}{\epsilon} \tag{5.68}$$

and the resulting derivative relation

$$\frac{d}{dR} = -\frac{1}{\epsilon} \frac{d}{dR_R} \tag{5.69}$$

all three state equations will have the signs of their derivative terms reversed. This sign reversal will, of course, appear in the Hamiltonian in the inner product of the costate vector with the differential equation of the state vector. As a result, all three costate equations will also have the signs of their derivative terms reversed. Third, since boundary conditions on M and Q are now specified at $R = R_f$, the constants of integration in the state and costate equations will be different from the corresponding values from the left side.

Despite these differences, the equation for the optimal u_0 in the

right boundary layer is identical to that of the left.

$$u_0^{IV} + u''_0 \left(2a - [2Z\sqrt{a}]^2\right) + a^2 u_0 = 0 \tag{5.70}$$

The equation for the optimal u_1 is also unchanged.

$$u_1^{IV} + u''_1 \left(2a - [2Z\sqrt{a}]^2\right) + a^2 u_1 = \frac{f_1^{IV} - af'''_2 + af''_3}{\lambda_{M0}\left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o} K}\right)} \tag{5.71}$$

The apparent difference in the sign of the forcing term af'''_2 is due to the fact that odd-power derivatives with respect to R_R have the opposite sign from the corresponding derivatives with respect to R_L. Expressing all derivatives with respect to R, no sign differences occur. Finally, it should be noted that the coefficients and the forcing terms in the above equations are defined exactly as they were in the left side boundary layer.

5.4 Matching the Optimal Initial Transient to Cruise

We proceed now to investigate the solution of the differential equation for the optimal u_0 and to determine the conditions under which solutions in the boundary layers can be matched to the cruise solution. The general solution for u_0 on the oscillatory branch (Fig. 5.1) is

$$u_0 = (u_{01} \cos \omega R_L + u_{02} \sin \omega R_L)\, e^{-\zeta\omega_n R_L} + (u_{03} \cos \omega R_L + u_{04} \sin \omega R_L)\, e^{+\zeta\omega_n R_L} \tag{5.72}$$

where ω_n and ζ, the undamped natural frequency and the damping ratio, are given in Eq. (5.49), (5.52), (5.53). The frequency of oscillation is

$$\omega = \omega_n \sqrt{1 - \zeta^2} \tag{5.73}$$

If matching is to be possible, then the outer expansion of the boundary layer solution must be finite for small values of ϵ. This means that the solution can not have positive exponential terms. Therefore, u_{03} and u_{04} must be zero. Now we proceed to take the outer expansion of u_0 from the boundary layer. First we transform the independent variable

$$R_L = \frac{R}{\epsilon} \tag{5.74}$$

and then we take the limit as $\epsilon \to 0$. It is seen that

$$\lim_{\epsilon \to 0} e^{-\zeta \omega_n R/\epsilon} = 0 \tag{5.75}$$

and hence the outer expansion of u_0 from the left boundary layer is simply zero. This will be written in notation similar to that of O'Malley

$$\left[u^{il}\right]^{ol} = 0 \tag{5.76}$$

The use of the superscript i denotes the inner solution associated with the initial boundary layer. The superscript f will signify the inner solution associated with the final boundary layer. The outer solution represents cruise and is identified by the superscript o.

At this point the inner expansion into the left boundary layer of all of the state and costate variables from cruise will be determined to first order in ϵ. Obviously, no transformation is necessary to determine inner expansions of variables that are constant in cruise. These include u_0, u_1, γ_0, γ_1, M_0, λ_{M0}, $\lambda_{\gamma 0}$ λ_{H0}, $\lambda_{\gamma 1}$ and λ_{H1}. The others require transforming the independent variable and applying the limiting processes described in Chapter II. We have

$$\left[\Delta\hat{H}^{o2}\left(\epsilon R_L\right)\right] = \left\{\gamma_1 \epsilon R_L + \epsilon^2 \frac{\gamma_1 R_L}{2x\sqrt{C_{D_o} K}} - \epsilon \frac{2x\gamma_1 (1 - x^2)}{\sqrt{C_{D_o} K}(1 + 6x^2 - 3x^4)}\right\} \tag{5.77}$$

from which

$$[\Delta\hat{H}^{o2}]^{i2} = \epsilon \left[\gamma_{1b} R_L - \frac{2x\gamma_{1b} (1 - x^2)}{\sqrt{C_{D_o} K}(1 + 6x^2 - 3x^4)}\right] \tag{5.78}$$

$$[\Delta\hat{H}_0{}^{o}]^{i} = 0 \tag{5.79}$$

$$[\Delta\hat{H}_1{}^{o}]^{i} = \gamma_{1b} R_L - \frac{2x\gamma_{1b} (1 - x^2)}{\sqrt{C_{D_o} K}(1 + 6x^2 - 3x^4)} \tag{5.80}$$

Also

$$[M^{c2}(\epsilon R_L)] = \left\{1 - e^{-\gamma_{1b} R_L \epsilon} + \epsilon^2 e^{-\gamma_{1b} R_L \epsilon}\left(\frac{\gamma_{1b}{}^2 R_L}{2x\sqrt{C_{D_o} K}}\right)\right\} \tag{5.81}$$

from which

$$[M_0{}^{o}]^{i} = 0 \tag{5.82}$$

$$[M_1{}^{o}]^{i} = \gamma_{1b} R_L \tag{5.83}$$

Finally,

$$[\lambda_M{}^{o2}(\epsilon R_L)] = \left\{\frac{1}{\gamma_{1b}} e^{\gamma_{1b} R_L \epsilon} + \epsilon^2 e^{\gamma_{1b} R_L \epsilon}\left(\frac{1 + x^2}{3x^2 - 1}\right) R_L\right\} \tag{5.84}$$

which yields

$$[\lambda_{M0}{}^{o}]^{i} = \frac{1}{\gamma_{1b}} \tag{5.85}$$

$$[\lambda_{m1}{}^{o}]^{i} = R_L \tag{5.86}$$

These values are summarized in Table 5.1.

Now u_0 from the left side boundary layer has been shown to have

Table 5.1 INNER EXPANSIONS OF CRUISE VARIABLES INTO THE LEFT SIDE BOUNDARY LAYER

Cruise Variable	Zero Order Expansion	First Order Expansion
γ	0	γ_{lb}
ΔH	0	$\gamma_{lb} R_L - \frac{2x\gamma_{lb}(1-x^2)}{\sqrt{C_{D_o} K}\,(1+6x^2-3x^4)} + M_{lb}(0)$
M	0	$\gamma_{lb} R_L + M_{lb}(0)$
u	0	0
λ_γ	0	$-\frac{1}{a}$
λ_H	0	$-\frac{1}{2x\sqrt{C_{D_o} K}}$
λ_M	$\frac{1}{\gamma_{lb}}$	$R_L + \frac{M_{lb}(0)}{\gamma_{lb}}$

an outer expansion of zero, thus matching the inner expansion of u_0 from cruise. We can write the boundary layer solution as

$$u_0 = (u_{01} \cos \omega R_L + u_{02} \sin \omega R_L)\, e^{-\zeta \omega_n R_L} \tag{5.87}$$

In this form, integrals of u_0 will have the same form as u_0, that is

$$\gamma_0 = (\gamma_{01} \cos \omega R_L + \gamma_{02} \sin \omega R_L)\, e^{-\zeta \omega_n R_L} + \gamma_{03} \tag{5.88}$$

Obviously, in order to match $\gamma_0 = 0$ from cruise, γ_{03} is required to be zero. Furthermore,

$$\Delta\hat{H}_0 = (\Delta\hat{H}_{01} \cos \omega R_L + \Delta\hat{H}_{02} \sin \omega R_L)\, e^{-\zeta \omega_n R_L} + \Delta\hat{H}_{03} \tag{5.89}$$

and $\Delta\hat{H}_{03} = 0$ to match cruise conditions.

From inspection of the costate equations in the boundary layer it is obvious that λ_{M0}, $\lambda_{\gamma 0}$ and λ_{H0} are all constrants. Matching to cruise conditions is simply a matter of equating these constant values to the corresponding cruise conditions shown in Table 5.1. The values required of $\lambda_{\gamma 0}$ and λ_{H0} agree with those values needed to minimize the zero-order Hamiltonian in the boundary layer. The first order Hamiltonian must also be zero everywhere along an optimal trajectory. If we consider its outer expansion for small values of ϵ we can take advantage of the fact that the outer expansions of γ_0, u_0 and $\Delta\hat{H}_0$ are all zero. The first order Hamiltonian then becomes

$$[\mathcal{H}_1^i]^o = -1 + \lambda_{M0}\, \gamma_{1b} = 0 \tag{5.90}$$

which implies that

$$\lambda_{M0} = +\frac{1}{\gamma_{1b}} \tag{5.91}$$

agreeing with our matching value.

Three first order variables in the left side boundary layer are completely determined by the zero order state and optimal control. These are M_1, λ_{H1} and λ_{M1}. Consider first the costates.

$$[\lambda_{H1}{}^{i}] = -\lambda_{M0}\int\left\{4x^2y\,\gamma_0 + \left(\gamma_{1b} + 8x^3y\sqrt{C_{D_o}K}\right)(u_0 + \Delta\hat{H}_0)\right\}dR_L \tag{5.92}$$

Every term of the integrand is of the form of u_0, a damped sinusoid, and hence integrates to the same form. Thus the outer expansion of λ_{H1} is simply the constant of integration. For matching with cruise, this constant must be $\frac{1}{2x\sqrt{C_{D_o}K}}$, from Table 5.1.

The solution of the canonical equation for λ_{M1} contains one secular term, and all other terms are damped sinusoids.

$$\lambda_{M1} = \lambda_{M0}\int\left\{\gamma_{1b} + u_0\left(8x^3y\sqrt{C_{D_o}K} + 2\gamma_{1b}\right)\right.$$
$$\left. + \left(4x^2y + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right)\left(\gamma_0 + 2x\sqrt{C_{D_o}K}\,\Delta\hat{H}_0\right)\right\}dR_L \tag{5.93}$$

The outer expansion of $[\lambda_{M1}{}^{i}]$ is given by

$$[\lambda_{M1}{}^{i}]^{o} = \lambda_{M0}\,\gamma_{1b}\,R_L + c \tag{5.94}$$

The coefficient of R_L is equal to one, because of the value previously assigned to λ_{M0}, and matching occurs if the constant of integration is $(M_{1b}(0)/\gamma_{1b})$.

Now consider the equation for M_1

$$M_1 = \int_0^{R_L} \left\{ \gamma_{1b} \left(1 + u_0 + \frac{\gamma_0}{2x\sqrt{C_{D_o} K}}\right) + \gamma_0^{\,2} \left(\frac{\gamma_{1b}}{2} + \frac{xy}{\sqrt{C_{D_o} K}}\right) \right.$$

$$\left. + \left(4x^3 y \sqrt{C_{D_o} K} + \frac{\gamma_{1b}}{2}\right)\left(u_0 + \Delta\hat{H}_0\right)^2 + 4x^2 y\, \gamma_0 \left(u_0 + \Delta\hat{H}_0\right)\right\} dR_L \qquad (5.95)$$

Again, all terms are damped sinusoids except for one secular term. The outer expansion of M_1 is

$$[M_1^{\,i}]^o = \gamma_{1b}\, R_L + c_i \qquad (5.96)$$

where c_i, the constant of integration, has a value such that the initial value of M_1 is zero. Thus the specified initial value of M is wholly satisfied by the initial value of M_0 in the boundary layer. Now if M from the boundary layer is to match with M from cruise (Table 5.1), then c_i must equal $M_{1b}(0)$.

If c_i happens to be zero, then the initial conditions that determine c_i will describe a locus of points from which transition to cruise can be accomplished at the same schedule of range and mass that would be experienced along a pure Breguet cruise. This locus will, of course, include the origin in γ_0 - $\Delta\hat{H}_0$ space. In general, c_i will not be zero and then $M_{1b}(0)$ represents the initial mass of the Breguet cruise that matches the transition. A negative $M_{1b}(0)$ represents the fuel penalty incurred in climbing to cruise and a positive $M_{1b}(0)$ represents a fuel saving as, for example, in transition to cruise from a higher altitude than cruising altitude.

The first order control, u_1, will have the same form as the zero order control, namely a damped sinusoid, but it also has forcing terms. The

outer expansions of the forcing terms must be determined before u_1 can be matched.

The first of these forcing terms is f_1^{iv} where

$$f_1 = -\lambda_{M1}\left[\gamma_{1b} + 4x^2 y\,\gamma_0 + \left(\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right)\left(u_0 + \Delta\hat{H}_0\right)\right]$$

$$+ \lambda_{M0} M_1 \left(2\gamma_{1b} + 8x^3 y\sqrt{C_{D_o}K}\right) \tag{5.97}$$

Notice first that the coefficient of M_1 is a constant. Also, from Table 5.1, the inner expansion of M_1 is a constant plus a secular term. Only two derivatives of M_1 will remove the effect of its secular term, and hence the contribution of M_1 to the outer expansion of f_1^{iv} is zero. The same is true of the term $\lambda_{M1}\gamma_{1b}$. The remaining terms involve a sum of damped sinusoids (within the brackets) multiplied by λ_{M1}, which is itself a damped sinusoid plus a secular term. Every term in this product will be multiplied by a decaying exponential and will have an outer expansion of zero. The other terms, f_2 and f_3, are of the same form as f_1 and since each of them is differentiated at least twice in the forcing function, they too will contribute outer expansions of zero to the forcing functions. Thus the outer expansion of u_1 will not be affected by the forcing functions. As was the case with u_0, the outer expansion of u_1 will be a constant and the value of that constant will be zero, to match the value in cruise.

Integrating u_1 gives us γ_1, whose outer expansion will be a constant. The value of this constant is seen from Table 5.1 to be the first-order flight path angle from cruise. Finally, integrating γ_1 gives us $\Delta\hat{H}_1$ which will have a secular term from the integral of γ_{1b}

$$[\Delta H_1{}^i]^o = \gamma_{1b} R_L + c \tag{5.98}$$

The constant is selected from Table 5.1, and matching of state, costate and control between the left side boundary layer and cruise is completed.

We now briefly consider the optimality condition that $\mathscr{H}_2 = 0$ over the entire optimal trajectory. Considering those variables whose outer expansions are non-zero we have

$$[\mathscr{H}_2{}^i]^o = \lambda_{M1}\,\gamma_{1b} + \lambda_{M0}\,\gamma_{1b}\left(-M_1 + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right) + \lambda_{H1}\gamma_1 = 0 \tag{5.99}$$

Substituting outer expansions we have

$$M_1(0) + \gamma_{1b}R_L - \gamma_{1b}R_L + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}} - M_1(0) + \lambda_{H1}\gamma_{1b} = 0 \tag{5.100}$$

$$\lambda_{H1} = -\frac{1}{2x\sqrt{C_{D_o}K}} \tag{5.101}$$

which is consistent with our previously determined matching conditions on λ_{H1}.

5.5 Matching the Optimal Final Transient to Cruise

Matching conditions at the right side boundary are similarly established. The development is a direct parallel of that of the left side boundary but with different values required for matching. Table 5.2 shows the inner expansions of state, control and costate variables from Breguet cruise when extended into the right side boundary layer. These values were developed by stretching the independent variable in the vicinity of R equals R_F for the Breguet solutions by means of the transformation

Table 5.2 INNER EXPANSIONS OF CRUISE VARIABLES INTO THE RIGHT SIDE BOUNDARY LAYER

Cruise Variable	Zero Order Expansion	First Order Expansion
γ	0	γ_{1b}
ΔH	$\gamma_{1b} R_{f0} + \ln x$	$\gamma_{1b}(R_{f1} - R_R) + \frac{\gamma_{1b}^2 R_{f0}}{2x\sqrt{C_{D_o} K}} + M_{1b}(0) - \frac{2x\gamma_1 (1 - x^2)}{\sqrt{C_{D_o} K}\,(1 + 6x^2 - 3x^4)}$
M	$1 - e^{-\gamma_{1b} R_{f0}}$	$e^{-\gamma_{1b} R_{f0}}\left[\gamma_{1b}\left(R_{f1} - R_R\right) + M_{1b}(0) + \frac{\gamma_{1b}^2 R_{f0}}{2x\sqrt{C_{D_o} K}}\right]$
u	0	0
λ_γ	0	$-\frac{1}{a}$
λ_H	0	$-\frac{1}{2x\sqrt{C_{D_o} K}}$
λ_M	$\frac{1}{\gamma_{1b}} e^{\gamma_{1b} R_{f0}}$	$e^{\gamma_{1b} R_{f0}}\left[R_{f1} - R_R + \frac{M_{1b}(0)}{\gamma_{1b}} + \frac{\gamma_{1b} R_{f0}}{2x\sqrt{C_{D_o} K}}\right]$

$$R = R_f - \epsilon R_R \tag{5.102}$$

and expanding the dependent variables for small ϵ. Also, since M_f is specified, we have at R_f

$$M_{0f} = M_f \tag{5.103}$$

$$M_{1f} = 0 \tag{5.104}$$

and the unspecified value of R_f can be expressed as an expansion

$$R_f = R_{0f} + \epsilon R_{1f} \tag{5.105}$$

thus accounting for the fact that since the first order solution does not change the final value of mass, it must change the final value of range.

In the right side boundary layer the optimal control on the oscillatory branch has the form

$$u_0 = e^{-\zeta\omega_n R_R} (u_{01} \cos \omega R_R + u_{02} \sin \omega R_R)$$
$$+ e^{+\zeta\omega_n R_R} (u_{03} \cos \omega R_R + u_{04} \sin \omega R_R) \tag{5.106}$$

The coefficients in the divergent term are taken as zero so that matching will be possible. The exponential term is transcendentally small for small ϵ

$$e^{-2\zeta\omega_n (R_{0f} + \epsilon R_{1f} - R)/\epsilon} \to 0 \tag{5.107}$$

It is seen that the outer expansion of u_0 from the rignt side boundary layer is zero, which matches the constant value of zero from cruise.

It follows directly that γ_0 in the boundary layer is a damped sinusoid and its constant of integration is zero. Furthermore, $\Delta\tilde{H}_0$ in the boundary layer is also a damped sinusoid and its constant of integration is also zero. That is, from the definition of $\Delta\tilde{H}_0$, namely

$$\Delta\tilde{H}_0 = \Delta H_0 - \ln x - \gamma_{1b} R_{0f} \tag{5.108}$$

we will have at $\Delta\tilde{H}_0$ equals zero

$$\Delta H_0 = \ln x + \gamma_{1b} R_{0f} \tag{5.109}$$

as shown in Table 5.2.

The form of the equation for $\tilde{M}$ is unchanged from the left side boundary. It is again a secular term plus a constant of integration. All secular terms in expansions from the right side boundary layer are opposite in sign from their counterparts in the left side boundary layer because of the previously noted sign difference in the differentials dR_L and dR_R. Continuing with $\tilde{M}$ we have

$$[M_f^{i1}]^o = - \gamma_{1b} R_R + \tilde{c}_f \tag{5.110}$$

$$[M_f^{i1}]^o = - \left(\frac{m_f}{m_i}\right) \gamma_{1b} R_R + c_f \tag{5.111}$$

In order to achieve a specified value of M_f we must have the constant value of M_0 equal to M_f and the constant, c_f, must be chosen such that the value of M_1 is zero when R_R equals R_f. We have

$$[M_f^{i2}] = M_f + \epsilon \left[- \left(\frac{m_f}{m_i}\right) \gamma_{1b} R_R + c_f\right] \tag{5.112}$$

This must match the expansion from cruise which is, from Table 5.2,

$$[M^{o2}] = 1 - e^{-\gamma_{1b}R_{0f}} + \epsilon e^{-\gamma_{1b}R_{0f}}\left[\gamma_{1b}(R_{1f}-R_R) + \frac{\gamma_{1b}^{2} R_{0f}}{2x\sqrt{C_{D_o}K}} + M_1(0)\right] \quad (5.113)$$

Now we have the following conditions for matching

$$1 - e^{-\gamma_{1b}R_{0f}} = M_f \quad (5.114)$$

$$e^{-\gamma_{1b}R_{0f}} = -\frac{m_f}{m_i} \quad (5.115)$$

$$c_f = e^{-\gamma_{1b}R_{0f}}\left[\frac{\gamma_{1b}^{2}R_{0f}}{2x\sqrt{C_{D_o}K}} + M_1(0) + \gamma_{1b}R_{1f}\right] \quad (5.116)$$

The first condition defines R_{0f} and is in fact a zero-order statement of the Breguet range equation. The second condition is equivalent to the first since by definition of M_f

$$1 - e^{-\gamma_{1b}R_{0f}} = M_f = 1 - \frac{m_f}{m_i} \quad 5.117)$$

The third condition relates c_f and R_{1f}. Since we know that M_{1f} equals zero, the equation for M_1 evaluated at R_f serves to define c in terms of the zero order terminal state (ΔH_{0f}, γ_{0f}). The third condition therefore defines R_{1f} in terms of c_f. The derivatiion of the required value of c_f is deferred until after matching has been established for the remaining variables.

The final value of mass on the matching Breguet cruise is

$$M_f^{o} = 1 - e^{-\gamma_{1b}R_{0f}} + \epsilon\left\{e^{-\gamma_{1b}R_{0f}}\left[M_{1b}(0) + \frac{\gamma_{1b}^{2}R_{0f}}{2x\sqrt{C_{D_o}K}} + \gamma_{1b}R_{1f}\right]\right\} \quad (5.118)$$

or

$$M_f^{o} = 1 - e^{-\gamma_{1b}R_{0f}} + \epsilon\, c_f \quad (5.119)$$

which does not in general equal the specified final value of mass. The difference is the first order term above and it is attributable to three sources. First, the initial value of mass on the matching Breguet cruise may have a non-zero value, $M_{1b}(0)$. Second is the fuel penalty paid to account for the climb at constant flight path angle that is characteristic of the Breguet cruise, $\frac{\gamma_{1b}^2 R_{0f}}{2x\sqrt{C_{D_o} K}}$. Third is the penalty paid in order to achieve a terminal state that may not be on the Breguet cruise, $\gamma_{1b} R_{1f}$. This is discussed more fully at the end of this chapter, where a more general range equation is developed.

As before, the costates $\lambda_{\gamma 0}$ and λ_{H0} are constants whose values are seen to be both zero. The value of λ_{M0}, another constant, is determined from the requirement that $\mathscr{H}_1 = 0$

$$[\mathscr{H}_1^f]^o = -1 + \lambda_{\tilde{M}0}\, \gamma_{1b} = 0 \tag{5.120}$$

$$\lambda_{\tilde{M}0} = \frac{1}{\gamma_{1b}} \tag{5.121}$$

But since

$$\lambda_{\tilde{M}0} = \frac{m_f}{m_i} \lambda_{M0} \tag{5.122}$$

we have

$$\lambda_{M0} = \frac{m_i}{m_f} \frac{1}{\gamma_{1b}} = \frac{1}{\gamma_{1b}} e^{+\gamma_{1b} P_f} \tag{5.123}$$

The solution of the equation for λ_{H1} again involves a damped sinusoid and a constant of integration. The constant is chosen as $-\frac{1}{2x\sqrt{C_{D_o} K}}$ and matching is achieved for λ_{H1}. The equation for λ_{M1} again yields a secular

term plus a constant of integration

$$[\lambda_{\tilde{M}1}{}^{f}]^{o} = -\lambda_{\tilde{M}0}\,\gamma_{1b}\,R_R + \tilde{c} \tag{5.124}$$

$$[\lambda_{M1}{}^{f}]^{o} = -e^{\gamma_{1b}R_f}\,R_R + c \tag{5.125}$$

The value of the constant is, from Table 5.2,

$$e^{+\gamma_{1b}R_{0f}}\left[R_{1f} + \frac{M_{1b}(0)}{\gamma_{1b}} + \frac{\gamma_{1b}\,R_{0f}}{2x\sqrt{C_{D_o}K}}\right]$$

The form of the forcing functions in the right side boundary layer is the same as it was in the left side. Again, they contribute nothing to the outer expansion of u_1. The constant of integration associated with u_1 must be zero. Now γ_1 has only a constant as its outer expansion and that constant must be γ_{1b}. The non-zero value of γ_{1b} introduces a secular term into the expression for $\Delta\tilde{H}_0$. We have

$$[\Delta\tilde{H}_0{}^{f}]^{o} = -\gamma_{1b}R_R + c \tag{5.126}$$

and the value of c is

$$c = \frac{\gamma_{1b}{}^{2}\,R_{0f}}{2x\sqrt{C_{D_o}K}} - \frac{2x\,\gamma_{1b}\,(1 - x^2)}{\sqrt{C_{D_o}K}\,(1 + 6x^2 - 3x^4)} + M_{1b}(0) + \gamma_{1b}R_{1f} \tag{5.127}$$

The optimality conditions on the Hamiltonian are consistent with the matching values of these variables. For $\mathscr{H}_0 = 0$ we have

$$\lambda_{\gamma 0} = \lambda_{H0} = 0 \tag{5.128}$$

From $\mathscr{H}_1 = 0$ we have already established that

$$\lambda_{M0} = \left(\frac{1}{\gamma_{1b}}\right) e^{\gamma_{1b}R_{0f}} \tag{5.129}$$

For $\mathcal{H}_2 = 0$ we have

$$[\lambda_{H1}{}^{f}]^{o} = -\lambda_{\tilde{M}1} - \lambda_{\tilde{M}0}\left[-\tilde{M}_1 + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\right] \tag{5.130}$$

$$[\lambda_{H1}{}^{f}]^{o} = -\left[R_{1f} - R_R + \frac{M_{1b}(0)}{\gamma_{1b}} + \frac{\gamma_{1b}\,R_{0f}}{2x\sqrt{C_{D_o}K}}\right]$$

$$-\frac{1}{\gamma_{1b}}\left[\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}} - \gamma_{1b}(R_{1f} - R_R) - M_{1b}(0) - \frac{\gamma_{1b}{}^{2}R_{0f}}{2x\sqrt{C_{D_o}K}}\right] \tag{5.131}$$

$$\lambda_{H1} = -\frac{1}{2x\sqrt{C_{D_o}K}} \tag{5.132}$$

which is the value required for matching, from Table 5.2.

5.6 Composite Matched Asymptotic Expansions and Costs in Transitions

Having established the conditions for matching cruise to both boundary layers it is now possible to express the optimal values of state, costate and control variables in matched asymptotic expansions. These will be be uniformly valid over the optimal trajectory between the point at which the trajectory leaves the constraint of maximum C_F and the point at which it meets the constraint of minimum SFC.

For this problem a matched asymptotic expansion of a variable will consist of the sum of the solutions for that variable in cruise as well as in the two boundary layers, minus the inner expansions of this variable as it passes from cruise into the boundary layers. Using the notation of O'Malley we have, for mass

$$M^{c2} = M^{i2} + M^{o2} + M^{f2} - (M^{o2})^{i2} - (M^{o2})^{f2} \tag{5.133}$$

In order to proceed with the expansion it is necessary first to solve the zero-order state differential equations (first order for M) in the boundary layers. Consider the expression for the optimal value of u_0 in the left side boundary layer as a damped sinusoid

$$u_0 = (u_{01} \cos \omega R_L + u_{02} \sin \omega R_L)\, e^{-\zeta\omega_n R_L} \tag{5.134}$$

Integrating twice and using the previously identified constants of integration

$$\gamma_0 = e^{-\zeta\omega_n R_L} \cdot \left\{\left(-\zeta\omega_n u_{01} - \omega u_{02}\right)\cos \omega R_L + \left(\omega u_{01} - \zeta\omega_n u_{02}\right)\sin \omega R_L\right\} \tag{5.135}$$

which can also be written as

$$\gamma_0 = e^{-\zeta\omega_n R_L} \left\{\gamma_{01} \cos \omega R_L + \gamma_{02} \sin \omega R_L\right\} \tag{5.136}$$

and

$$\Delta\hat{H}_0 = \frac{e^{-\zeta\omega_n R_L}}{\omega_n^2} \left\{-\left(\zeta\omega_n\gamma_{01} + \omega\gamma_{02}\right)\cos \omega R_L + \left(\omega\gamma_{01} - \zeta\omega_n\gamma_{02}\right)\sin \omega R_L\right\} \tag{5.137}$$

or

$$\Delta\hat{H}_0 = e^{-\zeta\omega_n R_L} \left\{\Delta\hat{H}_{01} \cos \omega R_L + \Delta\hat{H}_{02} \sin \omega R_L\right\} \tag{5.138}$$

The initial conditions on γ_0 and $\Delta\hat{H}_0$ are obviously γ_{01} and $\Delta\hat{H}_{01}$. The remaining constants, u_{01}, u_{02}, $\Delta\hat{H}_{02}$ and γ_{02} could be expressed in terms of γ_{01} and $\Delta\hat{H}_{01}$. However, a more useful relationship exists among u_0, γ_0 and $\Delta\hat{H}_0$. Consider the following definitions

$$\gamma_{01} = -\zeta\omega_n u_{01} - \omega u_{02} \tag{5.139}$$

$$\gamma_{02} = +\omega u_{01} - \zeta\omega_n u_{02} \rightarrow u_{01} = \left(\frac{1}{\omega}\right)\left(\gamma_{02} + \zeta\omega_n u_{02}\right) \tag{5.140}$$

$$\Delta\hat{H}_{01} = -\left(\frac{1}{\omega_n^2}\right)\left(\zeta\omega_n\gamma_{01} + \omega\gamma_{02}\right) \tag{5.141}$$

$$\Delta\hat{H}_{02} = \left(\frac{1}{\omega_n^2}\right)\left(\omega\gamma_{01} - \zeta\omega_n\gamma_{02}\right) \tag{5.142}$$

Substituting for γ_{01} and then u_{01} from (5.139) and (5.140) into (5.142) yeilds

$$\Delta\hat{H}_{02} + u_{02} = -2\left(\frac{\zeta\omega_n}{\omega_n^2}\right)\gamma_{02} \tag{5.143}$$

Substituting (5.139) and (5.140) into (5.141) yeilds

$$\Delta\hat{H}_{01} + u_{01} = -2\left(\frac{\zeta\omega_n}{\omega_n^2}\right)\gamma_{01} \tag{5.144}$$

and from these two results we have the composite result

$$\Delta\hat{H}_0 + u_0 = -2\left(\frac{\zeta}{\omega_n}\right)\gamma_0 \tag{5.145}$$

Now simply by regrouping terms, the M equation can be written as

$$M_1' = \gamma_{1b}\left(1 + u_0 + \frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\right) + \frac{\gamma_{1b}}{2}\left[\gamma_0^2 + \left(u_0 + \Delta\hat{H}_0\right)^2\right]$$

$$+ 4x^3 y\sqrt{C_{D_o}K}\left[\left(u_0 + \Delta\hat{H}_0\right) + \frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\right]^2 \tag{5.146}$$

Making use of our result for $u_0 + \Delta\hat{H}_0$ this becomes

$$M_1' = \gamma_{1b}\left(1 + u_0 + \frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\right)$$

$$+ \gamma_0^2\left\{\frac{\gamma_{1b}}{2}\left(1 + 4\frac{\zeta^2}{\omega_n^2}\right) + 4x^3 y\sqrt{C_{D_o}K}\left(\frac{1}{2x\sqrt{C_{D_o}K}} - \frac{2\zeta}{\omega_n}\right)^2\right\} \tag{5.147}$$

The first term is easily integrated by relating u_0 and γ_0 to the differential equations for γ_0 and $\Delta\hat{H}_0$ respectively to give

$$\gamma_{1b}\int_0^{R_L}\left(1+\frac{\gamma_0'}{a}+\frac{\Delta\hat{H}_0'}{2x\sqrt{C_{D_o}K}}\right)dR_L = \gamma_{1b}\left(R_L+\frac{\gamma_0}{a}+\frac{\Delta\hat{H}_0}{2x\sqrt{C_{D_o}K}}\right)+c \qquad (5.148)$$

To integrate the second term, express γ_0 as a damped sinusoid

$$\gamma_0 = e^{-\zeta\omega_n R_L}\left(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L\right) \qquad (5.149)$$

$$\gamma_0^2 = e^{-2\zeta\omega_N R_L}\left(\gamma_{01}^2\cos^2\omega R_L + \gamma_{02}^2\sin^2\omega R_L\right.$$

$$\left. + 2\gamma_{01}\gamma_{02}\sin\omega R_L\cos\omega R_L\right) \qquad (5.150)$$

$$\int\gamma_0^2\,dR_L = \frac{e^{-2\zeta\omega_n R_L}}{4\omega_n^2}\left[-\frac{\omega_n}{\zeta}\left(\gamma_{01}^2+\gamma_{02}^2\right)+\sin 2\omega R_L\left(\omega\gamma_{01}^2-\omega\gamma_{02}^2\right.\right.$$

$$\left.\left. - 2\zeta\omega_n\gamma_{01}\gamma_{02}\right)-\cos 2\omega R_L\left(\zeta\omega_n\gamma_{01}^2+\zeta\omega_n\gamma_{02}^2+2\omega\gamma_{01}\gamma_{02}\right)\right]$$

$$(5.151)$$

Summarizing

$$M_1 = \gamma_{1b}R_L + \gamma_{1b}\frac{\gamma_0}{a} + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\Delta\hat{H}_0 + \left[\frac{\gamma_{1b}}{2}\left(1+4\frac{\zeta^2}{\omega_n^2}\right)\right.$$

$$+ 4x^3y\sqrt{C_{D_o}K}\left(\frac{1}{2x\sqrt{C_{D_o}K}}-\frac{2\zeta}{\omega_n}\right)^2\Big]\left(\frac{1}{4\omega_n^2}\right)e^{-2\zeta\omega_n R_L}\Big[-\frac{\omega_n}{\zeta}\left(\gamma_{01}^2+\gamma_{02}^2\right)$$

$$+ \sin 2\omega R_L\left(\omega\gamma_{01}^2-\omega\gamma_{02}^2-2\zeta\omega_n\gamma_{01}\gamma_{02}\right)$$

$$\left. - \cos 2\omega R_L\left(\zeta\omega_n\gamma_{01}^2+\zeta\omega_n\gamma_{02}^2+2\omega\gamma_{01}\gamma_{02}\right)\right] + c_i \qquad (5.152)$$

The coefficient of the exponential is simplified by substituting for ζ^2 from its definition (5.49), (5.53) and then combining like terms. Then

after substituting the appropriate functions of R_L for γ_0 and $\Delta\hat{H}_0$, and substituting for γ_{02} in terms of γ_{0i} and $\Delta\hat{H}_{0i}$ we have

$$M_1 = \gamma_{1b}R_L + \frac{\gamma_{1b}}{a} e^{-\zeta\omega_n R_L}\left[\gamma_{0i}\cos\omega R_L - \frac{1}{\omega}\left(\omega_n^2\Delta\hat{H}_{0i} + \zeta\omega_n\gamma_{0i}\right)\sin\omega R_L\right]$$

$$+ \frac{\gamma_{1b}e^{-\zeta\omega_n R_L}}{2x\sqrt{C_{D_o}K}}\left[\Delta\hat{H}_{0i}\cos\omega R_L + \frac{1}{\omega}\left(\gamma_{0i} + \zeta\omega_n\Delta\hat{H}_{0i}\right)\sin\omega R_L\right]$$

$$+ \frac{e^{-2\zeta\omega_n R_L}}{4\omega^2}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}} - 8x^2y\frac{\zeta}{\omega_n}\right)\left[-\frac{\omega_n}{\zeta}\left(\gamma_{0i}^2 + \omega_n^2\Delta\hat{H}_{0i}^2 + 2\zeta\omega_n\gamma_{0i}\Delta\hat{H}_{0i}\right)\right.$$

$$+ \omega\sin 2\omega R_L\left(\gamma_{0i}^2 - \omega_n^2\Delta\hat{H}_{0i}\right)$$

$$\left. + \zeta\omega_n\cos 2\omega R_L\left(\gamma_{0i}^2 + \frac{2\omega_n}{\zeta}\Delta\hat{H}_{0i}\gamma_{0i} + \omega_n^2\Delta\hat{H}_{0i}^2\right)\right] + c_i \qquad (5.153)$$

Since the initial value of M_1 must be zero in the boundary layer solution, we have for the constant of integration

$$c_i = -\gamma_{1b}\left(\frac{\gamma_{0i}}{\omega_n^2} + \frac{\Delta\hat{H}_{0i}}{2x\sqrt{C_{D_o}K}}\right) + \left(\frac{1}{4\zeta\omega_n}\right)\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}} - 8x^2y\frac{\zeta}{\omega_n}\right)\left(\gamma_{0i}^2 + \omega_n^2\Delta\hat{H}_{0i}^2\right) \qquad (5.154)$$

A similar expression in terms of R_R, γ_{0f}, and $\Delta\tilde{H}_{0f}$ applies in the right side boundary layer. The signs of the expressions for γ_{01}, γ_{02}, and $\Delta\tilde{H}_{01}$ and $\Delta\tilde{H}_{02}$ are now opposite from those of the left side boundary layer. This leads to a change in sign in the expression relating the optimal control and the state

$$u_0 + \Delta\tilde{H}_0 = +2\left(\frac{\zeta}{\omega_n}\right)\gamma_0 \qquad (5.155)$$

The sign reversals are due to the reversal in sign of the derivatives

of the state variables in the right side boundary layer. We have for $\tilde{M}_1$

$$\tilde{M}_1 = -\gamma_{1b}R_R + \frac{\gamma_{1b}}{a}e^{-\zeta\omega_n R_R}\left[\gamma_{0f}\cos\omega R_R + \frac{1}{\omega}\left(\omega_n^2\,\Delta\tilde{H}_{0f} - \zeta\omega_n\gamma_{0f}\right)\sin\omega R_R\right]$$

$$+\frac{\gamma_{1b}e^{-\zeta\omega_n R_R}}{2x\sqrt{C_{D_o}K}}\left[\Delta\tilde{H}_{0f}\cos\omega R_R + \frac{1}{\omega}\left(\zeta\omega_n\Delta\tilde{H}_{0f} - \gamma_{0f}\right)\sin\omega R_R\right] - \frac{e^{-2\zeta\omega_n R_R}}{4\omega^2}\left(\gamma_{1b}\right.$$

$$\left.+\frac{2xy}{\sqrt{C_{D_o}K}} + 8x^2y\frac{\zeta}{\omega_n}\right)\left[-\frac{\omega_n}{\zeta}\left(\gamma_{0f}^2 + \omega_n^2\Delta\tilde{H}_{0f}^2 - 2\zeta\omega_n\Delta\tilde{H}_{0f}\gamma_{0f}\right) + \omega\sin 2\omega R_R\left(\gamma_{0f}^2\right.\right.$$

$$\left.\left.-\omega_n^2\Delta\tilde{H}_{0f}^2\right) + \zeta\omega_n\cos 2\omega R_R\left(\gamma_{0f}^2 - \frac{2\omega_n}{\zeta}\Delta\tilde{H}_{0f}\gamma_{0f} + \omega_n^2\Delta\tilde{H}_{0f}^2\right)\right] + \tilde{c}_f \qquad (5.156)$$

From this we can write

$$M_1 = e^{-\gamma_{1b}R_f}\left\{-\gamma_{1b}R_R + \frac{\gamma_{1b}}{a}e^{-\zeta\omega_n R_R}\left[\gamma_{0f}\cos\omega R_R + \frac{1}{\omega}\left(\omega_n^2\,\Delta\tilde{H}_{0f} - \zeta\omega_n\gamma_{0f}\right)\sin\omega R_R\right]\right.$$

$$+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}e^{-\zeta\omega_n R_R}\left[\Delta\tilde{H}_{0f}\cos\omega R_R + \frac{1}{\omega}\left(\zeta\omega_n\Delta\tilde{H}_{0f} - \gamma_{0f}\right)\sin\omega R_R\right]$$

$$-\frac{e^{-2\zeta\omega_n R_R}}{4\omega^2}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}} + 8x^2y\frac{\zeta}{\omega_n}\right)\left[-\frac{\omega_n}{\zeta}\left(\gamma_{0f}^2 + \omega_n^2\Delta\tilde{H}_{0f}^2 - 2\zeta\omega_n\Delta\tilde{H}_{0f}\gamma_{0f}\right)\right.$$

$$+\omega\sin 2\omega R_R\left(\gamma_{0f}^2 - \omega_n^2\Delta\tilde{H}_{0f}^2\right) + \zeta\omega_n\cos 2\omega R_R\left(\gamma_{0f}^2 - \frac{2\omega_n}{\zeta}\Delta\tilde{H}_{0f}\gamma_{0f}\right.$$

$$\left.\left.\left.+\omega_n^2\,\Delta\tilde{H}_{0f}^2\right)\right]\right\} + c_f \qquad (5.157)$$

The value of the constant is such that M_1 is zero at the final value of R (when R_R equals zero). The specified final value of M will be satisfied exactly by the final value of M_0. Accordingly we have

$$c_f = -e^{-\gamma_{1b}R_{0f}}\Big[\gamma_{1b}\Big(\frac{\gamma_{0f}}{\omega_n^2}+\frac{\Delta\tilde{H}_{0f}}{2x\sqrt{C_{D_o}K}}\Big)$$

$$+\frac{1}{4\zeta\omega_n}\Big(\gamma_{1b}+\frac{2xy}{\sqrt{C_{D_o}K}}+8x^2y\frac{\zeta}{\omega_n}\Big)\Big(\gamma_{0f}^2+\omega_n^2\Delta\tilde{H}_{0f}^2\Big)\Big] \qquad (5.158)$$

when the final state is fully specified.

Transforming both boundary layers to functions of R we can write out the constituent parts of the composite solution for M

$$M^{i2} = \gamma_{1b}R+\epsilon e^{-\zeta\omega_n R/\epsilon}\Big[\gamma_{1b}\Big(\frac{\gamma_{0i}}{\omega_n^2}+\frac{\Delta\hat{H}_{0i}}{2x\sqrt{C_{D_o}K}}\Big)\cos\Big(\frac{\omega R}{\epsilon}\Big)$$

$$+\frac{\gamma_{1b}}{\omega}\Big(\gamma_{0i}\Big[-\frac{\zeta}{\omega_n}+\frac{1}{2x\sqrt{C_{D_o}K}}\Big]-\Delta\hat{H}_{0i}\Big[1-\frac{\zeta\omega_n}{2x\sqrt{C_{D_o}K}}\Big]\Big)\sin\Big(\frac{\omega R}{\epsilon}\Big)\Big]$$

$$+\frac{\epsilon}{4\omega^2}\Big(\gamma_{1b}+\frac{2xy}{\sqrt{C_{D_o}K}}-8x^2y\frac{\zeta}{\omega_n}\Big)e^{-\zeta\omega_n R/\epsilon}\Big[-\frac{\omega_n}{\zeta}\Big(\gamma_{0i}^2+\omega_n^2\Delta\hat{H}_{0i}^2$$

$$+2\zeta\omega_n\gamma_{0i}\Delta\hat{H}_{0i}\Big)+\omega\Big(\gamma_{0i}^2-\omega_n^2\Delta\hat{H}_{0i}^2\Big)\sin\Big(\frac{2\omega R}{\epsilon}\Big)$$

$$+\zeta\omega_n\Big(\gamma_{0i}^2+\frac{2\omega_n}{\zeta}\Delta\hat{H}_{0i}\gamma_{0i}+\omega_n^2\Delta\hat{H}_{0i}^2\Big)\cos\Big(\frac{2\omega R}{\epsilon}\Big)\Big]$$

$$-\epsilon\gamma_{1b}\Big(\frac{\gamma_{0i}}{\omega_n^2}+\frac{\Delta\hat{H}_{0i}}{2x\sqrt{C_{D_o}K}}\Big)+\frac{\epsilon}{4\zeta\omega_n}\Big(\gamma_{1b}+\frac{2xy}{\sqrt{C_{D_o}K}}-8x^2y\frac{\zeta}{\omega_n}\Big)\Big(\gamma_{0i}^2+\omega_n^2\Delta\hat{H}_{0i}^2\Big) \qquad (5.159)$$

$$M^{o2} = 1-e^{-\gamma_{1b}R}+\epsilon e^{-\gamma_{1b}R}\Big[\frac{\gamma_{1b}^2R}{2x\sqrt{C_{D_o}K}}+M_{1b}(0)\Big] \qquad (5.160)$$

$$M^{f2} = 1 - e^{-\gamma_{1b} R_f}\left[1 + \gamma_{1b}\left(R_f - R\right)\right]$$

$$+ \epsilon e^{-\gamma_{1b} R_f - \zeta\omega_n\left(R_f - R\right)/\epsilon}\left[\gamma_{1b}\left(\frac{\gamma_{0f}}{\omega_n^2} + \frac{\Delta\tilde{H}_{0f}}{2x\sqrt{C_{D_o} K}}\right)\cos\left(\frac{\omega}{\epsilon}\left[R_f - R\right]\right)\right.$$

$$+ \frac{\gamma_{1b}}{\omega}\left(\Delta\tilde{H}_{0f}\left[1 + \frac{\zeta\omega_n}{2x\sqrt{C_{D_o} K}}\right] - \gamma_{0f}\left[\frac{\zeta}{\omega_n} + \frac{1}{2x\sqrt{C_{D_o} K}}\right]\right)\sin\left(\frac{\omega}{\epsilon}\left[R_f - R\right]\right)$$

$$- \frac{\epsilon}{4\omega^2}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o} K}} + 8x^2 y\frac{\zeta}{\omega_n}\right) e^{-\gamma_{1b} R_f - 2\zeta\omega_n\left(R_f - R\right)/\epsilon}\left[-\frac{\omega_n}{\zeta}\left(\gamma_{0f}^2\right.\right.$$

$$\left. + \omega_n^2\Delta\tilde{H}_{0f}^2 - 2\zeta\omega_n\Delta\tilde{H}_{0f}\gamma_{0f}\right) + \omega\left(\gamma_{0f}^2 - \omega_n^2\Delta\tilde{H}_{0f}^2\right)\sin\left(\frac{2\omega}{\epsilon}\left[R_f - R\right]\right)$$

$$\left.\left. + \zeta\omega_n\left(\gamma_{0f}^2 - \frac{2\omega_n}{\zeta}\Delta\tilde{H}_{0f}\gamma_{0f} + \omega_n^2\Delta\tilde{H}_{0f}^2\right)\cos\left(\frac{2\omega}{\epsilon}\left[R_f - R\right]\right)\right]\right]$$

$$- \epsilon\, e^{-\gamma_{1b} R_f}\left[\gamma_{1b}\left(\frac{\gamma_{0f}}{\omega_n^2} + \frac{\Delta\tilde{H}_{0f}}{2x\sqrt{C_{D_o} K}}\right)\right.$$

$$\left. + \frac{1}{4\zeta\omega_n}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o} K}} + 8x^2 y\frac{\zeta}{\omega_n}\right)\left(\gamma_{0f}^2 + \omega_n^2\Delta\tilde{H}_{0f}^2\right)\right] \tag{5.161}$$

$$\left[M^{o2}\right]^{i2} = \left[M^{i2}\right]^{o2} = \gamma_{1b} R + \epsilon M_{1b}(0) \tag{5.162}$$

$$\left[M^{o2}\right]^{f2} = \left[M^{f2}\right]^{o2} = 1 - e^{-\gamma_{1b} R_f}\left[1 + \gamma_{1b}\left(R_f = R\right)\right]$$

$$+ \epsilon\, e^{-\gamma_{1b} R_f}\left[\frac{\gamma_{1b}^2 R_f}{2x\sqrt{C_{D_o} K}} + M_{1b}(0)\right] \tag{5.163}$$

Finally, the composite solution for M to two terms is

$$M^{c2} = M^{i2} + M^{o2} + M^{f2} - \left[M^{o2}\right]^{i2} - \left[M^{o2}\right]^{f2} \tag{5.164}$$

At this point we can note that similar composite solutions can be expressed for u, γ, ΔH, and the costates, and the matching constants have already been evaluated. In all these cases, however, the results will depend on u_1. While u_1 is, in principle, evaluated quite directly, in fact the high order derivatives involved in the forcing functions make an analytic evaluation quite laborious. It is shown in Chapter VI that a very good representation of the optimal trajectories comes from considering a zero order boundary layer solution to match its corresponding cruise solution through first order in ϵ, and the labor involved in evaluating the first order corrections to the transients is not justified.

We have seen that the constants of integration associatedwith M^{i2} and M^{f2} are an indication of the difference between the fuel consumed in transition to and from cruise and the fuel that would have been used in a pure Breguet cruise over the same interval of range. The equations defining them have the same region of applicability in ΔH_0 - γ_0 space as do the state equations, and matching between cruise and transition to and from cruise is possible everywhere within that region. It is possible therefore, to assign a cost number for any point (ΔH_0, γ_0) in the region and to develop contours of constant cost. Each contour will define a locus of initial conditions from which the same cost is incurred in traversing a matching transition to cruise. The same could be done for final conditions on transitions from cruise. The cost is easily evaluated relative to the cost incurred in a pure Breguet cruise of the same range using equations (5.154) and (5.158).

The contours are not optimal trajectories. Trajectories, as we have seen, are damped sinusoids as a function of R. In $\Delta H_0 - \gamma_0$ space these become spirals. In traversing a spiral one passes through a region of higher as well as lower initial cost. This is because the spiral trajectory may include a region of negative ΔH wherein the airplane is climbing and requiring more thrust and hence a higher fuel consumption than for cruise, as well as a region of positive ΔH wherein the airplane could fly down to cruise at reduced thrust.

Since any point in $\Delta H_0 - \gamma_0$ space within the region of applicability of the equations can lie on an optimal transition trajectory, it is possible to assign to each point the throttle setting or thrust coefficent that the optimal trajectory would require as it passed through that point. Obviously, all optimal trajectories will include the origin, at which point the throttle setting equals that required for cruise. Moving away in one direction all throttle settings will ultimately reach maximum. In another direction all settings will reach minimum. These and other loci of constant thrust coefficient are discussed in Chapter VI in connection with some numerical examples.

5.7 Non-Oscillatory Optimal Control

Thus far we have considered only the oscillatory form of the optimal control. It is, of course, possible that the roots of the characteristic equation of the differential equation for the optimal control will all be real. We then have

$$u_0 = u_{01} e^{+r_1 R} + u_{02} e^{+r_2 R} + u_{03} e^{+r_3 R} + u_{04} e^{+r_4 R} \tag{5.165}$$

The roots are symmetrically located about the imaginary axis so that we have

$$r_1 = -r_3 \tag{5.166}$$

$$r_2 = -r_4 \tag{5.167}$$

The development of the boundary layer solutions and the matching of these solutions to cruise proceeds in almost identical fashion to the oscillatory case. If there is to be matching, the coefficients of the two divergent exponentials (say u_{03} and u_{04}) must be zero. Then, by directly integrating the zero order differential equations for γ_0 and $\Delta\hat{H}_0$ we have

$$\gamma_0 = +a\left[\frac{u_{01}}{r_1} e^{+r_1 R} + \frac{u_{02}}{r_2} e^{+r_2 R}\right] \tag{5.168}$$

$$\Delta\hat{H}_0 = +a\left[\frac{u_{01}}{r_1^2} e^{+r_1 R} + \frac{u_{02}}{r_2^2} e^{+r_2 R}\right] \tag{5.169}$$

We would like to express the optimal control, u_0, as a linear combination of the two elements of the zero order state, γ_0 and $\Delta\hat{H}_0$. To do so we solve the above two equations for $u_{01}e^{r_1 R}$ and $u_{02}e^{r_2 R}$ in terms of γ_0 and $\Delta\hat{H}_0$ and then express their sum, u_0, as

$$u_0 = -\left(\frac{r_1 r_2}{a}\right)\Delta\hat{H}_0 + \left(\frac{r_1 + r_2}{a}\right)\gamma_0 \tag{5.170}$$

This is the optimal control for initial transition. For final transition (away from cruise) the control law is slightly different due to the change in sign associated with the state equations in the right side boundary layer. The control law for transition from cruise is

$$u_0 = -\left(\frac{r_1 r_2}{a}\right)\Delta\hat{H}_0 - \left(\frac{r_1 + r_2}{a}\right)\gamma_0 \tag{5.171}$$

In order to derive an analytic expression for the cost in fuel associated with transition we consider the state equations in the form

$$\gamma_0 = \gamma_{01} e^{r_1 R} + \gamma_{02} e^{r_2 R} \tag{5.172}$$

$$\Delta\hat{H}_0 = \Delta\hat{H}_{01} e^{r_1 R} + \Delta\hat{H}_{02} e^{r_2 R} \tag{5.173}$$

Initial conditions are defined as

$$\gamma_{0i} = \gamma_{01} + \gamma_{02} \tag{5.174}$$

$$\Delta\hat{H}_{0i} = \Delta\hat{H}_{01} + \Delta\hat{H}_{02} \tag{5.175}$$

Now from a comparison of equations (5.168) and (5.172) we can write for equation (5.174)

$$\gamma_{0i} = \frac{a}{r_1} u_{01} + \frac{a}{r_2} u_{02} \tag{5.176}$$

Similarly, equations (5.169) and (5.173) allow us to write for equation (5.175)

$$\Delta\hat{H}_{0i} = \frac{a}{r_1^2} u_{01} + \frac{a}{r_2^2} u_{02} \tag{5.177}$$

The right side terms of equation (5.176) will change sign for descent. Now equations (5.176) and (5.177) can be solved to express u_{01} and u_{02} in terms of the initial conditions, γ_{0i} and $\Delta\hat{H}_{0i}$.

$$u_{01} = + \left(\frac{r_1^2}{a}\right) \frac{\left(\gamma_{0i} - r_2 \Delta\hat{H}_{0i}\right)}{\left(r_1 - r_2\right)} \tag{5.178}$$

$$u_{02} = + \left(\frac{r_2^2}{a}\right) \frac{\left(r_1 \Delta\hat{H}_{0i} - \gamma_{0i}\right)}{\left(r_1 - r_2\right)} \tag{5.179}$$

For descent, equation (5.179) is unchanged but equation (5.178) becomes

$$u_{01} = -\left(\frac{r_1^2}{a}\right)\frac{\left(\gamma_{0i}+r_2\Delta\hat{H}_{0i}\right)}{\left(r_1-r_2\right)} \tag{5.180}$$

From a comparison of equation (5.172) with equation (5.168) we have

$$\gamma_{01} = \pm\left(\frac{a}{r_1}\right)u_{01} \tag{5.181}$$

$$\gamma_{02} = \pm\left(\frac{a}{r_2}\right)u_{02} \tag{5.182}$$

with the negative signs applying in final transition (descent), and from equation (5.169) and (5.173) we have

$$\Delta\hat{H}_{01} = \left(\frac{a}{r_1^2}\right)u_{01} \tag{5.183}$$

$$\Delta\hat{H}_{02} = \left(\frac{a}{r_2^2}\right)u_{02} \tag{5.184}$$

The fuel cost associated with the initial transition is expressed by the equation

$$M_1(R_L) = \gamma_{1b}R_L + \frac{\gamma_{1b}}{a}\gamma_0 + \frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\Delta\hat{H}_0 + \int_0^{R_L}\left\{\frac{\gamma_{1b}}{2}\left[\gamma_0^2+\left(u_0+\Delta\hat{H}_0\right)^2\right]\right.$$
$$\left. + 4x^3y\sqrt{C_{D_o}K}\left[\left(u_0+\Delta\hat{H}_0\right)^2+\frac{\gamma_0^2}{4x^2C_{D_o}K}+\frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\left(u_0+\Delta\hat{H}_0\right)\right]\right\}dR_L \tag{5.185}$$

For descent

$$M_1(R_R) = -e^{-\gamma_{1b}R_R}\left[\gamma_{1b}R_R+\frac{\gamma_{1b}}{a}\gamma_0+\frac{\gamma_{1b}}{2x\sqrt{C_{D_o}K}}\Delta\tilde{H}_0+\int_0^{R_R}\left\{\frac{\gamma_{1b}}{2}\left[\gamma_0^2+\left(u_0+\Delta\tilde{H}_0\right)^2\right]\right.\right.$$
$$\left.\left. + 4x^3y\sqrt{C_{D_o}K}\left[\left(u_0+\Delta\tilde{H}_0\right)^2+\frac{\gamma_0^2}{4x^2C_{D_o}K}+\frac{\gamma_0}{2x\sqrt{C_{D_o}K}}\left(u_0+\Delta\tilde{H}_0\right)\right]\right\}dR_R\right] \tag{5.186}$$

To evaluate these equations we need the following expressions

$$\gamma_0^2 = \gamma_{01}^2 e^{2r_1R} + \gamma_{02}^2 e^{2r_2R} + 2\gamma_{01}\gamma_{02}e^{(r_1+r_2)R} \tag{5.187}$$

$$u_0^2 = u_{01}^2 e^{2r_1R} + u_{02}^2 e^{2r_2R} + 2u_{01}u_{02} e^{(r_1+r_2)R} \tag{5.188}$$

$$\Delta\hat{H}_0^2 = \Delta\hat{H}_{01}^2 e^{2r_1R} + \Delta\hat{H}_{02}^2 e^{2r_2R} + 2\Delta\hat{H}_{01}\Delta\hat{H}_{02} e^{(r_1+r_2)R} \tag{5.189}$$

$$2u_0\Delta\hat{H}_0 = 2u_{01}\Delta\hat{H}_{01} e^{2r_1R} + 2u_{02}\Delta\hat{H}_{02} e^{2r_2R}$$

$$+ 2(u_{01}\Delta\hat{H}_{02} + u_{02}\Delta\hat{H}_{01}) e^{(r_1+r_2)R} \tag{5.190}$$

$$\gamma_0 u_0 = u_{01}\gamma_{01} e^{2r_1R} + u_{02}\gamma_{02} e^{2r_2R} + (u_{01}\gamma_{02} + u_{02}\gamma_{01}) e^{(r_1+r_2)R} \tag{5.191}$$

$$\gamma_0\Delta\hat{H}_0 = \Delta\hat{H}_{01}\gamma_{01} e^{2r_1R} + \gamma_{02}\Delta\hat{H}_{02} e^{2r_2R} + (\gamma_{01}\Delta\hat{H}_{02} + \gamma_{02}\Delta\hat{H}_{01}) e^{(r_1+r_2)R} \tag{5.192}$$

Now define the following constants

$$\hat{a}_0 = \Big\{\left(u_{01} + \Delta\hat{H}_{01}\right)\left(u_{02} + \Delta\hat{H}_{02}\right)\left(\gamma_{1b} + 8x^3y\sqrt{C_{D_o}K}\right) + \gamma_{01}\gamma_{02}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}\right)$$

$$+4x^2y\left[\left(u_{01} + \Delta\hat{H}_{01}\right)\gamma_{02} + \left(u_{02} + \Delta\hat{H}_{02}\right)\gamma_{01}\right]\Big\} \tag{5.193}$$

$$\hat{a}_1 = \Big\{\left(u_{01} + \Delta\hat{H}_{01}\right)^2\left(\frac{\gamma_{1b}}{2} + 4x^3y\sqrt{C_{D_o}K}\right) + \gamma_{01}\Big[\left(\frac{\gamma_{1b}}{2} + \frac{xy}{\sqrt{C_{D_o}K}}\right)\gamma_{01}$$

$$+4x^2y\left(u_{01} + \Delta\hat{H}_{01}\right)\Big]\Big\} \tag{5.194}$$

$$\hat{a}_2 = \Big\{\left(u_{02} + \Delta\hat{H}_{02}\right)^2\left(\frac{\gamma_{1b}}{2} + 4x^3y\sqrt{C_{D_o}K}\right) + \gamma_{02}\Big[\left(\frac{\gamma_{1b}}{2} + \frac{xy}{\sqrt{C_{D_o}K}}\right)\gamma_{02}$$

$$+4x^2y\left(u_{02} + \Delta\hat{H}_{02}\right)\Big]\Big\} \tag{5.195}$$

The ascent cost can now be written as

$$M_1 = \gamma_{1b} R_L + \frac{\gamma_{1b}}{a} \gamma_0 + \frac{\gamma_{1b}}{2\sqrt{C_{D_o} K x}} \Delta\hat{H}_0 + \frac{\hat{a}_0}{r_1+r_2} e^{(r_1+r_2)R_L}$$

$$+ \frac{\hat{a}_1}{2r_1} e^{2r_1 R_L} + \frac{\hat{a}_2}{2r_2} e^{2r_2 R_L} + c_i \tag{5.196}$$

where

$$c_i = -\frac{\gamma_{1b}}{a} \gamma_{0i} - \frac{\gamma_{1b}}{2\sqrt{C_{D_o} K x}} \Delta\hat{H}_{0i} - \frac{\hat{a}_0}{r_1+r_2} - \frac{\hat{a}_1}{2r_1} - \frac{\hat{a}_2}{2r_2} \tag{5.197}$$

The corresponding cost for descent transition is

$$M_1 = e^{-\gamma_{1b} R_f} \Big[\gamma_{1b} R_R + \frac{\gamma_{1b}}{a} \gamma_0 + \frac{\gamma_{1b}}{2\sqrt{C_{D_o} K x}} \Delta\tilde{H}_0$$

$$+ \frac{\tilde{a}_0}{r_1+r_2} e^{(r_1+r_2)R_R} + \frac{\tilde{a}_1}{2r_1} e^{2r_1 R_R} + \frac{\tilde{a}_2}{2r_2} e^{2r_2 R_R} \Big] + c_f \tag{5.198}$$

$$c_f = e^{-\gamma_{1b} R_f} \Big[\frac{\gamma_{1b}}{a} \gamma_{0f} + \frac{\gamma_{1b}}{2\sqrt{C_{D_o} K x}} \Delta\tilde{H}_{0f} + \frac{\tilde{a}_0}{r_1+r_2} + \frac{\tilde{a}_1}{2r_1} + \frac{\tilde{a}_2}{2r_2} \Big] \tag{5.199}$$

5.8 Solution Without a Cruise Section

If the specified final value of mass is sufficiently large the solution will not achieve cruise altitude. The end conditions on altitude and flight path angle may even lie within the initial boundary layer. For slightly smaller specified values of final mass there will be two boundary layers but they will coalesce, and no cruise section will exist. In the absence of a cruise section matching is not required. As a result it is necessary to use the most general form of the optimal control in evaluating a solution. Those constants associated with the positive exponential, which were required to be zero in order to achieve matching, may now be non-zero.

The optimal control, flight path angle, and altitude difference are now expressed as

$$u_0 = e^{-\zeta\omega_n R_L}(u_{01}\cos\omega R_L + u_{02}\sin\omega R_L) + e^{+\zeta\omega_n R_L}(u_{03}\cos\omega R_L + u_{04}\sin\omega R_L) \tag{5.200}$$

$$\gamma_0 = e^{-\zeta\omega_n R_L}(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L) + e^{+\zeta\omega_n R_L}(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L) \tag{5.201}$$

$$\Delta\hat{H}_0 = e^{-\zeta\omega_n R_L}(\Delta\hat{H}_{01}\cos\omega R_L + \Delta\hat{H}_{02}\sin\omega R_L) + e^{+\zeta\omega_n R_L}(\Delta\hat{H}_{03}\cos\omega R_L + \Delta\hat{H}_{04}\sin\omega R_L) \tag{5.202}$$

The twelve unknowns in these equations can all be identified in terms of the specified initial and final values of the state variables. First, by differentiating the above expressions for $\Delta\hat{H}_0$ and γ_0 with respect to R_L and equating the results to γ_0 and au_0 respectively we have

$$\begin{aligned}\frac{d\Delta\hat{H}_0}{dR_L} = \gamma_0 = e^{-\zeta\omega_n R_L}\Big[&(\omega\Delta\hat{H}_{02} - \zeta\omega_n\Delta\hat{H}_{01})\cos\omega R_L \\ &- (\omega\Delta\hat{H}_{01} + \zeta\omega_n\Delta\hat{H}_{02})\sin\omega R_L\Big] \\ &+ e^{+\zeta\omega_n R_L}\Big[(\omega\Delta\hat{H}_{04} + \zeta\omega_n\Delta\hat{H}_{03})\cos\omega R_L \\ &+ (-\omega\Delta\hat{H}_{03} + \zeta\omega_n\Delta\hat{H}_{04})\sin\omega R_L\Big]\end{aligned} \tag{5.203}$$

$$\frac{d\gamma_0}{dR_L} = au_0 = e^{-\zeta\omega_n R_L}\Big[(\omega\gamma_{02} - \zeta\omega_n\gamma_{01})\cos\omega R_L$$

$$- (\omega\gamma_{01} + \zeta\omega_n\gamma_{02})\sin\omega R_L\Big]$$

$$+ e^{+\zeta\omega_n R_L}\Big[(\omega\gamma_{04} + \zeta\omega_n\gamma_{03})\cos\omega R_L$$

$$+ (-\omega\gamma_{03} + \zeta\omega_n\gamma_{04})\sin\omega R_L\Big] \tag{5.204}$$

Now by comparing terms from these differentiated expressions with the original equations for $\underline{\gamma}_0$ and $\underline{u}_0$ we have

$$\underline{\gamma}_0 = A\,\Delta\hat{\underline{H}}_0 \tag{5.205}$$

$$\Delta\hat{\underline{H}}_0 = A^{-1}\,\underline{\gamma}_0 \tag{5.206}$$

$$\underline{u}_0 = \frac{1}{a} A\,\underline{\gamma}_0 \tag{5.207}$$

where the matrix A is defined as

$$A = \begin{vmatrix} -\zeta\omega_n & \omega & 0 & 0 \\ -\omega & -\zeta\omega_n & 0 & 0 \\ 0 & 0 & \zeta\omega_n & \omega \\ 0 & 0 & -\omega & \zeta\omega_n \end{vmatrix} \tag{5.208}$$

and

$$A^{-1} = \begin{vmatrix} -\frac{\zeta}{\omega_n} & -\frac{\omega}{\omega_n^2} & 0 & 0 \\ \frac{\omega}{\omega_n^2} & -\frac{\zeta}{\omega_n} & 0 & 0 \\ 0 & 0 & \frac{\zeta}{\omega_n} & -\frac{\omega}{\omega_n^2} \\ 0 & 0 & \frac{\omega}{\omega_n^2} & \frac{\zeta}{\omega_n} \end{vmatrix} = \frac{1}{\omega_n^2} A^T \tag{5.209}$$

We now have equations to express the eight elements of the vectors $\underline{u}_0$ and $\Delta\hat{\underline{H}}_0$ in terms of the four elements of the vector $\underline{\gamma}_0$. Two more equations are available from the initial conditions

$$\gamma_i = \gamma_{01} + \gamma_{03} \tag{5.210}$$

$$\Delta\hat{H}_i = \Delta\hat{H}_{01} + \Delta\hat{H}_{03} \tag{5.211}$$

and two from the final conditions

$$\gamma_f = e^{-\zeta\omega_n R_f}(\gamma_{01}\cos\omega R_f + \gamma_{02}\sin\omega R_f) + e^{+\zeta\omega_n R_f}(\gamma_{03}\cos\omega R_f + \gamma_{04}\sin\omega R_f) \tag{5.212}$$

$$\Delta\hat{H}_f = e^{-\zeta\omega_n R_f}(\Delta\hat{H}_{01}\cos\omega R_f + \Delta\hat{H}_{02}\sin\omega R_f) + e^{+\zeta\omega_n R_f}(\Delta\hat{H}_{03}\cos\omega R_f + \Delta\hat{H}_{04}\sin\omega R_f) \tag{5.213}$$

These final conditions introduce another unknown, R_f, and hence require another equation. The required equation is that of M_f, the specified final value of M

$$M_f = \left\{\int_0^{R_L}\left[\gamma_{1b}\left(1+u_0+\frac{\gamma_0}{2\sqrt{C_{D_o}Kx}}\right)+\frac{\gamma_{1b}}{2}\left(\gamma_0^{\,2}+\left[u_0+\Delta\hat{H}_0\right]^2\right)\right.\right.$$
$$\left.\left.+\,4x^3y\sqrt{C_{D_o}K}\left(u_0+\Delta\hat{H}_0+\frac{\gamma_0}{2\sqrt{C_{D_o}Kx}}\right)^2\right]dR_L\right\}_{R_L=R_f} \tag{5.214}$$

This expression can be integrated directly. Some of the tedium is removed by referring to the manner in which this expression was integrated in the initial and final boundary layers for matching with cruise. First, it is useful to compare the sum of u_0 and $\Delta\hat{H}_0$ with γ_0

$$u_0 + \Delta\hat{H}_0 = e^{-\zeta\omega_n R_L}\left[(u_{01} + \Delta\hat{H}_{01})\cos\omega R_L + (u_{02} + \Delta\hat{H}_{02})\sin\omega R_L\right]$$

$$+ e^{+\zeta\omega_n R_L}\left[(u_{03} + \Delta\hat{H}_{03})\cos\omega R_L + (u_{04} + \Delta\hat{H}_{04})\sin\omega R_L\right] \qquad (5.215)$$

$$u_0 + \Delta\hat{H}_0 = e^{-\zeta\omega_n R_L}\left[\left(-\frac{\zeta}{\omega_n}\gamma_{01} + \frac{\omega}{\omega_n^2}\gamma_{02} - \frac{\zeta}{\omega_n}\gamma_{01} - \frac{\omega}{\omega_n^2}\gamma_{02}\right)\cos\omega R_L\right.$$

$$\left. + \left(-\frac{\zeta}{\omega_n}\gamma_{02} - \frac{\omega}{\omega_n^2}\gamma_{01} + \frac{\omega}{\omega_n^2}\gamma_{01} - \frac{\zeta}{\omega_n}\gamma_{02}\right)\sin\omega R_L\right]$$

$$+ e^{+\zeta\omega_n R_L}\left[\left(\frac{\zeta}{\omega_n}\gamma_{03} + \frac{\omega}{\omega_n^2}\gamma_{04} + \frac{\zeta}{\omega_n}\gamma_{03} - \frac{\omega}{\omega_n^2}\gamma_{04}\right)\cos\omega R_L\right.$$

$$\left. + \left(\frac{\zeta}{\omega_n}\gamma_{04} - \frac{\omega}{\omega_n^2}\gamma_{03} + \frac{\omega}{\omega_n^2}\gamma_{03} + \frac{\zeta}{\omega_n}\gamma_{04}\right)\sin\omega R_L\right] \qquad (5.216)$$

$$u_0 + \Delta\hat{H}_0 = -2\frac{\zeta}{\omega_n}\left\{e^{-\zeta\omega_n R_L}(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)\right.$$

$$\left. + e^{-\zeta\omega_n R_L}(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)\right\} \qquad (5.217)$$

Now we can evaluate the combinations of products of the terms γ_0 and $u_0 + \Delta\hat{H}_0$ that appear in the integrand in terms of the components of $\underline{\gamma}_0$. We have

$$\gamma_0^2 = e^{-2\zeta\omega_n R_L}(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)^2$$

$$+ e^{+2\zeta\omega_n R_L}(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)^2$$

$$+ 2(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L) \qquad (5.218)$$

$$(u_0+\Delta\hat{H}_0)^2 = 4\frac{\zeta^2}{\omega_n^2}\Big\{e^{-2\zeta\omega_n R_L}(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)^2$$

$$+ e^{+2\zeta\omega_n R_L}(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)^2$$

$$- 2(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)\Big\} \qquad (5.219)$$

$$\gamma_0(u_0+\Delta\hat{H}_0) = 2\frac{\zeta}{\omega_n}\Big\{e^{-2\zeta\omega_n R_L}(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)^2$$

$$- e^{+2\zeta\omega_n R_L}(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)^2\Big\} \qquad (5.220)$$

Combining terms we can now express M as

$$M = \int_0^{R_L}\Big\{\gamma_{1b}\Big(1+u_0+\frac{\gamma_0}{2\sqrt{C_{D_o}K}\,x}\Big)$$

$$+ e^{-2\zeta\omega_n R_L}(\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)^2\Big[\frac{\gamma_{1b}}{2}\Big(1+4\frac{\zeta^2}{\omega_n^2}\Big)$$

$$+ \frac{xy}{\sqrt{C_{D_o}K}} + 8x^2y\frac{\zeta}{\omega_n}\Big(1+2x\sqrt{C_{D_o}K}\frac{\zeta}{\omega_n}\Big)\Big]$$

$$+ e^{+2\zeta\omega_n R_L}(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)^2\Big[\frac{\gamma_{1b}}{2}\Big(1+4\frac{\zeta^2}{\omega_n^2}\Big)$$

$$+ \frac{xy}{\sqrt{C_{D_o}K}} - 8x^2y\frac{\zeta}{\omega_n}\Big(1-2x\sqrt{C_{D_o}K}\frac{\zeta}{\omega_n}\Big)\Big]$$

$$+ (\gamma_{01}\cos\omega R_L + \gamma_{02}\sin\omega R_L)(\gamma_{03}\cos\omega R_L + \gamma_{04}\sin\omega R_L)\Big[\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}}$$

$$- 8\frac{\zeta^2}{\omega_n^2}\Big(\frac{\gamma_{1b}}{2} + 4x^3y\sqrt{C_{D_o}K}\Big)\Big]\Big\}\,dR_L \qquad (5.221)$$

The following integrals can now be used, in the appropriate combinations, to express the integrated form of M

$$\int_0^{R_L} \gamma_{1b}\left(1+u_0+\frac{\gamma_0}{2\sqrt{C_{D_o}K}\,x}\right)dR_L = \gamma_{1b}R_L + \frac{\gamma_{1b}}{a}\gamma_0\Big|_{R_L=0}^{R_L=R_L} + \frac{\gamma_{1b}\omega\hat{H}_0}{2\sqrt{C_{D_o}K}\,x}\Big|_{R_L=0}^{R_L=R_L} \tag{5.222}$$

$$\int_0^{R_L} e^{\pm 2\zeta\omega_n R_L}\cos^2\omega R_L\, dR_L = \frac{e^{\pm 2\zeta\omega_n R_L}}{4\omega_n^2}\left[\pm\frac{\omega_n}{\zeta} + \omega\sin 2\omega R_L \pm \zeta\omega_n\cos 2\omega R_L\right] \tag{5.223}$$

$$\int_0^{R_L} e^{\pm 2\zeta\omega_n R_L}\sin^2\omega R_L\, dR_L = \frac{e^{\pm 2\zeta\omega_n R_L}}{4\omega_n^2}\left[\pm\frac{\omega_n}{\zeta} - \omega\sin 2\omega R_L \pm \zeta\omega_n\cos 2\omega R_L\right] \tag{5.224}$$

$$\int_0^{R_L} e^{\pm 2\zeta\omega_n R_L}\sin\omega R_L\cos\omega R_L\, dR_L = \frac{e^{\pm 2\zeta\omega_n R_L}}{4\omega_n^2}\left[\pm\frac{\omega_n}{\zeta} \pm \zeta\omega_n\sin 2\omega R_L - \omega\cos 2\omega R_L\right] \tag{5.225}$$

$$\int_0^{R_L}\cos^2\omega R_L\, dR_L = \frac{R_L}{2} + \frac{\sin 2\omega R_L}{4\omega} \tag{5.226}$$

$$\int_0^{R_L}\sin^2\omega R_L\, dR_L = \frac{R_L}{2} - \frac{\sin 2\omega R_L}{4\omega} \tag{5.227}$$

$$\int_0^{R_L}\sin\omega R_L\cos\omega R_L\, dR_L = -\frac{1}{4\omega}\cos 2\omega R_L \tag{5.228}$$

The two coefficients have previously been simplified as follows

$$\frac{\gamma_{1b}}{2}\left(1+4\frac{\zeta^2}{\omega_n^2}\right) + \frac{xy}{\sqrt{C_{D_o}K}} \pm 8x^2y\frac{\zeta}{\omega_n}\left(1\pm 2x\sqrt{C_{D_o}K}\,\frac{\zeta}{\omega_n}\right) = \gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o}K}} \pm 8x^2y\frac{\zeta}{\omega_n} \tag{5.229}$$

Finally we have our equation for the specified value of M_f in terms of the four components of $\underline{\gamma}_0$ and R_{Lf}

$$
\begin{aligned}
M_f = {} & \gamma_{1b} R_{Lf} + \frac{\gamma_{1b}}{\omega_n^2}(\gamma_{0f} - \gamma_{0i}) + \frac{\gamma_{1b}}{2\sqrt{C_{D_o} K}\, x}(\angle\hat{H}_{0f} - \Delta\hat{H}_{0i}) \\
& + \frac{e^{-2\zeta\omega_n R_{Lf}}}{4\omega_n^2}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o} K}} - 8x^2 y \frac{\zeta}{\omega_n}\right)\left\{-\frac{\omega_n}{\zeta}\left(\gamma_{01}^2 + \gamma_{02}^2\right) + \left[\omega\left(\gamma_{01}^2 - \gamma_{02}^2\right)\right.\right. \\
& \left.\left. - 2\zeta\omega_n\gamma_{01}\gamma_{02}\right]\sin 2\omega R_{Lf} - \left[\zeta\omega_n\left(\gamma_{01}^2 - \gamma_{02}^2\right) + 2\omega\gamma_{01}\gamma_{02}\right]\cos 2\omega R_{Lf}\right\} \\
& + \frac{e^{+2\zeta\omega_n R_{Lf}}}{4\omega_n^2}\left(\gamma_{1b} + \frac{2xy}{\sqrt{C_{D_o} K}} + 8x^2 y \frac{\zeta}{\omega_n}\right)\left\{\frac{\omega_n}{\zeta}\left(\gamma_{03}^2 + \gamma_{04}^2\right)\right. \\
& \left. + \left[\omega\left(\gamma_{03}^2 - \gamma_{04}^2\right) + 2\zeta\omega_n\gamma_{03}\gamma_{04}\right]\sin 2\omega R_{Lf} + \left[\omega_n\zeta\left(\gamma_{03}^2 - \gamma_{04}^2\right) - 2\omega\gamma_{03}\gamma_{04}\right]\right\} \\
& + \left\{\gamma_{1b}\left(1 - 4\frac{\zeta^2}{\omega_n^2}\right) + \left(\frac{1}{2C_{D_o} K x^2} - 8\frac{\zeta^2}{\omega_n^2}\right)\right\}\left[\frac{R_{Lf}}{2}\left(\gamma_{01}\gamma_{03} + \gamma_{02}\gamma_{04}\right)\right. \\
& \left. + \frac{\sin 2\omega R_{Lf}}{4}(\gamma_{01}\gamma_{03} - \gamma_{02}\gamma_{04}) - \frac{\cos 2\omega R_{Lf}}{4}(\gamma_{01}\gamma_{04} + \gamma_{02}\gamma_{03})\right] \qquad (5.230)
\end{aligned}
$$

Since this equation is highly non-linear, an exact solution is impossible. An iterative solution can be obtained by guessing a value of R_{Lf}, solving for values of the components of $\underline{\gamma}_0$, then evaluating M_f from the above equation and repeating for a new guessed value of R_{Lf} until a satisfactory agreement with the specified value of M_f is achieved.

To evaluate the components of $\underline{\gamma}_0$ we can easily simplify our set of eight equations in eight unknowns ($\underline{u}_0$ elements are not required).

From our equations for $\Delta\hat{H}_{01}$ and γ_{01} we can say

$$
\gamma_{02} = -\frac{\omega_n^2}{\omega}\left(\Delta\hat{H}_{01} + \frac{\zeta}{\omega_n}\gamma_{01}\right) \qquad (5.231)
$$

$$\Delta\hat{H}_{02} = +\frac{1}{\omega}\left(\gamma_{01} + \zeta\omega_n \Delta\hat{H}_{01}\right) \tag{5.232}$$

Similarly

$$\gamma_{04} = -\frac{\omega_n^2}{\omega}\left(\Delta\hat{H}_{03} - \frac{\zeta}{\omega_n}\gamma_{03}\right) \tag{5.233}$$

$$\Delta\hat{H}_{04} = \frac{1}{\omega}\gamma_{03} + \frac{\zeta\omega_n}{\omega}\Delta\hat{H}_{03} \tag{5.234}$$

If we eliminate γ_{03} and $\Delta\hat{H}_{03}$ from the latter two equations by means of our initial condition equations

$$\gamma_{03} = \gamma_{0i} - \gamma_{01} \tag{5.235}$$

$$\Delta\hat{H}_{03} = \Delta\hat{H}_{0i} - \Delta\hat{H}_{01} \tag{5.236}$$

and then substitute for γ_{02}, γ_{04}, $\Delta\hat{H}_{02}$ and $\Delta\hat{H}_{04}$ in our end-condition equations we have two equations in two unknowns

$$\begin{aligned}\gamma_{0f} - e^{+\zeta\omega_n R_{Lf}}&\left[\gamma_{01}\cos\omega R_{Lf} + \frac{1}{\omega}\left(\zeta\omega_n\gamma_{01} - \omega_n^2\Delta\hat{H}_{01}\right)\sin\omega R_{Lf}\right]\\ &= -2\left[\cos\omega R_{Lf}\sinh\zeta\omega_n R_{Lf} + \frac{\zeta\omega_n}{\omega}\sin\omega R_{Lf}\cosh\zeta\omega_n R_{Lf}\right]\gamma_{01}\\ &+ 2\left(\frac{\omega_n^2}{\omega}\sin\omega R_{Lf}\sinh\zeta\omega_n R_{Lf}\right)\Delta\hat{H}_{01}\end{aligned} \tag{5.237}$$

$$\begin{aligned}\Delta\hat{H}_{0f} - e^{+\zeta\omega_n R_{Lf}}&\left[\Delta\hat{H}_{0i}\cos\omega R_{Lf} + \frac{1}{\omega}\left(\gamma_{0i} - \zeta\omega_n\Delta\hat{H}_{0i}\right)\sin\omega R_{Lf}\right]\\ &= -2\left[\cos\omega R_{Lf}\sinh\zeta\omega_n R_{Lf} - \frac{\zeta\omega_n}{\omega}\sin\omega R_{Lf}\cosh\zeta\omega_n R_{Lf}\right]\Delta\hat{H}_{01}\\ &- \frac{2}{\omega}\left(\sin\omega R_{Lf}\sinh\zeta\omega_n R_{Lf}\right)\gamma_{01}\end{aligned} \tag{5.238}$$

Now the solution proceeds as follows

1. Guess a value of R_{Lf}
2. Solve equations (5.237) and (5.238) for γ_{01} and $\Delta\hat{H}_{01}$
3. Solve equations (5.235) and (5.236) for γ_{03} and $\Delta\hat{H}_{03}$
4. Solve equations (5.231) and (5.233) for γ_{02} and γ_{04}
5. Solve equation (5.230) for M_f and subtract the specified value of M_f. Plot the difference against R_{Lf}. If the difference is not sufficiently close to zero, return to step 1, guess a new value of R_{Lf} and repeat until the difference at step 5 is sufficiently close to zero. Since the sign of dM/dR_L must always be positive, the process of locating the zero crossing on the plot should not involve many trials.

So far in this section we have considered solutions in which both the positively and negatively damped exponential terms make non-negligible contributions over the entire trajectory. If the specified final value of M_f is steadily increased, the effects of the negative exponentials on the final state and of the positive exponentials on the initial state will approach zero. There results

$$\gamma_{0i} = \gamma_{01} \tag{5.239}$$

$$\Delta\hat{H}_{0i} = \Delta\hat{H}_{01} \tag{5.240}$$

$$\gamma_{0f} = e^{+\zeta\omega_n R_{Lf}}\left[\gamma_{03}\left(\cos\omega R_{Lf} + \frac{\zeta\omega_n}{\omega}\sin\omega R_{Lf}\right) - \Delta\hat{H}_{03}\left(\frac{\omega_n^2}{\omega}\sin\omega R_{Lf}\right)\right] \tag{5.241}$$

$$\Delta\hat{H}_{0f} = e^{+\zeta\omega_n R_{Lf}}\left[\Delta\hat{H}_{03}\left(\cos\omega R_{Lf} + \frac{\zeta\omega_n}{\omega}\sin\omega R_{Lf}\right) + \frac{\gamma_{03}}{\omega}\sin\omega R_{Lf}\right] \tag{5.242}$$

The term with the negative exponential factor now no longer appears in the equation for M_f but otherwise that equation is unchanged. The solution proceeds in the same way as before, but the equations are somewhat simpler.

The equations of the state and optimal control now have the following form in the vicinity of the initial point

$$u_0 = e^{-\zeta\omega_n R_L} (u_{01} \cos \omega R_L + u_{02} \sin \omega R_L) \tag{5.243}$$

$$\gamma_0 = e^{-\zeta\omega_n R_L} (\gamma_{01} \cos \omega R_L + \gamma_{02} \sin \omega R_L) \tag{5.244}$$

$$\Delta\hat{H}_0 = e^{-\zeta\omega_n R_L} (\Delta\hat{H}_{01} \cos \omega R_L + \Delta\hat{H}_{02} \sin \omega R_L) \tag{5.245}$$

From the general expression for the sum of u_0 and $\Delta\hat{H}_0$ we see that the control law in the vicinity of the initial point is

$$u_0 = - \Delta\hat{H}_0 - \frac{\zeta}{\omega_n} \gamma_0 \tag{5.246}$$

which is identical to the control law in the left side boundary layer in the matched asymptotic problem. Similarly, in the vicinity of the final point we have

$$u_0 = e^{+\zeta\omega_n R_L} (u_{03} \cos \omega R_L + u_{04} \sin \omega R_L) \tag{5.247}$$

$$\gamma_0 = e^{+\zeta\omega_n R_L} (\gamma_{03} \cos \omega R_L + \gamma_{04} \sin \omega R_L) \tag{5.248}$$

$$\Delta\hat{H}_0 = e^{+\zeta\omega_n R_L} (\Delta\hat{H}_{03} \cos \omega R_L + \Delta\hat{H}_{04} \sin \omega R_L) \tag{5.249}$$

Referring again to the general expression for the sum of u_0 and $\Delta\hat{H}_0$ we find the control law to be

$$u_0 = -\Delta\hat{H}_0 + 2\frac{\zeta}{\omega_n}\gamma_0 \tag{5.250}$$

which is identical to the control law in the right side boundary layer in the matched asymptotic case.

The optimal control must pass from its initial form to its final form at some intermediate state. By inspection of the two forms it is obvious that the only requirement for that state is

$$\gamma_0 = 0 \tag{5.251}$$

The altitude difference need not be zero since it has the same sign and coefficient in both forms of the control law. Thus in some short range problems the trajectory may not be required to reach cruise altitude.

It is useful to compare this short range problem to the matched asymptotic problem. If in the short range problem the positive and negative exponential terms interact, it is as if the two boundary layers of the asymptotic problem were so close as to overlap. If the positive and negative exponential terms do not interact, it is as if the two boundary layers matched asymptotically to each other without an intermediate (cruise) section.

Both forms of the short range problem are singular perturbation problems and require the transformation

$$R_L = \frac{R}{\epsilon} \tag{5.252}$$

to become regular. This transformation allowed investigation of an interval along the R-axis of width on the order of ϵ and is consistent with our restriction to short ranges. Without the restriction to short range we must consider the cruise problem, which is of course singular. Thus by further extending the range, or final value of mass, we are led to the matched asymptotic problem with its initial and final boundary layers matching an intermediate solution representing cruise.

5.9 Breguet Range Including Corrections for Transitions

In concluding this chapter we can evaluate an expression for final range. We have already modified the Breguet range equation to account for the gradual increase in altitude and the resulting increase in fuel consumption. We now can incorporate first order corrections to account for the possibilities that the initial and final values of γ_0 and ΔH_0 may not correspond to values on a Breguet cruise.

From our terminal boundary layer matching condition we have

$$1 - e^{-\gamma_{1b} R_{f0}} = M_f \tag{5.253}$$

That is, the specified value of M_f determines the zero order value of the unknown final range

$$R_{f0} = -\frac{1}{\gamma_{1b}} \ln (1 - M_f) \tag{5.254}$$

The first order correction to final range comes from the other matching condition

$$R_{f1} = -\frac{1}{\gamma_{1b}}\left[c_i + \frac{\gamma_{1b}^2 R_{f0}}{2x\sqrt{C_{D_o}K}} - c_f e^{+\gamma_{1b}R_{f0}}\right] \tag{5.255}$$

The value of c_i is defined as a function of initial conditions on γ_0 and $\Delta\hat{H}_0$ by (5.154) for $\zeta \leq 1$ and by (5.197) for $\zeta > 1$. Likewise c_f is defined as a function of final conditions on γ_0 and $\Delta\tilde{H}_0$ by (5.158) for $\zeta \leq 1$ and by (5.199) for $\zeta > 1$. We now have for the final range

$$R_f = R_{f0} + \epsilon R_{f1} \tag{5.256}$$

$$R_f = \frac{-1}{\gamma_{1b}} \ln(1 - M_f) + \frac{\epsilon}{\gamma_{1b}}\left[c_f e^{\gamma_{1b}R_{f0}} - c_i + \frac{\gamma_{1b}}{2\sqrt{C_{D_o}K}\,x} \ln(1 - M_f)\right] \tag{5.257}$$

$$R_f = -\left(\frac{1}{\gamma_{1b}}\right) \ln(1 - M_f)\left(1 - \frac{\epsilon\,\gamma_{1b}}{2\sqrt{C_{D_o}K}\,x}\right) - \frac{\epsilon}{\gamma_{1b}} c_i + \frac{\epsilon}{\gamma_{1b}} c_f e^{\gamma_{1b}R_{f0}} \tag{5.258}$$

The first term is seen to be identical to the Breguet range equation as it was developed in Chapter II: a correction factor appears, decreasing final range, to account for increasing altitude at constant flight path angle in cruise. The second term represents the change in range due to transition from an initial state that is not on the Breguet cruise. It can be related to a non-zero value of mass on the matching Breguet cruise at R_L equals zero. It can be expressed in terms of the initial values of γ_0 and $\Delta\hat{H}_0$ and is zero if they are also zero (i.e., if they are on Breguet cruise).

The third term represents the change in range due to transition to a terminal state that is not on a Breguet cruise. It can be related to the mass difference (at R equals R_f) between specified final mass and final

mass on the matching Breguet cruise. It can be expressed in terms of the final values of γ_0 and $\Delta\tilde{H}_0$ and is zero if they lie on a Breguet cruise.

Finally, by making use of Eq. (4.45), (4.81), (4.83) and (5.253), the range equation (5.258) can be identified with the more familiar form of the Breguet range equation

$$(r_f)_{max} = \left[\frac{v}{SFC_0}\left(\frac{L}{D}\right)_0\right]_{max}\left\{\ln\left(\frac{m_i}{m_f}\right)\left(1-\frac{\epsilon\,\gamma_{lb}}{2\sqrt{C_{D_o}Kx}}\right)-\epsilon c_i+\epsilon c_f\left(\frac{m_i}{m_f}\right)\right\} \qquad (5.259)$$

Both of the range correction terms can be positive, negative or zero, depending on whether the average thrust required for transition is less than, more than or equal to that required on a Breguet cruise over the same range. A positive value of c_i will result in a reduced final range. This is to be expected, since a positive c_i means that the Breguet cruise to which the initial transition matches has a fuel budget that is reduced at R equals zero by $\epsilon\, c_i$ Similarly a negative value of c_f means that the matching Breguet cruise terminates at a value of mass that is less than the specified value, M_f, and hence translates into a loss of range incurred in diverging from cruise to meet the specified final values of γ_0 and $\Delta\tilde{H}_0$. Both of these terms are zero if the initial and final state are on the Breguet cruise and in that case the range equation is identical to that derived from Breguet cruise in Chapter IV.

CHAPTER VI

COMPUTATIONAL RESULTS

6.1 Introduction

This Chapter gives the results of some computational studies of transitions and the costs associated with them. Several non-optimal control policies are described and comparisons are made between them and the optimal policy. The comparisons are made for three aircraft. The Boeing SST is used to represent aircraft that cruise at supersonic speed. The Boeing 707 represents aircraft that cruise at transonic speed. The McDonnell Douglas F-4 is used to study transonic cruise in an aircraft that is capable of supersonic flight. Finally, some comments are made about how the optimal policy might be implemented in a flight control system.

6.2 Alternate Control Policies

It is useful to compare an optimal initial transition trajectory and its cost with a series of trajectories which use non-optimal controls. These transitions assume that the aircraft has already accelerated to its cruise speed and is attempting to reach cruise altitude and level off. The first non-optimal control assumes that the aircraft maintains maximum power setting until it reaches cruise altitude and then assumes its cruise power setting and levels off in zero time. This trajectory will be called the $(C_F)_{max}$ trajectory. In the second, the power setting is first set at its cruise value and the aircraft then eventually levels off at its cruise altitude. This trajectory will be called the $(C_F)_{cruise}$ trajectory. The

third is a constant rate of climb trajectory, identified as γ_{const}. Here again the power setting and attitude are assumed to change instantaneously to cruise values as cruise altitude is reached. The fourth trajectory is the optimal trajectory, u_{opt}.

All of the suboptimal controls represent a constraint on one variable in the equation for constant velocity. The exact equation may be written

$$\left(\frac{C_F}{C_{D_o}} - 1\right) \frac{e^{-2\Delta\hat{H}}}{x^2} - \frac{e^{-\Delta\hat{H}_\gamma}}{x\sqrt{C_{D_o} K}} = (1 + u)^2 \tag{6.1}$$

If C_F is a constant, whether $(C_F)_{max}$ or $(C_F)_{cruise}$ or any other value, we must have for u

$$u = -1 \pm \mathrm{SQRT}\left[\left(\frac{C_F}{C_{D_o}} - 1\right) \frac{e^{-2\Delta\hat{H}}}{x^2} - \frac{e^{-\Delta\hat{H}_\gamma}}{x\sqrt{C_{D_o} K}}\right] \tag{6.2}$$

If γ is a constant, then u must be zero and we have for C_F

$$C_F = C_{D_o}\left[1 + \frac{x^2}{e^{-2\Delta\hat{H}}}\left(1 + \frac{e^{-\Delta\hat{H}_\gamma}}{x\sqrt{C_{D_o} K}}\right)\right] \tag{6.3}$$

If u is u_{opt} for initial transients

$$u = -\Delta\hat{H} - 2\frac{\zeta}{\omega_n}\gamma \tag{6.4}$$

and we must have for C_F

$$C_F = C_{D_o}\left\{1 + \frac{x^2}{e^{-2\Delta\hat{H}}}\left[(1-\Delta\hat{H} - 2\frac{\zeta}{\omega_n}\gamma)^2 + \frac{e^{-2\Delta\hat{H}\gamma}}{x\sqrt{C_{D_o}K}}\right]\right\} \quad (6.5)$$

For the $(C_F)_{const}$ trajectories there is obviously a limit on the values of $\Delta\hat{H}$ and γ such that the argument of the square root (6.2) remains positive. This is easily identified in $\Delta\hat{H}$ - γ space. It corresponds to the condition

$$u = -1 \quad (6.6)$$

and from our definition of u this is equivalent to saying that lift is zero. Obviously, at a particular value of $\Delta\hat{H}$ there is a maximum value of γ at which constant velocity flight can be maintained. That situation corresponds to minimum drag since lift, and hence the induced component of drag, is zero.

It is also true that in constant rate of climb trajectories there will be a maximum climb angle above which constant velocity flight cannot be maintained. This value is determined from considering the maximum value of C_F required for such flights.

The maximum value cannot occur below cruise altitude, or it will be impossible to maintain the constant rate of climb, and the trajectory will become a $(C_F)_{max}$ trajectory. Accordingly, using $(C_F)_{max}$ and cruise altitude ($\Delta\hat{H} = 0$) we have

$$\left(\frac{1}{x^2}\left[\frac{(C_F)_{max} - C_{D_o}}{C_{D_o}}\right] - 1\right)\left(x\sqrt{C_{D_o}K}\right) = \gamma_{max} \quad (6.7)$$

The optimal trajectory will be limited by a locus of points corresponding to a maximum value of C_F with u satisfied by the optimal control law. A similar locus will exist for the minimum value of C_F.

Now using the zero order equations for altitude and flight path angle, (5.21) and (5.22), and the first order equation for mass, (5.23), it is possible to evaluate zero order trajectories and first order costs associated with them for these four control laws. The equations are integrated using a fourth order Runge-Kutta routine. The stopping condition for u_{opt} and $(C_F)_{cruise}$ is the attaining of steady state altitude. For the other two controls the stopping condition is the event of altitude exceeding its cruise value.

These equations are the state equations for the linear-quadratic problem and small initial values are chosen so as not to violate the assumptions inherent in the equations.

6.3 Boeing SST

For our first cost comparison we choose an initial altitude of

$$\Delta\hat{H}_{0i} = -0.30 \tag{6.8}$$

For various values of initial flight path angle up to the respective maxima, trajectories have been calculated for the four controls and the resulting costs plotted in Fig. 6.1. The airplane used in these calculations was the Boeing SST (Appendix A) in supersonic cruise. The variable plotted on the vertical scale of Fig. 6.1 is the difference in the first order mass term between the indicated transition climb trajectory and a pure Breguet cruise of the same range, that is $M_1 - \gamma_{1b}\dot{R}_L$. It is based on equation (5.23).

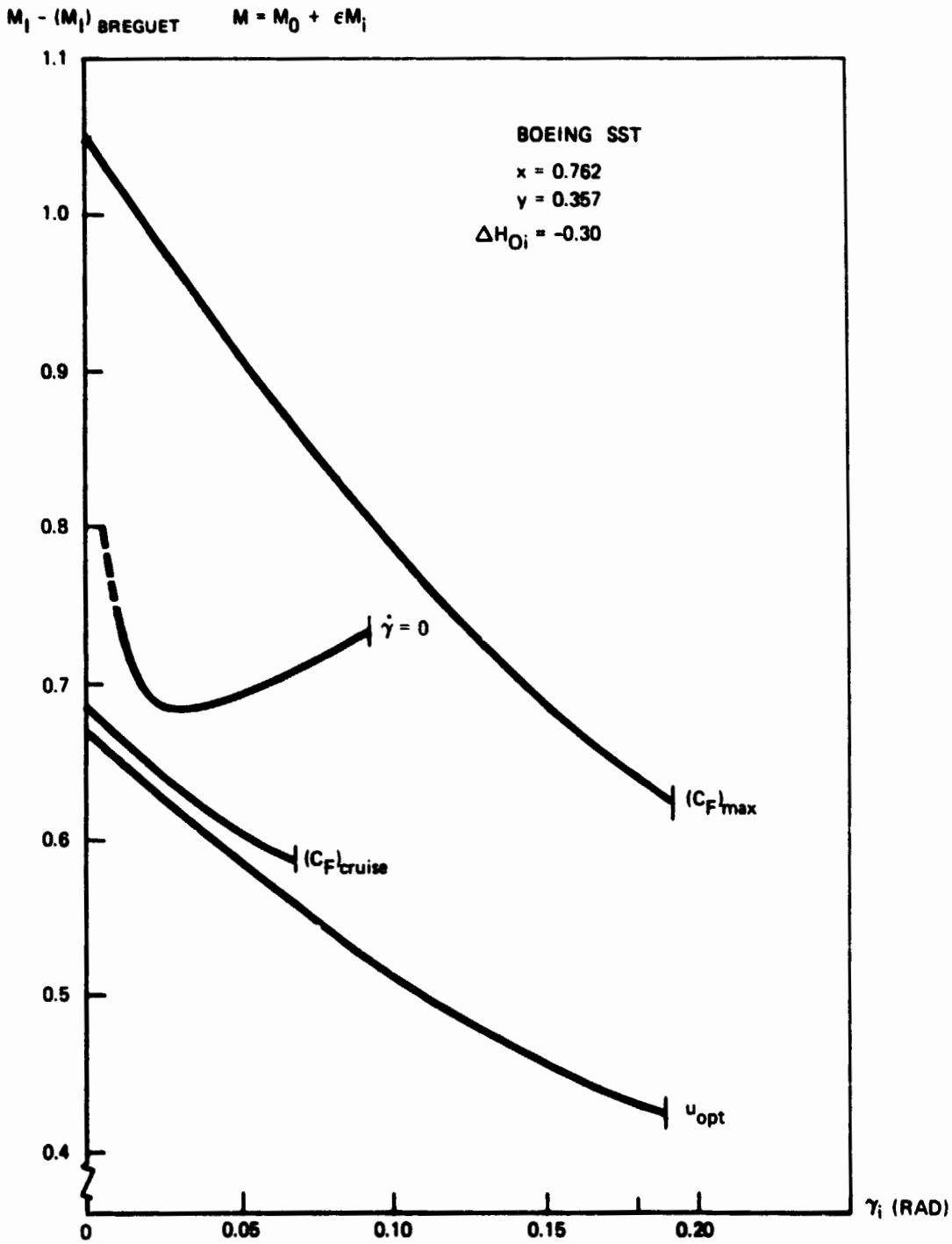

Figure 6.1. Initial Transition Cost for Various Controls

It is seen that $(C_F)_{cruise}$ is a good approximation to minimum cost but its range of feasible initial flight path angles is severely restricted. It can be concluded that if $(C_F)_{cruise}$ were identical to $(C_F)_{max}$ then these two trajectories would be identical to each other and their cost would be virtually the same as that for the u_{opt} trajectory.

In a narrow range of angles the γ_{const} trajectories also compare well with minimum cost. This comparison worsens as γ_i approaches its maximum feasible value and worsens rapidly as γ_i approaches zero.

For realistic attitudes associated with climb to cruise, the constant velocity transition must be either u_{opt} or $(C_F)_{max}$. In general the recommended procedure for a pilot to follow in flying his transition to cruise is to accelerate and climb at $(C_F)_{max}$ until cruise speed is reached, then to climb at $(C_F)_{max}$ and constant speed until cruise altitude is reached, and then to level off at cruise altitude and speed in an unspecified manner [22, 23, 24]. It is seen from Fig. 6.1 that in this comparison the cost improvement in terms of M_1 ranges from 0.38 to 0.20. To convert this number to a weight it is necessary to multiply by ϵ and by the initial cruise weight. For the SST this converts to a weight of from 540 to 285 pounds of fuel.

Figure 6.2 is a comparison of the zero order trajectories for the four controls in R - $\Delta\hat{H}$ space. All start from an initial flight path angle of -0.05 radians and an initial $\Delta\hat{H}_0$ of -0.30 or -6,240 feet. The optimal trajectory overshoots in $\Delta\hat{H}_0$ by 0.00127 which is equivalent to 26 feet. Figure 6.3 shows the same trajectories in $\Delta\hat{H}_0$ - γ space. This is essentially a phase plane, and optimal trajectories spiral into the origin.

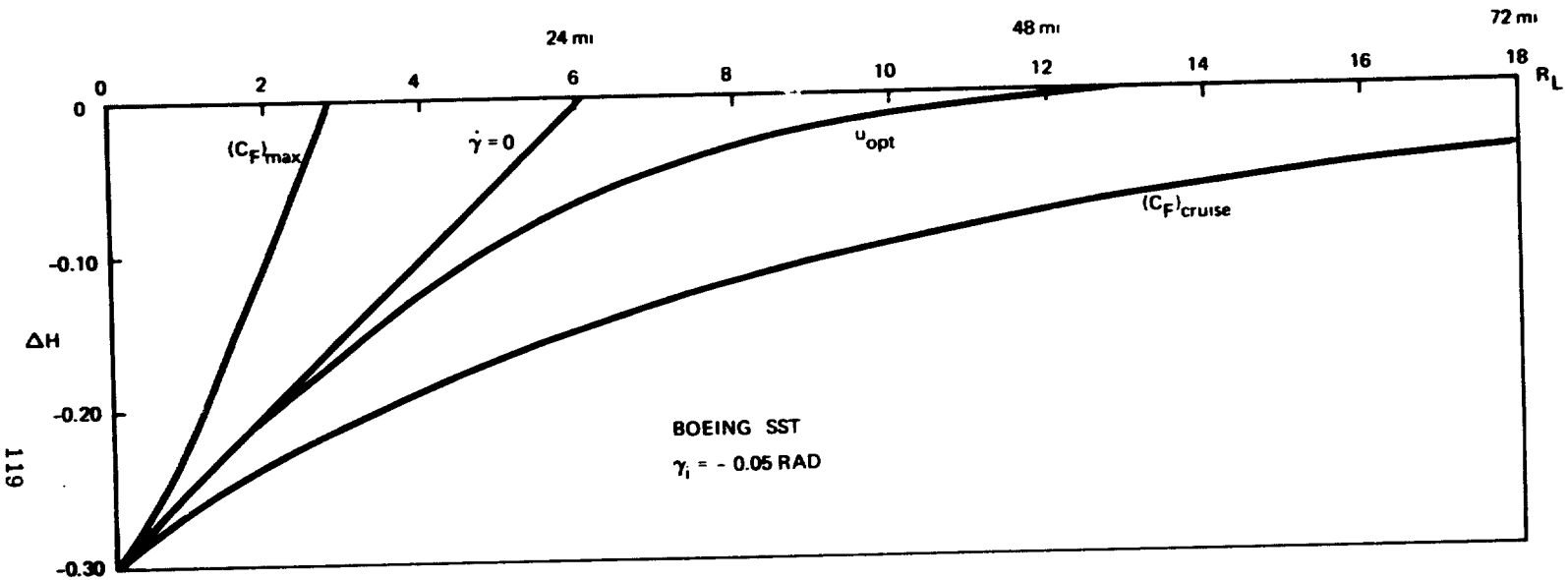

Figure 6.2. Altitude Transient for Various Controls

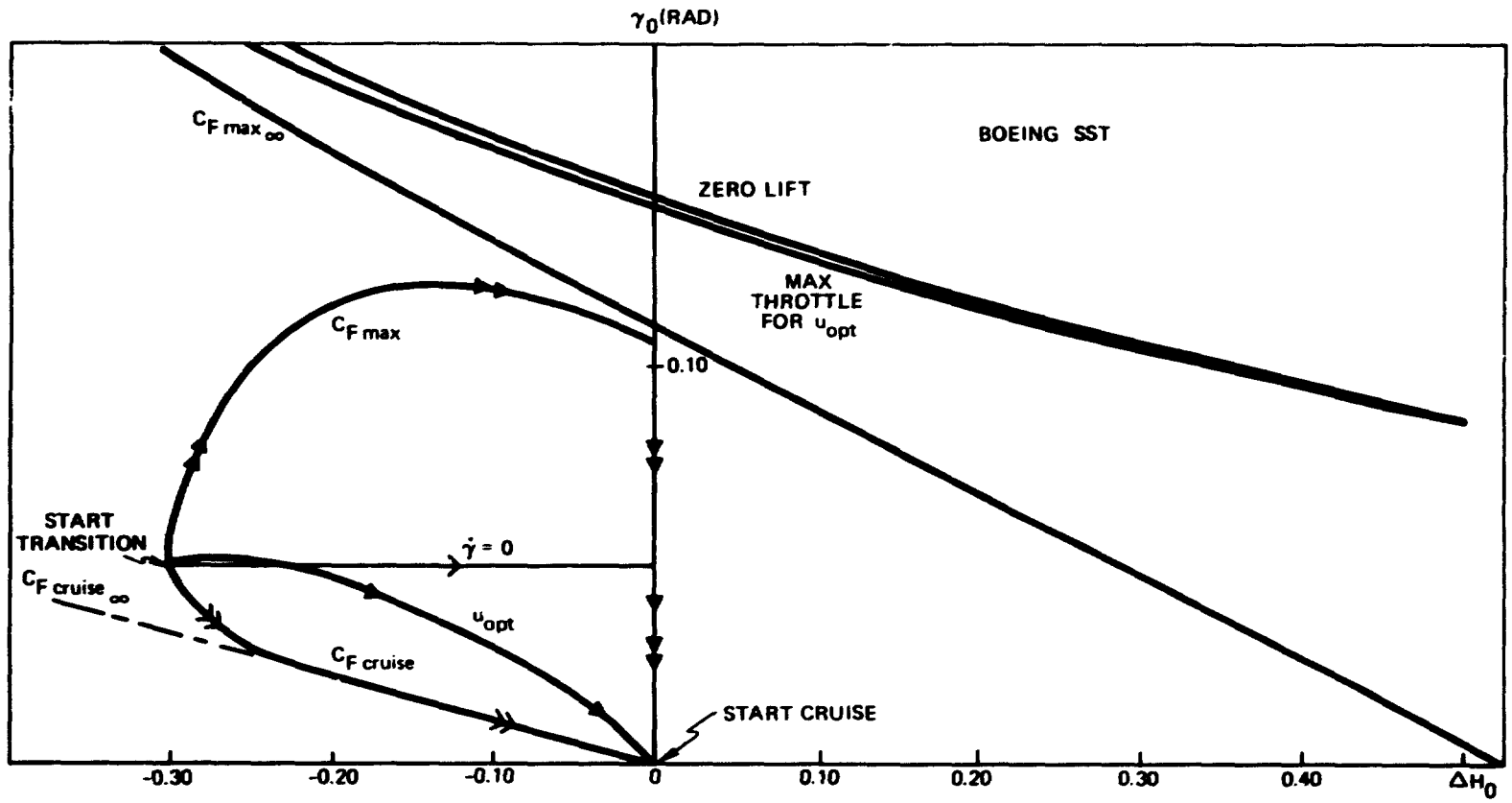

Figure 6.3. Altitude Transients in ΔH_0 - γ_0 Space for Various Controls

Figure 6.3 also shows a $(C_F)_{max}$ trajectory approaching from some much lower altitude than cruise. If pursued to its limit it would settle at the maximum cruise altitude of the aircraft. This trajectory acts as a separatrix for all other $(C_F)_{max}$ trajectories originating at other initial conditions. All other $(C_F)_{max}$ trajectories will fare smoothly into this separatrix and continue on to maximum altitude. None will cross it. This also applies to trajectories from higher altitudes than the maximum cruise altitude. They would fly down to the maximum cruise altitude remaining on one side or the other of the extension of the trajectory from infinity (separatrix). The separatrix for ascent at $(C_F)_{cruise}$ is also shown.

For negative values of $\Delta \hat{H}_0$ the separatrix follows fairly closely a locus of zero lift at maximum C_F. This locus is also indicated in Fig. 6.3 It is evaluated by equating u to -1.0 in Eq (6.2). Above this locus the flight path angle would be too steep to maintain constant velocity flight.

Figure 6.3 also indicates a locus of points at which throttle setting is maximum if the optimal control is used. This locus comes from setting C_F to $(C_F)_{max}$ in Eq (6.5). Above this locus the throttle setting required for a constant velocity range-optimal transition would be greater than the maximum throttle setting. If one follows the $(C_F)_{max}$ separatrix backwards to lower altitude, eventually the separatrix will be above the locus of maximum throttle for u_{opt}. The point at which this intersection takes place is interesting because in ascending at $(C_F)_{max}$ from some large initial altitude difference this will be the point at which it is possible

to begin using the optimal control. Figure 6.4 is a sketch of that intersection and an optimal trajectory from it. Approaching from some lower altitude at $(C_F)_{max}$ one follows the separatrix (A - A′) until point C. There one begins the optimal spiral into 0, down-throttling all the way. Consider now point B′ as an initial condition. The optimal spiral requires upthrottling initially and the throttle saturates at B′. The extension of this optimal trajectory is indicated in a dashed line. From B′ a $(C_F)_{max}$ arc fares into the separatrix and eventually comes out of saturation at, or very near, C, from which it follows an optimal spiral to 0.

For the Boeing SST the separatrix and the locus of $(C_F)_{max}$ at u_{opt} essentially overlap in the vicinity of their intersection. From studying a digital computer print-out of the trajectories in the vicinity of their intersection, the point (-0.745, -0.300) in $\Delta\hat{H}_0 - \gamma_0$ space was taken as the intersection. Since this point would be well outside the linear-quadratic region, a comparison was made using the full state equations (5.3, 5.4, 5.5) and the linear-quadratic optimal control (5.137) which, for these equations, becomes a sub-optimal control. The comparative trajectories are shown in Fig. 6.5. The optimal trajectory overshoots the Breguet cruise by a $\Delta\hat{H}$ of about 0.03 (624 feet) and returns to meet the Breguet cruise at a value of R_L of about 15. This corresponds to a range of about 60 miles and would require about two minutes to complete. The difference in M between the two trajectories would be 0.00076 which corresponds to a fuel weight savings of 487 pounds. This can be converted by the Breguet range equation to a range improvement of 5.44 n. mi. In Fig. 6.6, u and C_F are plotted as functions of R_L for the two transitions of Fig. 6.5.

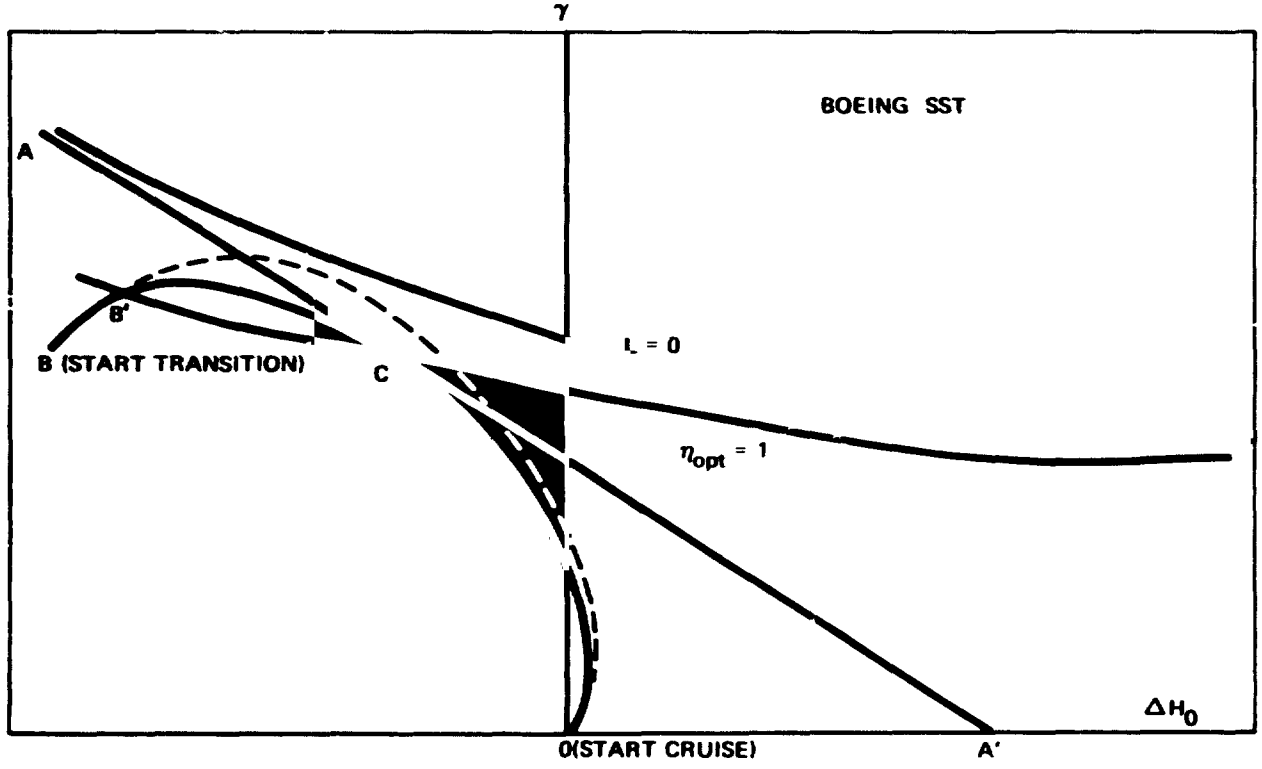

Figure 6.4. An Optimal Trajectory with Throttle Limiting

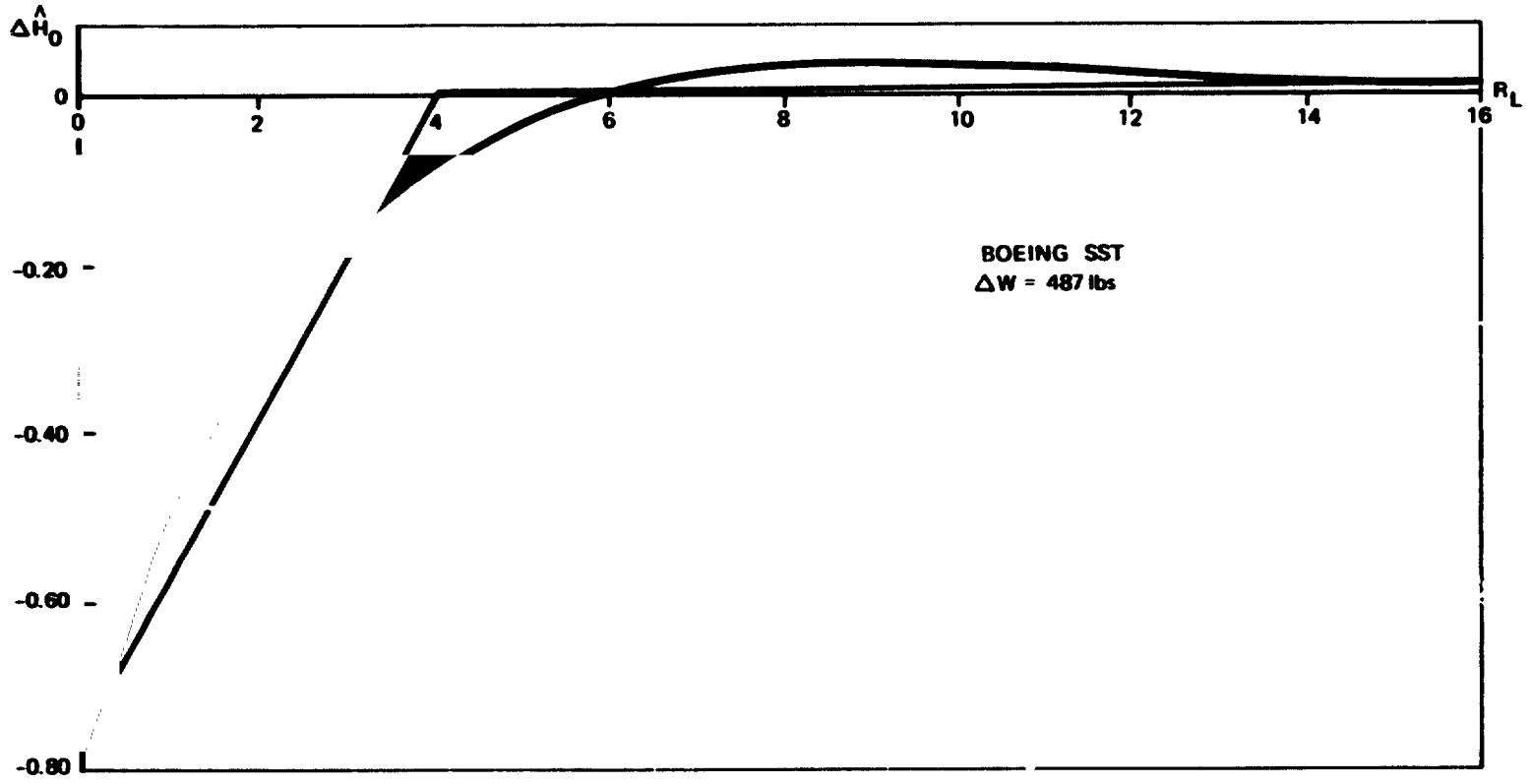

Figure 6.5. Altitude Transient for u_{opt} as a Sub-Optimal Control

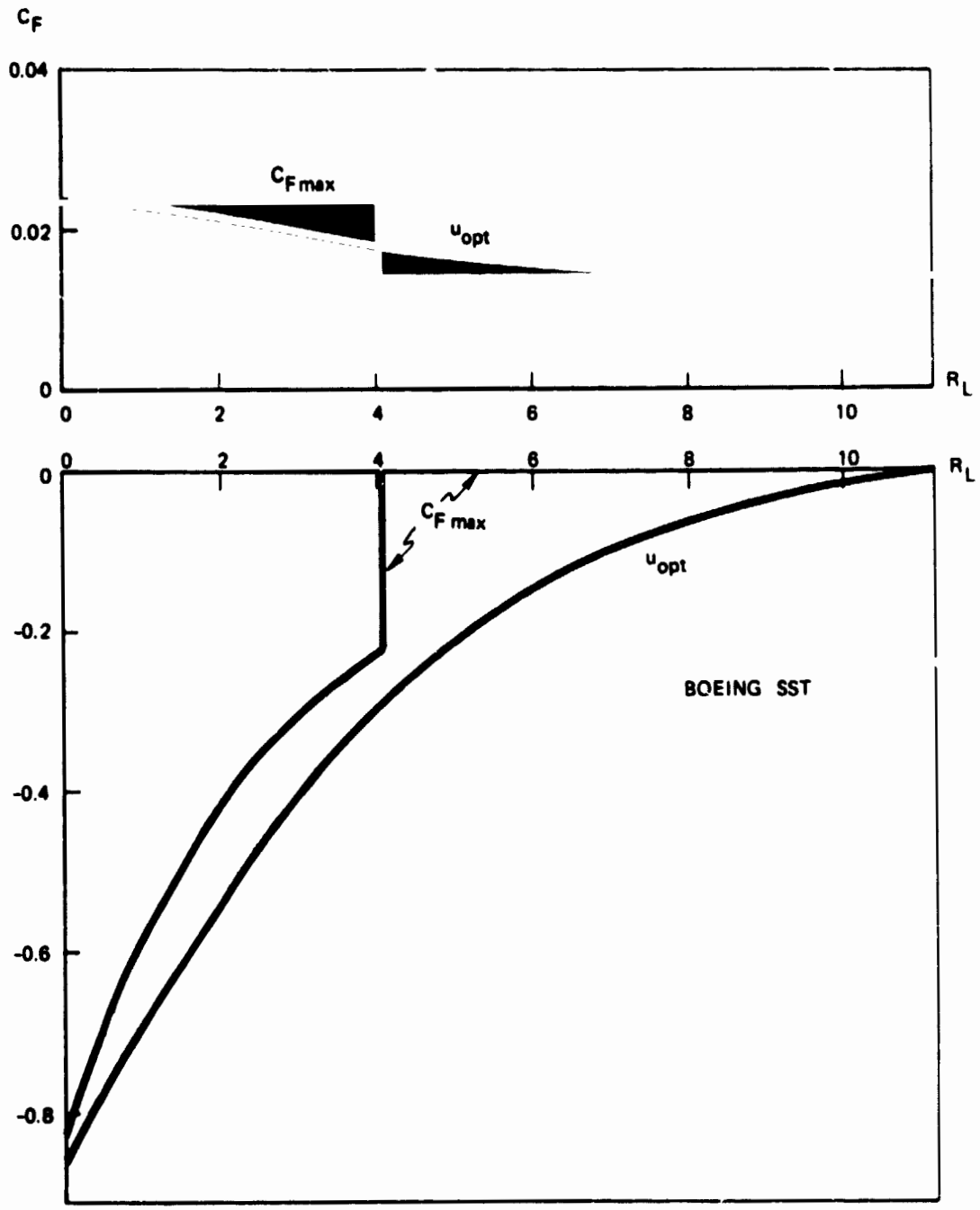

Figure 6.6. C_{Fmax} and u Transients for the Trajectory of Figure 6.5

Figure 6.7 shows in ΔH_0 - γ_0 space a comparison of two complete climb-cruise-descent trajectories. Single arrowheads denote the u_{opt} transitions and double arrowheads denote the $(C_F)_{max}$, $(C_F)_{min}$ transitions. The cruise segment (ΔH_0 axis from 0.0 to +0.10) is common to both trajectories. Initial and final values of γ_0 are zero. Initial altitude is the same as final altitude. The initial value of $\Delta \hat{H}_0$ is taken as -0.20 and the increase in ΔH_0 during cruise as +0.10. This means that the final value of $\Delta \tilde{H}_0$ is -0.30. These altitude values are kept small in order not to exceed the assumptions inherent in the linear-quadratic problem. The increase in ΔH_0 during cruise can be related to a zero order final value of range through the constant cruise flight path angle and then to a final value of mass which must be the specified final value of mass.

In this presentation one can see that the zero order ascent and descent and the first order cruise are the most significant parts of the trajectory. The first order corrections to ascent and descent would be of order ϵ smaller and would not make an observable change in the figure. The zero order cruise, on the other hand, would be represented by the origin alone and would not fairly represent cruise. It is possible to speculate, therefore, that one could make a simpler approximation to the analytic representation of the solution by asymptotically matching the zero order boundary layer solutions to the first order cruise.

We now proceed to evaluate some numeric results related to Fig. 6.7 in order to show the relationship to the Breguet range of the first order corrections to it due to initial and final transitions and to non-zero cruise flight path angle. The numeric values used for the SST flight parameters are shown in Appendix A.

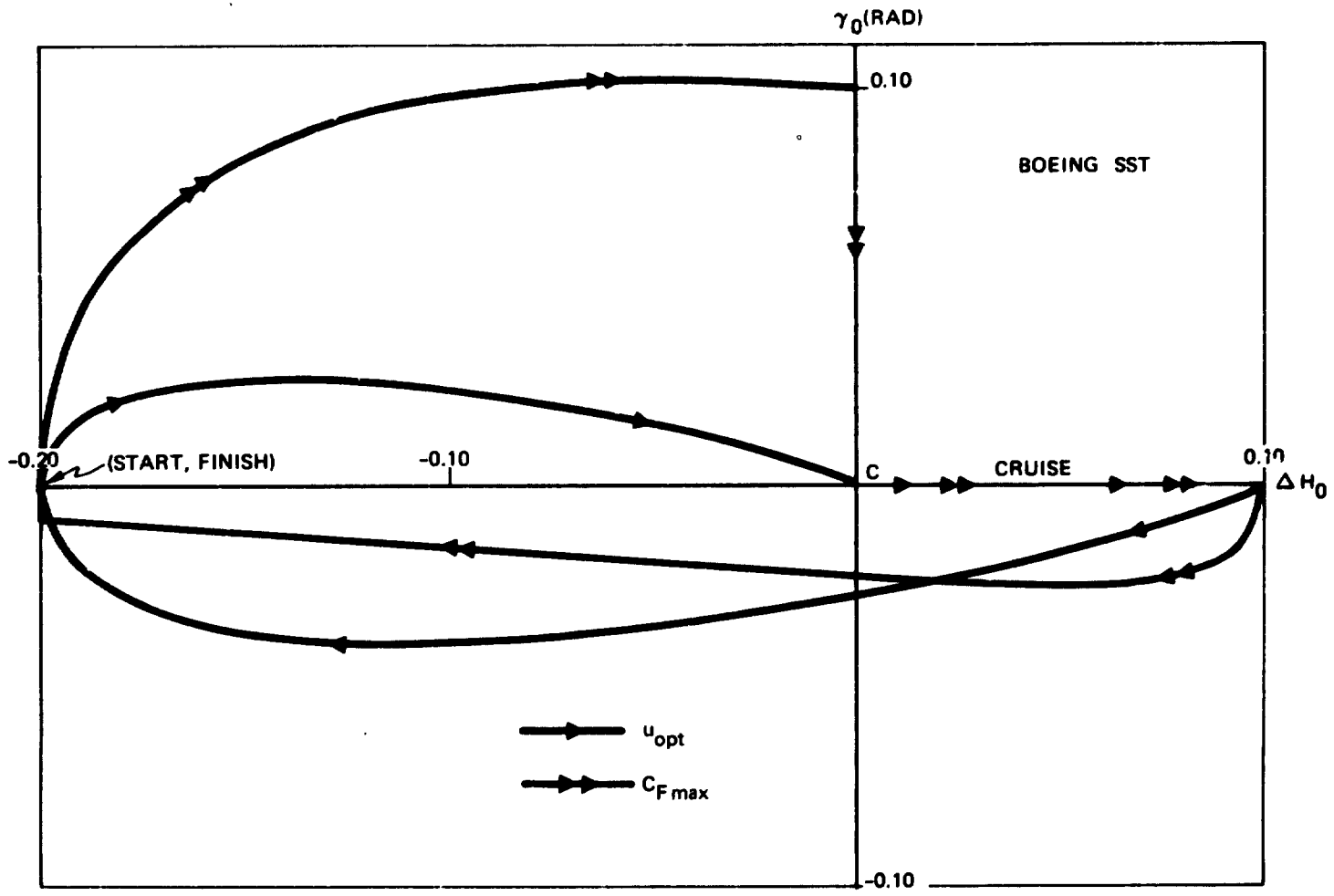

Figure 6.7. Climb, Cruise, Descent Trajectories for u_{opt} and C_{Fmax} Transitions

The altitude difference re. ing from cruise is +0.10. This implies a zero order range of

$$R_{0f} = \frac{\Delta H_0}{\gamma_{1b}} = 0.4649 \tag{6.9}$$

and a final mass of

$$M_{0f} = 1 - e^{-\gamma_{1b} R_{0f}} = 0.09516 \tag{6.10}$$

The zero order range in nautical miles is

$$\text{range} = (R_{0f})(\frac{v}{c_0})(\frac{1}{6076}) = 715.8 \text{ n. mi.} \tag{6.11}$$

The complete first order correction to range has been shown to be

$$R_{1f} = -\frac{c_i}{\gamma_{1b}} + \frac{c_f}{\gamma_{1b}} e^{\gamma_{1b} R_{0f}} + \frac{1}{2\sqrt{C_{D_o} K}\, x} \ln(1 - M_{0f}) \tag{6.12}$$

The third term is the Breguet correction due to non-zero cruise flight path angle. Its value is

$$\frac{1}{2x\sqrt{C_{D_o} K}} \ln(1 - M_{0f}) = -0.9903 \tag{6.13}$$

$$\Delta \text{range} = -.9903\ (\frac{v}{c_0})(\frac{\epsilon}{6076}) = -3.4 \text{ n. mi.} \tag{6.14}$$

Thus the range achieved on a pure Breguet cruise for which the final value of M is 0.09516 is

$$\text{range} = 712.4 \text{ n. mi.} \tag{6.15}$$

Since initial and final conditions are not on a Breguet cruise there will be increments in fuel or range associated with meeting the initial and final state. From digital solutions using the linear-quadratic problem we have

	u_{opt}	$(C_F)_{max}$	
c_i	0.44124	0.76852	
c_f	0.53041	0.48846	(6.16)

The first order correction to the u_{opt} problem due to ascent and descent transitions is

$$R_{1f} = -2.0513 + 2.2312 = +0.1799 \tag{6.17}$$

$$\Delta \text{range} = +0.6 \text{ n. mi.} \tag{6.18}$$

for a total of 713.0 n. mi. For the $(C_F)_{max} - (C_F)_{min}$ problem the correction is

$$\Delta R_{1f} = -3.5728 + 2.0548 = -1.5180 \tag{6.19}$$

$$\Delta \text{range} = -5.2 \text{ n. mi.} \tag{6.20}$$

for a total range of 707.2 n. mi. and the saving of u_{opt} over $(C_F)_{max} - (C_F)_{min}$ is +5.8 n. mi.

Looking at the components of the first order corrections for u_{opt} we see that the amount of range lost from a pure Breguet cruise because of optimal transition to cruise from a lower altitude is

$$\Delta R = -\epsilon \frac{c_i}{\gamma_{1b}} \tag{6.21}$$

$$\Delta \text{range} = -7.0 \text{ n. mi.} \tag{6.22}$$

The amount of range increase over a pure Breguet cruise as a result of optimal transition from cruise to a lower altitude is

$$\Delta R = -\epsilon \frac{c_f}{\gamma_{1b}} e^{\gamma_{1b} R_{0f}} \tag{6.23}$$

$$\Delta \text{range} = +7.6 \text{ n. mi.} \tag{6.24}$$

The sum of the increments is +0.6 n. mi. as has already been shown.

The u_{opt} transitions will of course require less fuel than the $(C_F)_{max}$ - $(C_F)_{min}$ transitions. The amount of this fuel saving is calculated from c_i and c_f. In ascent

$$\Delta W = \Delta c_i W_i \epsilon = 466 \text{ lb.} \tag{6.25}$$

and in descent

$$\Delta W = \Delta c_f W_i \epsilon \, e^{-\gamma_{1b} R_{0f}} = 54 \text{ lb.} \tag{6.26}$$

for a combined weight saving of 520 lbs. The minimum value of C_F was taken as 0.011 instead of 0.012 so that the entire flight could be made with the afterburner on, that is, with a uniform engine description throughout the flight.

Figure 6.8 shows the additional range realized by using u_{opt} instead of $(C_F)_{max}$ in an initial transition to cruise. The increment in range is plotted as a function of the initial value of γ_0, with the initial value of $\Delta \hat{H}_0$ taken as -0.30. Incremental savings in mass, from Fig. 6.1, are

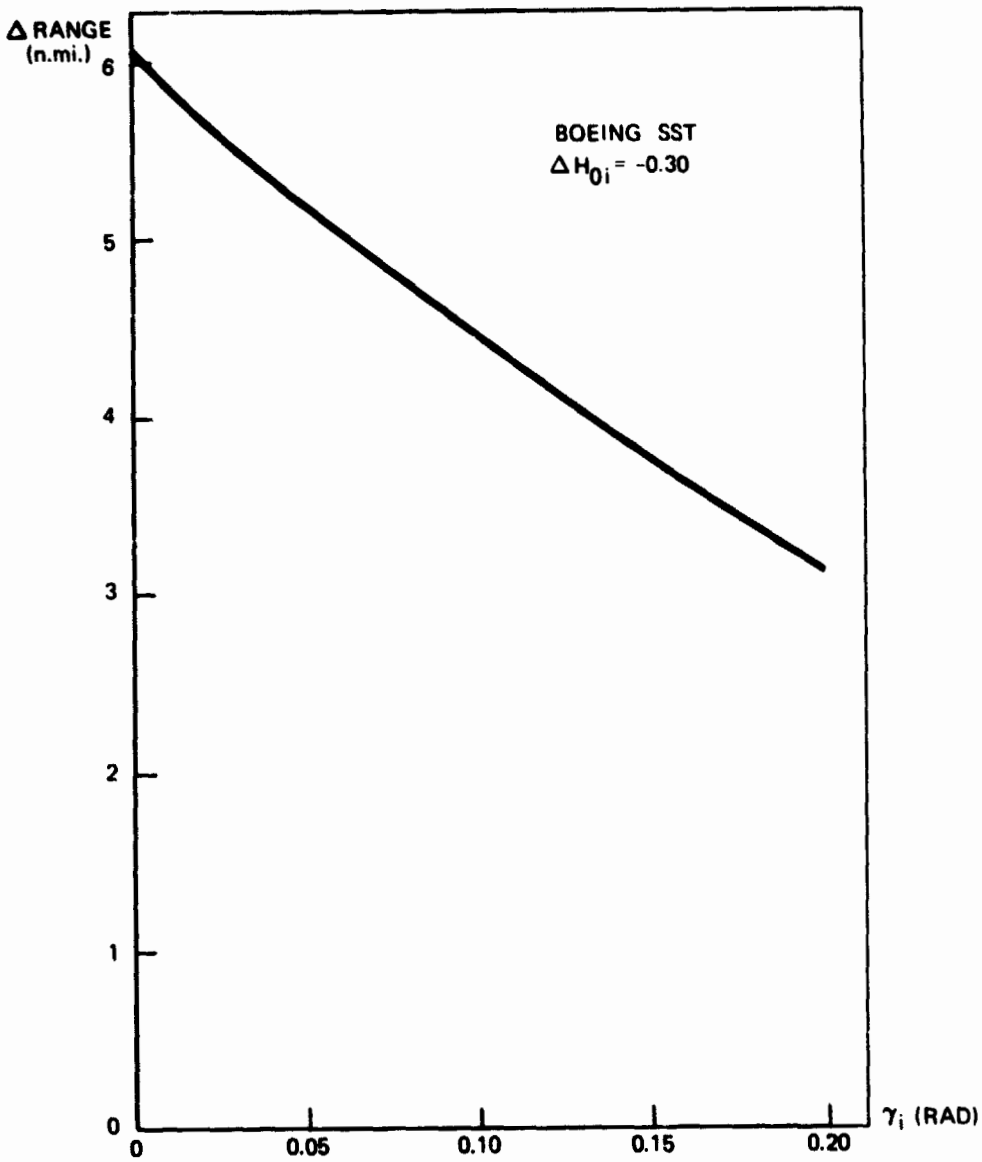

Figure 6.8. Range Increase Between u_{opt} and C_{Fmax} Transitions From Various Initial Conditions

converted to range savings by the modified Breguet range equation. The additional range in these transitions is comparable in size to the increase claimed (3.8 n. mi.) for complete trajectories with climb and descent transitions in [10].

The demonstrated fuel savings are, of course, a small part of the total weight of the aircraft. Indeed, the use of singular perturbation methods implies that weight saving relative to the total weight will be on the order of ϵ in comparison to one. So will the resulting increase in range when compared to the total range. But as a percent of payload the saving is not insignificant, since the percentage of payload to gross weight for an SST may be only on the order of 5% [25]. Furthermore, flight experience with the first operational SST, the Concorde, has shown it to have fuel reserves only on the order of 24,000 lbs after a flight of 3400 n. mi. (equivalent to a Paris to Washington, D.C. flight) carrying a payload that also happened to be 24,000 lbs [26].

6.4 McDonnell Douglas F-4

We next consider an early version of the McDonnell F-4 (Appendix A). This aircraft is capable of supersonic cruise, but we shall consider it only on transonic cruise. The principal reference for this aircraft [1] assumes that it has constnat SFC. The authors recognize a weakness in their assumption but justify it on the fact that better data were not available to them. We shall use this aircraft to observe the effect on cost of various values of the parameter y which is proportional to the slope of the curve of SFC vs C_F in the vicinity of the cruise value of SFC. We assume that the cruise value of SFC does not change as y changes. There-

fore, since

$$SFC = c_0 + c_1 C_F \tag{6.27}$$

$$= c_0 \left[1 + y \frac{C_F}{C_{D_o}} \right] \tag{6.28}$$

and to zero order in cruise

$$C_F = C_{D_o}(1 + x^2) \tag{6.29}$$

we have

$$SFC = c_0 [\ 1 + y(1 + x^2)\ ] \tag{6.30}$$

The value of x is determined solely from y. Since SFC at cruise is to remain constant, the value of c_0 must change with y. Changing c_0 will affect the value of ϵ since

$$\epsilon = \frac{c_0}{\beta v} \tag{6.31}$$

Finally, from the equation for SFC, we see that the maximum value of SFC will increase as y increases.

Figures 6.9 through 6.13 show a comparison of the cost between a full throttle climb to transonic cruise and the optimal cost for five values of y. First notice that if y is zero the cost of both trajectories is less than it would be for any other value of y. Then as y increases, the cost of both trajectories increases but the difference between them becomes greater. The largest value chosen for y is slightly larger than the value based on the Boeing SST data. The middle value of y corresponds to the cruise value of the Boeing 707-320B which uses the PW JT3D turbofan

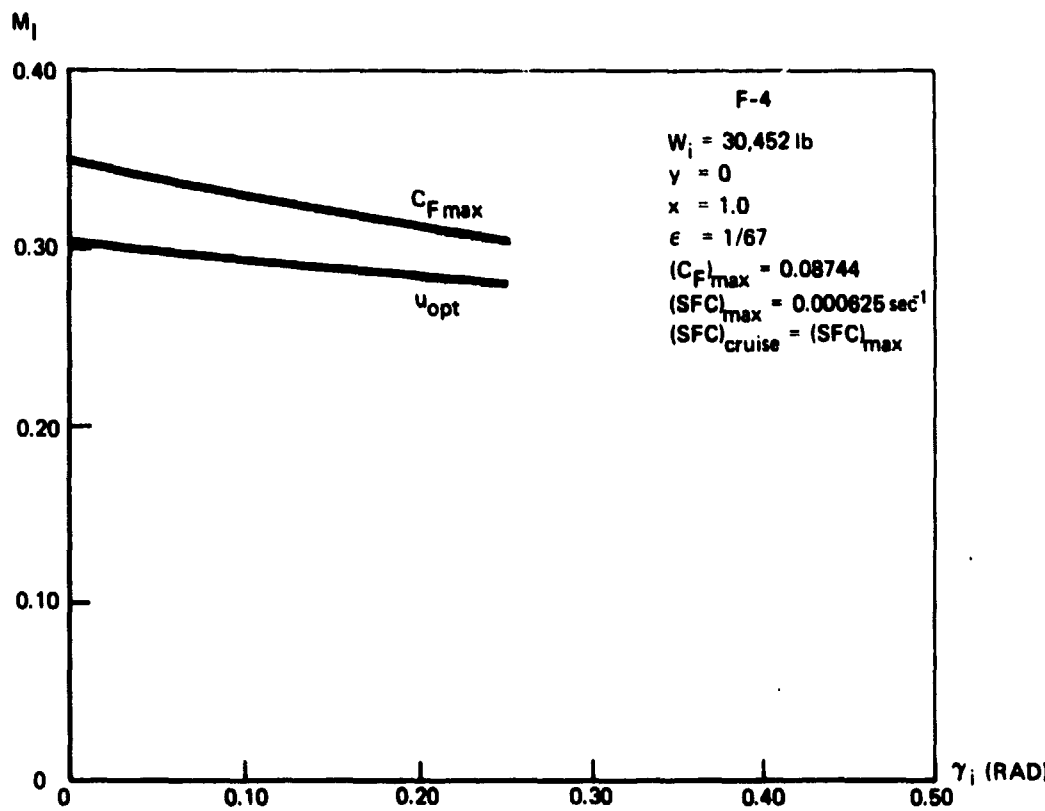

Figure 6.9. Fuel Cost for u_{opt} and C_{Fmax} Initial Transitions for Various Initial Conditions at y = 0

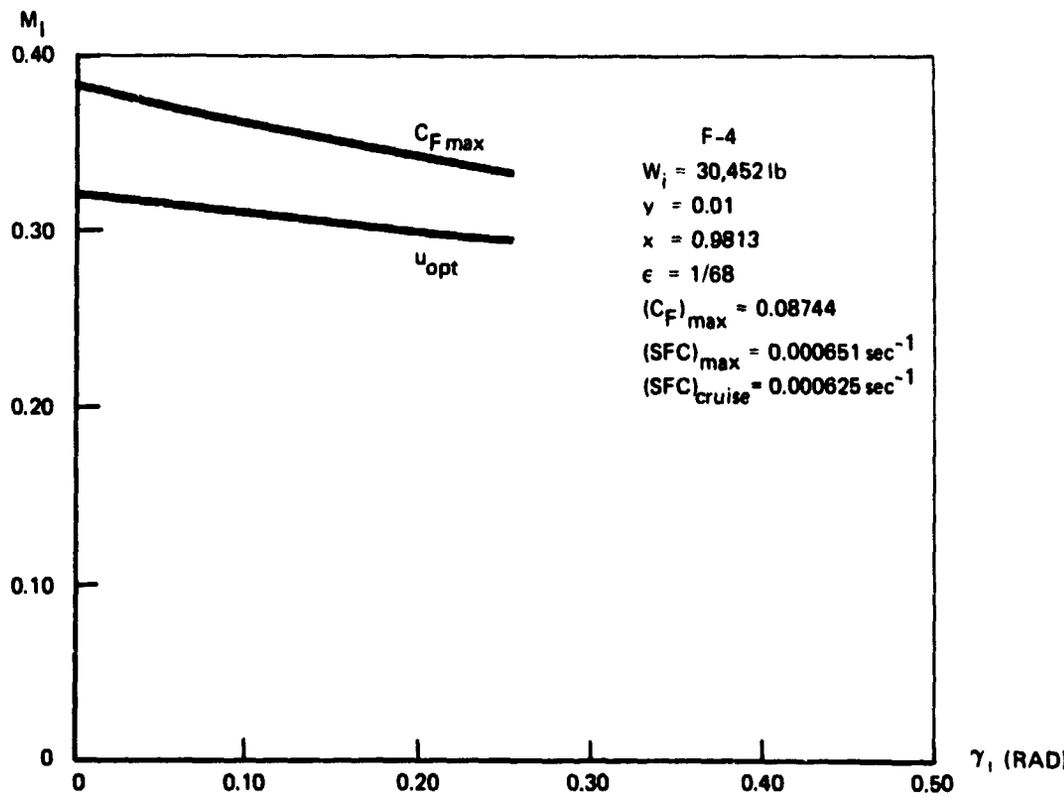

Figure 6.10. Fuel Cost for u_{opt} and C_{Fmax} Initial Transitions for Various Initial Conditions at y = 0.01

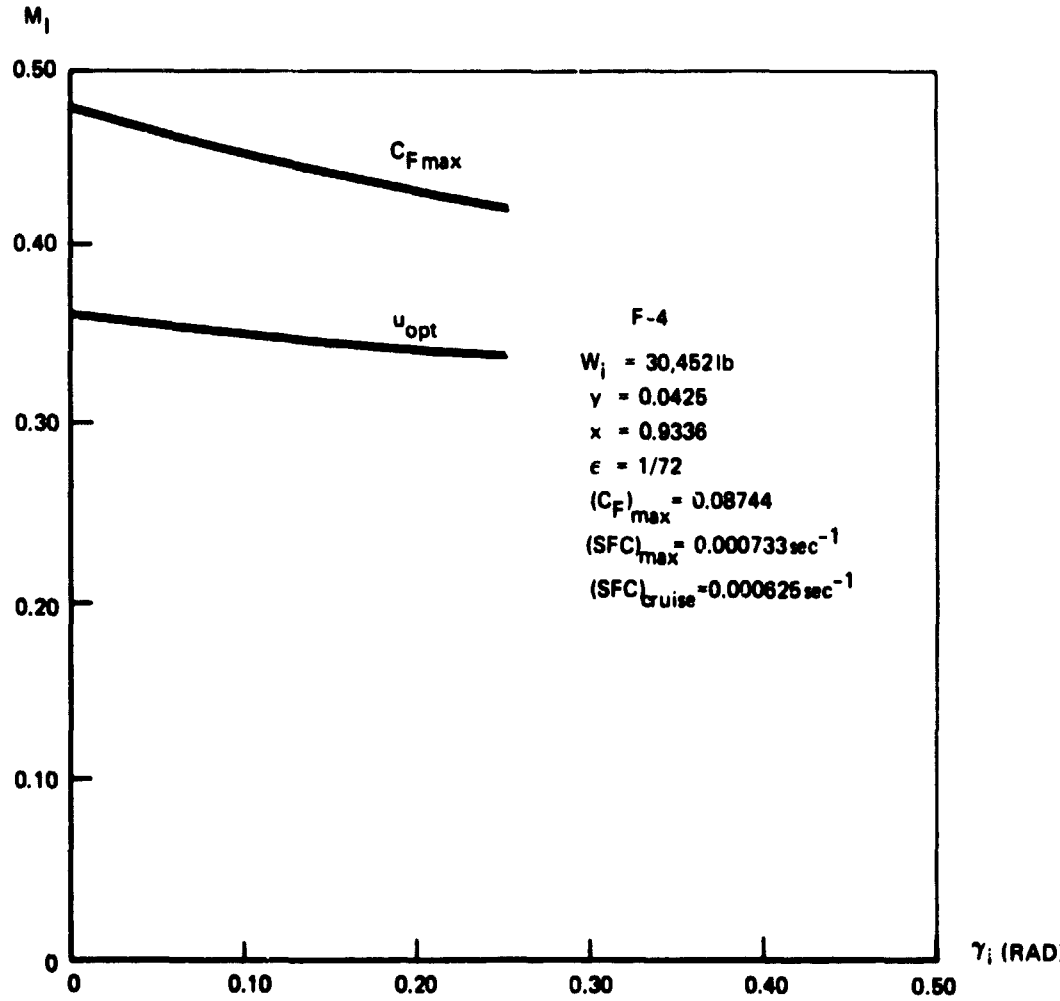

Figure 6.11. Fuel Cost for u_{opt} and C_{Fmax} Initial Transitions for Various Initial Conditions at y = 0.0425

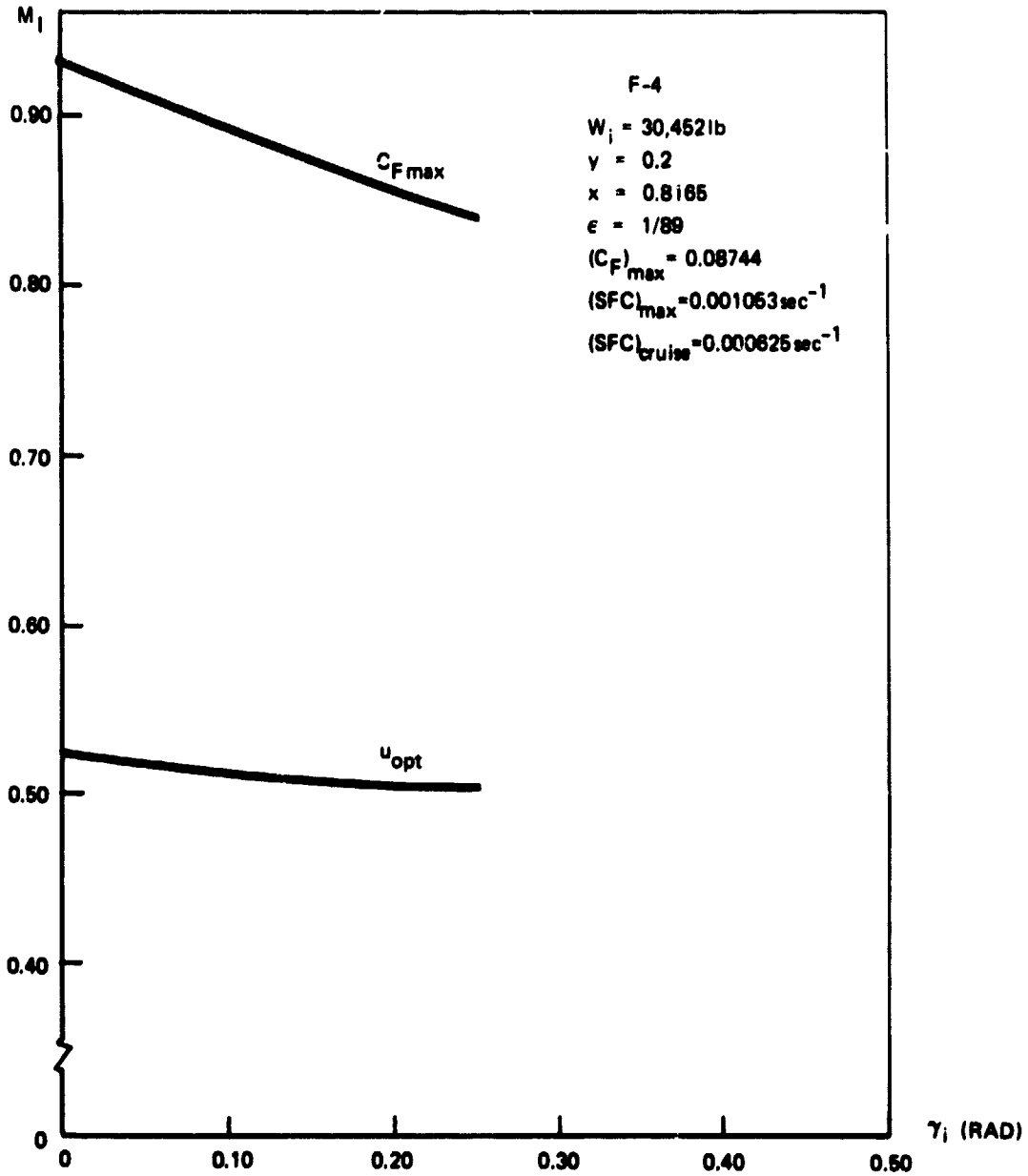

Figure 6.12. Fuel Cost for u_{opt} and C_{Fmax} Initial Transitions for Various Initial Conditions at y = 0.2

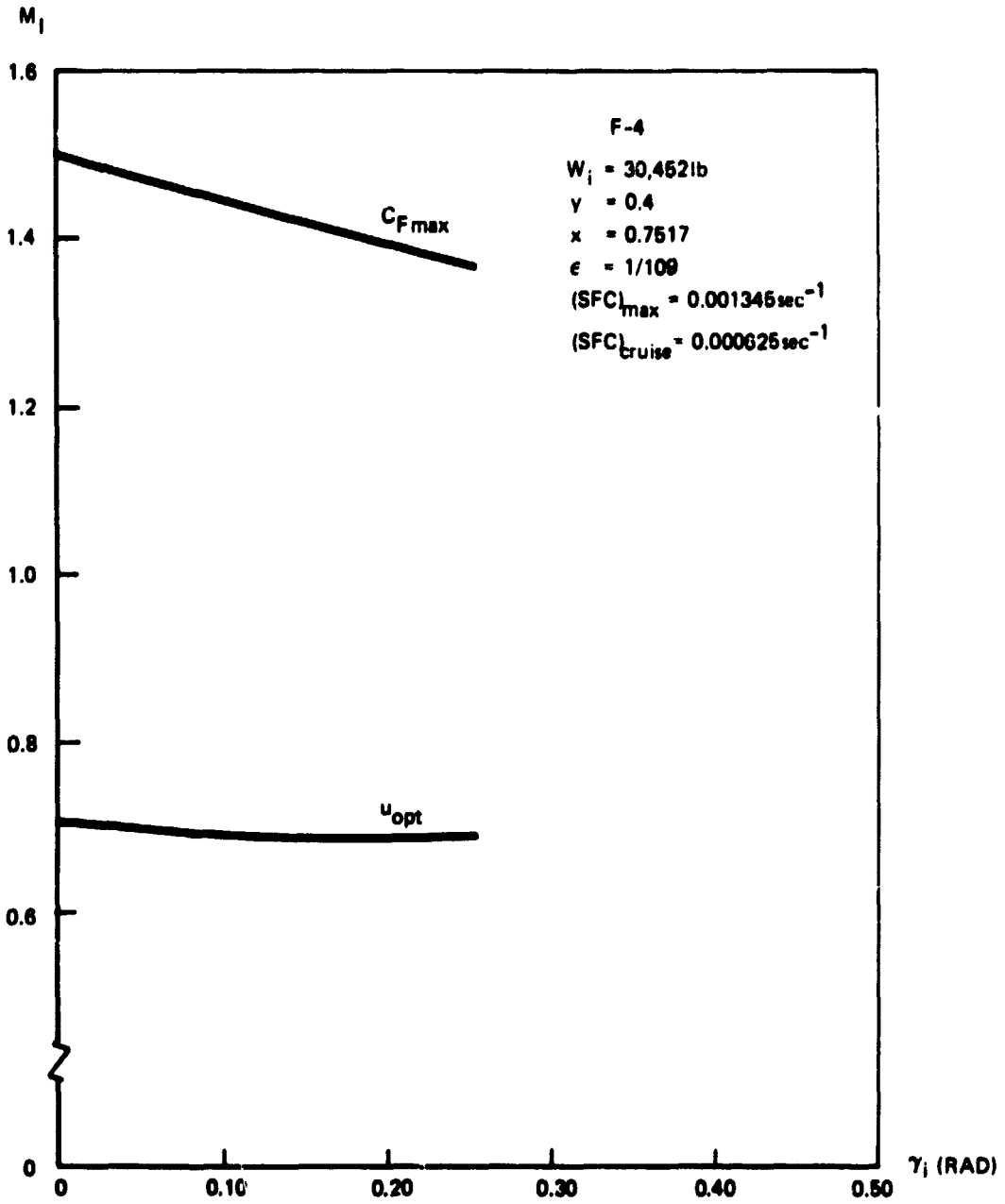

Figure 6.13. Fuel Cost for u_{opt} and C_{Fmax} Initial Transitions for Various Initial Conditions at y = 0.4

engine. The effect of increasing y is to increase the value of SFC at maximum thrust. As y increases, the cost of operating above cruise thrust is increased. The optimal trajectories therefore tend to become much more heavily damped as y increases. This relationship of y to damping was mentioned in Chapter V. Figure 6.14 is a comparison of the transient responses in ΔH_0 as a function of R_L. It shows clearly the relationship between y and damping.

Figure 6.15 shows the fuel saving in pounds associated with the various values of y. It is a restatement of the data of Figs. 6.9 through 6.13 for an initial weight of 30,452 lbs. It is seen that if y is zero the weight saving is only on the order of 10 to 20 lbs. However, for larger (but not unrealistic) values of y substantial savings in fuel can be realized.

In Fig. 6.16 the effect of the parameter y on the range improvement for the u_{opt} initial transition over that for $(C_F)_{max}$ is shown. The fuel weight savings from Fig. 6.15 are converted to increments of range by the Breguet relation

$$\Delta \text{range} = -\frac{1}{\gamma_{1b}} \ln(1 - \Delta M)\left(\frac{v}{c_0}\right)\left(\frac{1}{6076}\right) \text{ n. mi.} \tag{6.32}$$

The first order cruise flight path angle has been shown to be

$$\gamma_{1b} = 2x\sqrt{C_{D_o} K}\ \frac{(1 + x^2)}{(3x^2 - 1)} \tag{6.33}$$

The parameter c_0 changes with y so that SFC at cruise is constant. Its value is also plotted in Fig. 6.16. Fuel savings are based on the data of Fig. 6.15 at an initial flight path angle of zero. It is seen from Fig. 6.16

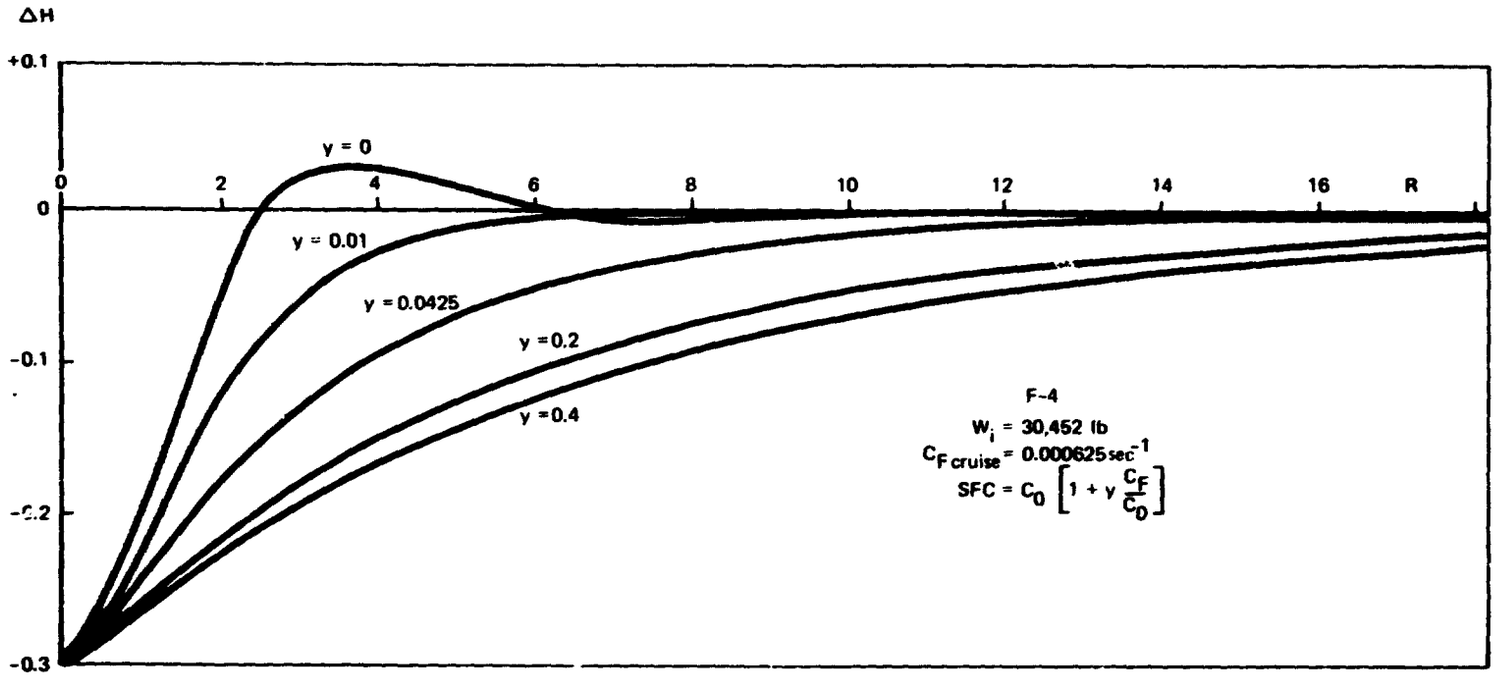

Figure 6.14. Transient Response for Various y at Constant Value of $(C_F)_{cruise}$

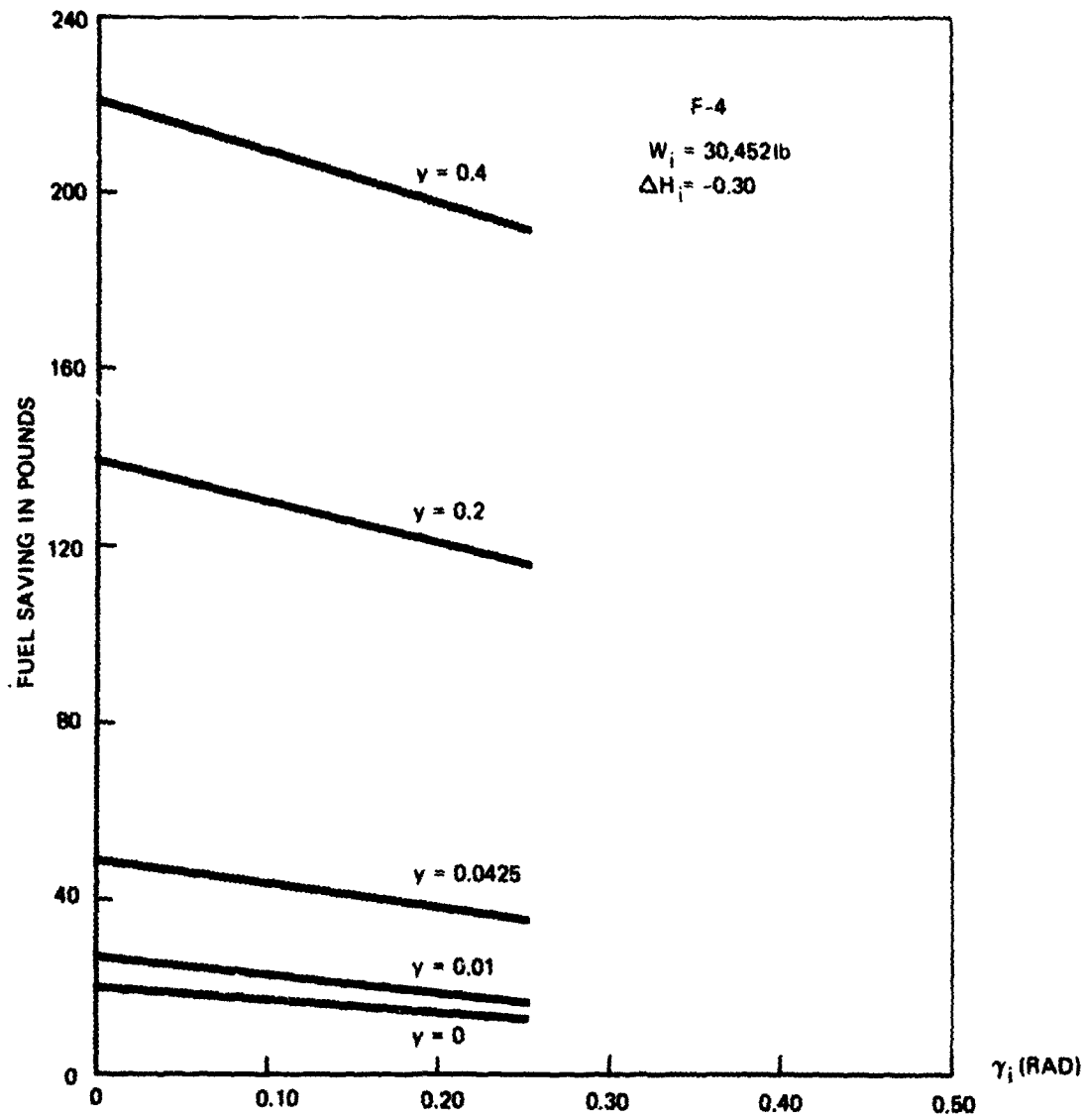

Figure 6.15. Fuel Saving Between u_{opt} and C_{Fmax} Initial Transitions for Various Initial Conditions

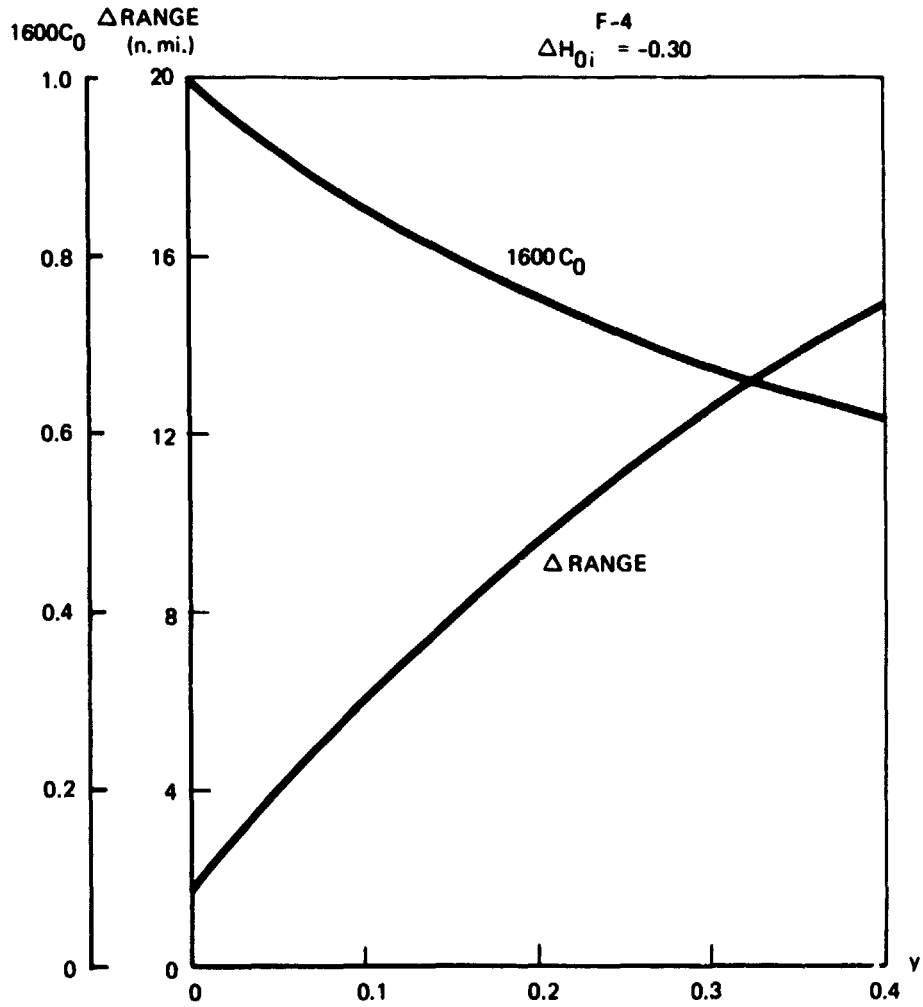

Figure 6.16. Range Increment Between u_{opt} and C_{Fmax} Initial Transitions as a Function of y

that for values of y that are almost all less than that of the SST (i.e., 0.357), the F-4 achieves range increases that are generally much better than that of the SST.

The result of this parametric study is to enable us to describe the conditions under which maximum range transitions to cruise can produce a worth-while saving in fuel. First it has been seen that the thrust required in cruise must not be too close to the maximum thrust capability of the aircraft. If required cruise thrust approaches maximum thrust, fuel saving will approach zero. Second, the specific fuel consumption associated with maximum thrust must be greater than that required for cruise. If it is not, then the potential fuel saving will be negligibly small even though the maximum thrust may be much greater than the required cruise thrust. In summary there must be sufficient thrust capability over and above that required for cruise and there must be a cost associated with using it.

A large number of aircraft do meet these conditions but it is also important to note that a large number fail to meet these conditions. Most of the commercial aircraft currently in service with the airlines fly at nearly their maximum thrust and in a fairly flat part of the curve of SFC vs C_F. This holds for the PW JT3D which powers the Boeing 707 and the McDonnell Douglas DC-8, and also for the PW JT8D which powers the Boeing 727 and 737 and the McDonnell Douglas DC-9 [29]. Both of these engines have a value of y on the order of 0.04 and cruise at approximately 80% of maximum thrust. For the 707 in an initial transition to cruise, there appears to be no first order difference in cost between the u_{opt} policy and any of the other policies. For example, from an initial

state of (-.30, +.10) in $\Delta\hat{H}_0-\gamma_0$ space, the fuel cost associated with optimal transition to Breguet cruise is $0.360\epsilon W_i$. The costs of the other three policies are $0.362\epsilon W_i$, $0.372\epsilon W_i$, and $0.374\epsilon W_i$ for $(C_F)_{cruise}$, $(C_F)_{max}$, and γ_{const}, respectively.

6.5 Implementation of the Optimal Control Policy

This thesis has developed the optimal control policy in terms of incremental changes in lift away from its cruise value. The control thus developed is a linear combination of the elements of the state vector and hence is suitable for implementation as a feedback control. It is also possible to implement an open-loop or programmed control system. Since analytic solutions are obtainable it is necessary only to program one of the zero-order state varibales, γ_0 or ΔH_0, or the control variable, u_0, as a function of range. It would even be convenient to program ΔH_0, which is $\gamma_0 R$, as a function of range.

It is probable, however, that to follow closely an optimal transition trajectory would require a degree of accuracy in the measurement of $\Delta\hat{H}_0$, $\Delta\tilde{H}_0$ or γ_0 that could only be achieved by an inertial unit. Certainly the transition described for the SST in Fig. 6.5, requiring an overshoot in altitude of 624 ft in a transition requiring 60 n. mi. and two minutes to complete, could probably not be duplicated by the pilot using a clock and a rate of climb meter for his cues. Exact duplication, however, may not be required.

Simulation of transitions with a pilot in the loop would be necessary to determine how well a pilot could follow an optimal trajectory, what cues

he would require, and how sensitive the cost is to deviations from the optimal. In the case of the SST, pilots have encountered difficulties in trying simultaneously to maintain constant Mach number and to level off to cruise altitude [24]. These difficulties were observed both in ground-based simulations and in flight simulations of the SST climb profile. Pilots had difficulty in avoiding overshoot in altitude and in Mach number while monitoring Mach error and pitch attitude. No data were available on fuel cost associated with the overshoots.

Cost savings achievable through optimal transitions would have to be weighed against the dollar and weight cost associated with the cues needed to implement the optimal transitions. For an aircraft that already has an inertial navigation system on board, it would be a simple matter to implement the optimal cruise transition policies of this thesis. Individual cost determinations would have to be made for other aircraft.

CHAPTER VII

CONTRIBUTIONS, CONCLUSIONS, RECOMMENDATIONS FOR FURTHER WORK

7.1 Contributions

This thesis has contributed an approximate analytic solution to one of a class of airplane performance optimization problems for which even computational solutions have been extremely difficult to obtain [8, 11]. The analytic solution was obtained by the use of singular perturbation techniques in conjunction with the minimum principle of Pontryagin. Solutions were obtained in cruise and in transitions to and from cruise and then these three distinct segments were matched asymptotically.

Inclusion of the normal acceleration equation made possible the inclusion of maneuvering lift effects on induced drag, an effect previously appearing only in computational solutions. Singular perturbation techniques allowed the drag force, which can not be linearized in any meaningful way, to be expressed as a quadratic function of state and control vector elements. The costate for mass, which is associated with these quadratic terms in the variational Hamiltonian, was shown to be a constant, to zero order in ϵ, thus producing a solvable linear-quadratic optimal control problem.

Optimal control laws have been developed for constant velocity transition to and from cruise in three dimensional state space (altitude, flight path angle, and mass), and expressions for the cost associated with

them have also been developed. These cost expressions can serve as a lower bound for purposes of evaluating other techniques of performing these transitions.

7.2 Conclusions

Two basic conclusions can be drawn from this thesis, one from the analytic point of view and the other from the practical point of view.

The first conclusion is that singular perturbation methods offer the possibility of approximate analytic solutions to certain optimal control problems that would otherwise have to be solved by computational methods. An important class of such problems is airplane performance problems in which equations involving aerodynamic drag (mass and velocity equations) can be neglected as a zero order approximation to the solution in the neighborhood of the singularities. The analytic results should be useful in themselves but should also provide clues toward finding methods of easing the computational difficulties associated with higher order versions of these problems.

Second, for some aircraft the optimal control policy for transition developed in this thesis offers the possibility of significant fuel savings. These are aircraft that do not cruise at or near their maximum power setting (including transonic cruise for aircraft that are capable of supersonic cruise) and for which the specific fuel consumption at maximum power setting is somewhat greater than that for cruise.

7.3 Recommendations for Further Work

The first recommendation would be to apply the techniques demonstrated in this thesis to the solution of other similarly structured optimal control problems, for example, problems in which velocity varies slowly in the boundary layer and is constant to zero order in ϵ.

The solution obtained in this thesis should be studied in conjunction with higher order computational solutions to try to gain insights into the nature of the optimal solutions and thereby to determine how best to approach computational solutions to this and similar problems with higher order state vectors.

One could also expand the present study to consider the nature of the control if the slope of SFC as a function of C_F were not merely a straight line in the vicinity of cruise but a series of connected straight line segments. Such a representation would be more accurate in the case of most power plants. The value assigned to the slope in a linear approximation directly affects the damping inherent in the optimal control through the parameter y (Fig. 5.2), and y determines x, the altitude parameter which represents the difference between optimal cruise altitude and the altitude for maximum lift-drag ratio. Since the altitude for maximum lift-drag ratio remains constant, a change in x represents a change in the optimal cruise altitude. Thus as $d(SFC)/d(C_F)$ changes discretely from one value to another, so also do two important aspects of the optimal control problem: damping in transition and optimal cruise altitude. There appears to be no point in using higher order functions of C_F to describe SFC. If a quadratic function were used, then the mass equation would be third

order in thrust. The third order effects would be lost in forming the linear-quadratic optimal control problem.

The descent from cruise could also be studied further. The possibility of decelerating flight at minimum thrust or at zero thrust has not been considered here. Nevertheless the cruise-glide solution is a very real possibility for maximum range flight. Optimal gliding flight is easily described by energy state methods. The transition from optimal cruise conditions when the engines are shut off to optimal gliding flight could possibly be set up as a boundary layer problem. Higher dimensional glides might be established as perturbations about the single variable optimal glide from energy-state methods.

It is also possible to study the maximum range problem from take-off through cruise and to landing by combining energy-state methods for acceleration and climb and for deceleration and descent with the optimal transitions and cruise developed in this thesis. This would require patching of solutions as opposed to matching. The energy climb (computational solutions) would be followed until cruise velocity is attained. This state would become the initial condition for transitions to cruise. Similarly the state at the end of cruise when the engines are shut down becomes the initial condition for the transition to optimal glide.

The nature of the most general optimal transition from cruise to descent would in itself be an interesting study. Is there a throttling solution that is superior to an instantaneous zeroing of thrust? Should transition from cruise to the higher altitude for optimal glide be made while throttling or at constant thrust (including zero)?

It is hoped that this thesis will stimulate others to pursue these and other related topics of research in optimal airplane performance and the application of singular perturbation techniques thereto.

APPENDIX A

CHARACTERISTICS OF TYPICAL SUPERSONIC AND TRANSONIC CRUISING AIRPLANES

This appendix presents those parameters necessary to describe the airframe and power-plant of three aircraft that cruise at constant Mach number in the stratosphere.

A. 1 Boeing SST

The first is the Boeing supersonic transport. Data is from [10]. For the airframe we have

W_i	640,640 lbs	initial cruise weight
S	7578 sq ft	wing area
v	2479 ft/sec	cruise speed
M	2.56	cruise Mach number
C_{D_o}	0.00878	profile drag coefficient
K	0.5	induced drag coefficient

Table A. 1: Boeing SST Airframe Characteristics

The power plant consists of four turbojets with afterburners, and in supersonic cruise the afterburners are on. Specific Fuel Consumption as a function of thrust coefficient is shown in Fig. A. 1. As a linear approximation to this function in the afterburning region we can write

$$SFC = c_0 + c_1 C_F \tag{A. 1}$$

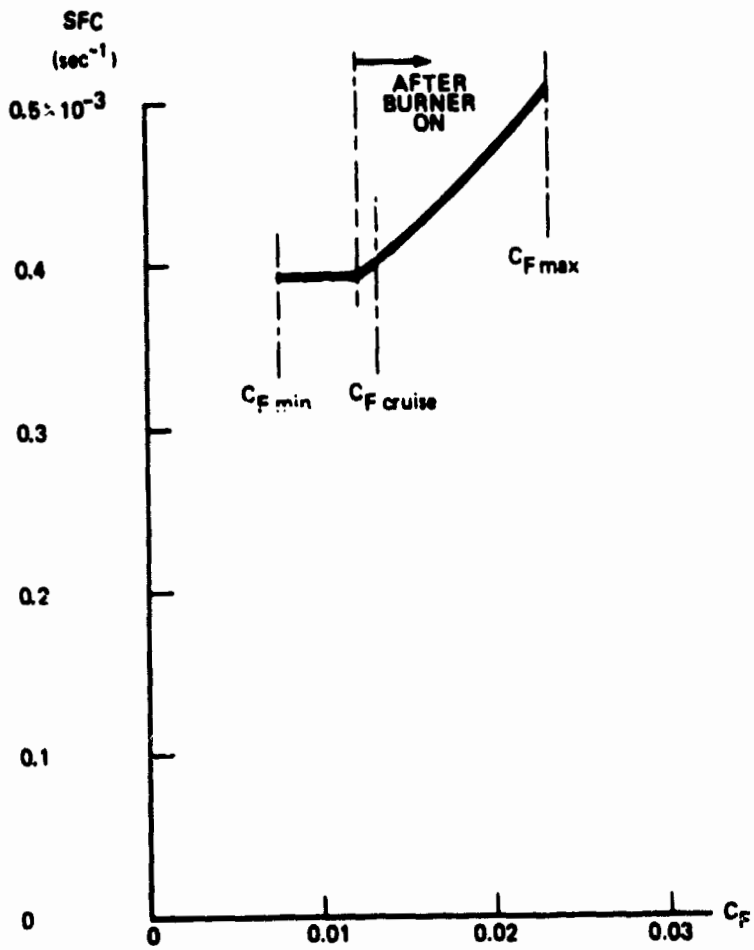

Figure A.1. SFC vs C_F for Boeing SST

$$SFC = 0.000265 + 0.010789\, C_F \tag{A.2}$$

with dimensions of inverse seconds.

Now certain parameters which are defined in Chapter III can be evaluated for the SST. If we write

$$SFC = c_0 \left[1 + y \frac{C_F}{C_{D_o}} \right] \tag{A.3}$$

$$y = \frac{c_1}{c_0} C_{D_o} \tag{A.4}$$

then we have

$$y = 0.357 \tag{A.5}$$

Since x is determined solely by y we have

$$x(y) = 0.762 \tag{A.6}$$

The first order value of the cruise flight path angle becomes

$$\gamma_{1b}(x, C_{D_o}, K) = 0.2151 \tag{A.7}$$

The cruise value of C_F becomes

$$C_F(x, C_{D_o}) = 0.01388 \tag{A.8}$$

which is 60% of its maximum value.

Finally, the parameter ϵ becomes

$$\epsilon(v, c_0, \beta) = (1/450) \tag{A.9}$$

In the stratosphere the scale height of the atmosphere is

$$\beta^{-1} = 20,800 \text{ ft} \tag{A.10}$$

The range-optimal control for this airplane is a damped sinusoid in R. It has the following natural frequency and damping ratio, respectively

$$\omega_n = 0.33 \tag{A.11}$$

$$\zeta = 0.8762 \tag{A.12}$$

The parameter a, which appears in the γ equation, is equal to the square of ω_n.

$$a = 0.1089 \tag{A.13}$$

A.2 McDonnell Douglas F-4

The second airplane is an early version of the McDonnell Douglas F-4. It was used by Bryson, et al [1] and recurs frequently in later literature. This aircraft is capable of supersonic cruise but we consider it in transonic cruise. For the airframe

W_i	30,452 lbs	Initial cruise weight
S	530 sq ft	wing area
v	872 ft/sec	cruise velocity
M	0.9	cruise Mach number
C_{D_o}	0.014	profile drag coefficient
K	0.2095	induced drag coefficient

Table A.2: F-4 Airframe Characteristics

The power plant consists of two GE J-79 turbojets with afterburners. Specific Fuel Consumption is taken nominally as 0.000625 sec^{-1}. That is, if

$$SFC = c_0 \left[1 + y \frac{C_F}{C_{D_o}} \right] \tag{A.14}$$

then y is assumed to be zero. In the parametric study of Chapter VI, y is allowed to assume various constant values while SFC at cruise remains constant. Figure A.2 shows the extremes of this function. It is obvious that c_0, the intercept on the SFC-axis, changes and hence the perturbation parameter, ϵ, changes too. The cruise value of C_F is not allowed to vary. It holds constant at 0.028, which is 32% of its maximum value.

The parameters of Chapter III all depend on the value assigned to y. As a result they are presented in Table A.3 for various values of y. The roots of the range-optimal control are shown in Fig. A.3.

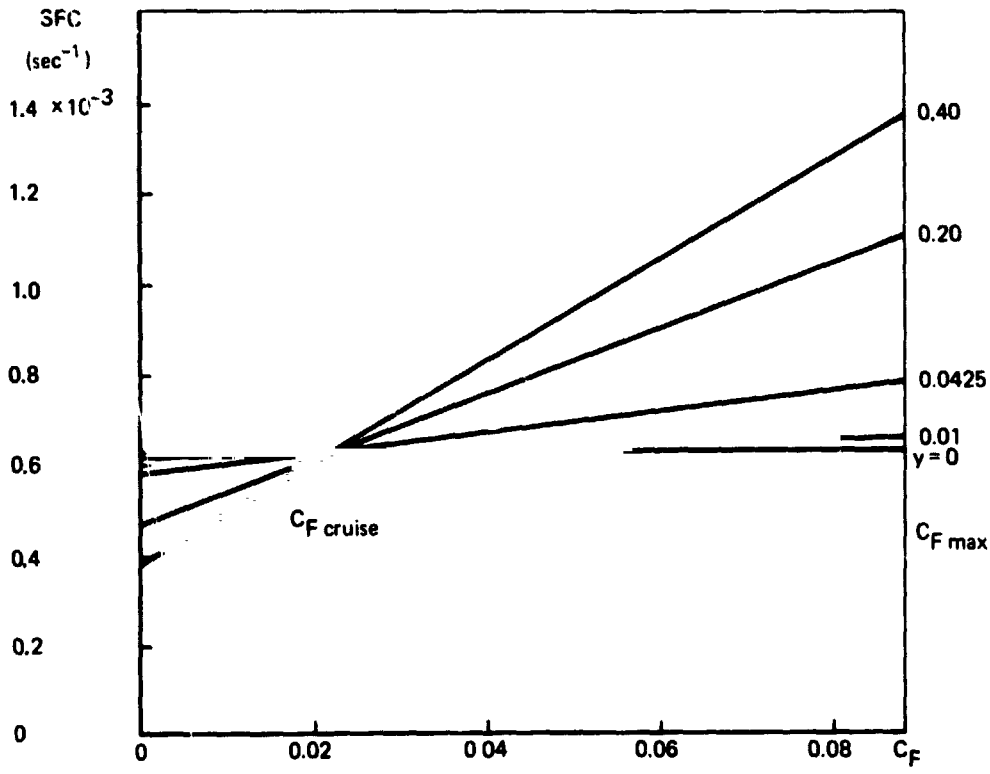

Figure A.2. SFC vs C_F for Various Values of y for F-4

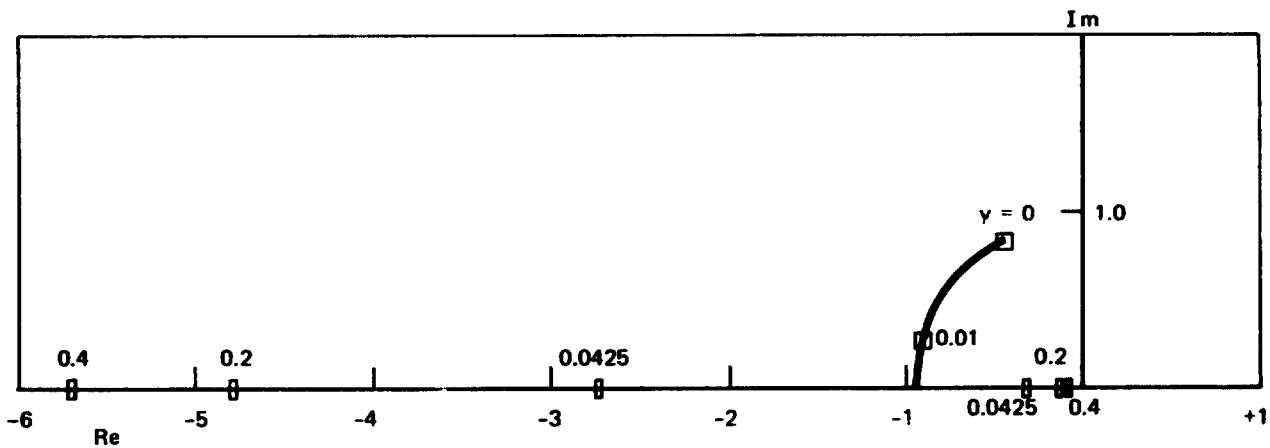

Figure A.3. Roots of u_{opt} Equation for F-4 as y \aries

y	0	0.01	0.0425	0.2	0.4
x	1	0.9813	0.9336	0.8165	0.7517
γ_{1b}	0.1083	0.1104	0.1172	0.1474	0.1833
$1600c_0$	1	0.9807	0.9263	0.7500	0.6150
ϵ^{-1}	67.0	68.0	72.0	89.0	109.0

Table A.3: F-4 Cruise Parameters for Values of y

A.3 Boeing 707-320B

The third airplane is the intercontinental version of the Boeing 707. The airframe parameters are from [27], with drag coefficients extracted from information on cruise thruse in [27] and [28]. We have

W_i	270,000 lbs	Initial cruise weight
S	2892 sq ft	wing area
v	775 ft/sec	cruise velocity
M	0.8	cruise Mach number
C_{D_o}	0.0114	profile drag coefficient
K	0.062	induced drag coefficient

Table A.4: 707-320B Airframe Characteristics

The power plant consists of four PW JT3D turbofan engines without duct-burning capability. Specific Fuel Consumption as a function of thrust coefficient is shown, for cruise in the stratosphere, in Fig. A.4. The plot was developed from data in [20]. The aircraft cruises in the

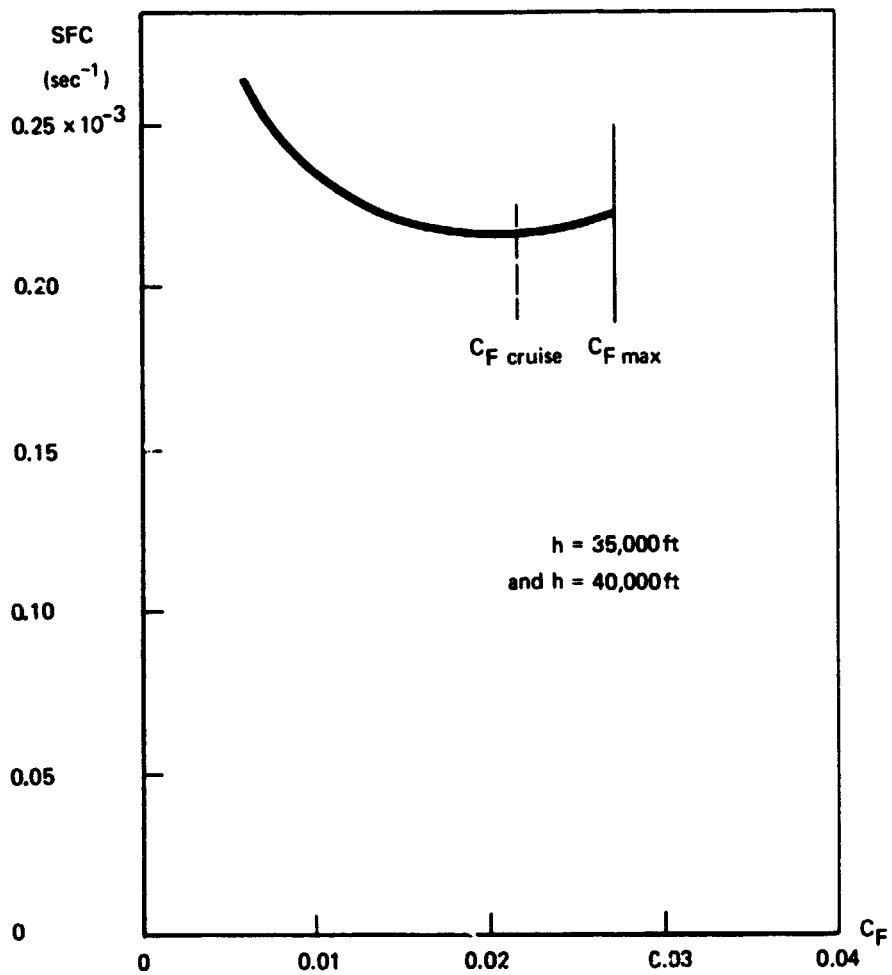

Figure A. 4. SFC vs C_F for JT3-D

positive slope region close to maximum thrust coefficient. Note that a range-optimal solution will not use the negative slope region which would increase SFC as C_F is decreased. Instead the range optimal solution would resort to chattering: minimum SFC would be maintained as thrust was reduced below the value for SFC_{min} by alternately using zero thrust and thrust for SFC_{min}. The duty cycle would be determined by the amount of thrust required.

As a linear approximation to the function of C_F in the vicinity of its cruise value we can write for SFC, from (A. 1)

$$SFC = 0.0002014 + 0.0007508\, C_F \tag{A. 15}$$

with dimensions of inverse seconds. The parameter y is

$$y(c_0, c_1, C_{D_o}) = 0.0425 \tag{A. 16}$$

The other parameters of Chapter III are

$$x(y) = 0.934 \tag{A. 17}$$

$$\gamma_{1b}(x, C_{D_o}, K) = 0.0575 \tag{A. 18}$$

and

$$(C_F)_{cruise} = 0.02134 \tag{A. 19}$$

which is 80% of its maximum value. Finally, the perturbation parameter is

$$\epsilon = (1/185) \tag{A. 20}$$

The range-optimal control for this airplane has two real roots. The values are

$$p_1 = -7.493 \tag{A.21}$$

$$p_2 = -0.149 \tag{A.22}$$

The parameter a has the value

$$a = 1.116 \tag{A.23}$$

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Biography

Joseph Taffe O'Connor was born [redacted] He attended St. Patrick's Grammar School in Lawrence and St. John's Preparatory School in Danvers, Mass., graduating in June, 1950. He then enrolled at Bowdoin College, Brunswick, Maine, in the Combined Plan of Study with the Massachusetts Institute of Technology. In June, 1955, he received the degree of Bachelor of Arts, cum laude, from Bowdoin and the degree of Bachelor of Science in Aeronautical Engineering from MIT and was commissioned a second lieutenant in the U. S. Air Force Reserve.

Following graduation he accepted a position as a staff engineer at the MIT Instrumentation Laboratory (now the Charles Stark Draper Laboratory, Inc.). He worked for the Flight Control and TAC Groups performing analytic and simulation studies of airplane and missile autopilot systems. He served on active duty in the U.S. Air Force from June, 1956, to January, 1958, as the aircraft requirements coordinator for electronics research at the Cambridge Research Center, L. G. Hanscom Field, Bedford, Mass.

After returning to the Instrumentation Laboratory he completed his requirements for the degree of Master of Science in Aeronautical Engineering and received that degree from MIT in June, 1959.

In October, 1963, he transferred within the Instrumentation Laboratory to the Apollo project, where he worked in the Space Guidance Analysis Group on the design and simulation of the digital autopilot for the Lunar

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In September, 1969, he entered the Graduate School of MIT in the Department of Aeronautics and Astronautics as a candidate for the degree of Doctor of Philosophy. He continued to work in the Hybrid Simulation Division as a Research Assistant.

Mr. O'Connor is a member of Sigma Gamma Tau and Sigma Xi. He is married to the former Jane [redacted] of Roslindale, Mass. They live in Newton, Mass., with their daughters, Caitlin and Maeve, and their son, Neal.