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FLAP-LAG-TORSIONAL DYNAMIC MODELLING OF ROTOR BLADES IN HOVER AND IN FORWARD FLIGHT, INCLUDING THE EFFECT OF CUBIC NON-LINEARITIES

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ABSTRACT

The differential equations of motion, and boundary conditions, describing the flap-lead/lag-torsional motion of a flexible rotor blade with a pre-cone angle and a variable pitch angle $\theta(x,t)$, which incorporates a pre-twist $\theta_{pt}(x)$, are derived via Hamilton's principle. The equations are valid for both extensional and inextensional blades, and the meaning of inextensionality is formally discussed. The equations are reduced to a set of three integro partial differential equations by elimination of the "extension" variable. Both cases of hover and of forward flight are addressed. The generalized aerodynamic forces are modeled using Greenberg's extension of Theodorsen's strip theory. After the equations of motion are obtained, they are systematically expanded into polynomial non-linearities with the objective of retaining all terms up to third-degree so that the influence of such terms on the motion of the system may be evaluated. The blade is modelled as a long, slender, initially straight beam of isotropic Hookean material. Offsets from the blade's elastic axis through its shear center and the axes for the mass, area and aerodynamic centers, as well as radial non-uniformities of the blade's stiffnesses and cross section properties are taken into account. The effect of warp of the cross section is also included in the formulation.

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CHAPTER I - INTRODUCTION

An important problem associated with helicopter dynamics is the determination of the dynamic response and aeroelastic stability associated with the rotor blades. Considerable attention has been given to rotary-wing aeroelasticity problems, as evidenced by the numerous publications in this area, e.g. [1-60]. As a result of this effort, it is now recognized that such problems are inherently non-linear and thus special attention should also be given to the development of consistent mathematical models to address them. Due to their flexibility, rotor blades are able to bend in two principal directions as well as twist. This fact alone introduces several important non-linear terms in the differential equations of motion. It is also well known that considerable complexity is further introduced due to the effect of rotation, through the addition of centrifugal, inertia and Coriolis forces, and of the aerodynamic forces. Comprehensive reviews of the subject have been written by several investigators, e.g., [1-3]. The work reported in [1] by Loewy reviews the literature up to 1969, while the recent review presented by Friedmann [3] includes most of the work done up to 1976/1977. Several aspects of helicopter modelling have been discussed by Ormiston [4].

In 1958, Houbolt and Brooks [14] derived a comprehensive set of linear differential equations to describe the coupled flap-lead/lag-torsional (or, in short, flap-lag-torsional) dynamics of non-uniform rotating pre-twisted blades. Expressions for the aerodynamic loads were not specified in [14]. Further work by several investigators established that non-linear terms not considered in [14] play a fundamental role in the response and the stability of both hingeless (cantilevered) and articulated (pinned) rotors. The non-linear equations of motion are quite complex, and efforts are still devoted today toward their formulation. Hodges and Ormiston [15] analyzed the coupled flap-lag behavior, which can exhibit instabilities, by restricting themselves to torsionally rigid blades. A set of integro

partial differential equations of motion with quadratic non-linearities were developed taking into account the effects of the blade's pitch angle and of a small pre-cone angle. A study of the stability of the flap-lag motions of a hingeless rotor in hover was then performed by making use of such equations. A comparison between the results of the stability analysis for elastic and rigid blade models was also presented in [15]. Work with rigid blade models has also been done by several other investigators, e.g. [22-31].

Hodges and Dowell [32] developed a comprehensive set of differential equations with quadratic non-linearities for the flap-lag-torsional dynamics of rotating non-uniform extensional blades. The presence of pre-twist and of a small pre-cone angle, and the effects of offsets between the mass and area centroids from the elastic axis through the shear center were considered. Three-axes Euler angles were used to describe the orientation of the principal axes of the blade's cross section -- an approach that facilitates the development of the equations. This approach was also used by Peters and Ormiston [33] when considering the effects of the second-order non-linearities on the angle of attack of hingeless rotor blades. The equations in [32] were used in [34] to investigate the stability of a uniform rotor blade in the hovering flight condition. The equations of motion developed in [32] were also used by Dowell, Traybar and Hodges [36] to correlate the experimental and theoretical results obtained for the static deflections and bending natural frequencies of a cantilever beam with a tip mass, after the appropriate terms due to the inclusion of the tip mass were added to such equations. Reasonably good agreement was reported for the cases where the basic assumption in [32] of retaining only quadratic non-linear terms in the equations could be justified. A similar study was also recently presented by Rosen and Friedmann [37].

The equations developed in [32,34] were further extended by Hodges [38] to include not only the effects of twist and pre-cone but also of droop and sweep angles, torque and blade root offsets, and pitch-link flexibility simulated by a

torsional root spring.

The coupled flap-lag-pitch response of hingeless blades for both cases of hover and forward flight were considered by Friedmann and Tong [40,41]. A set of equations to describe the flap-lag-torsional motions of a pre-twisted cantilevered blade in hover, considering root torsional flexibility, was also derived by Friedmann [42]. The derivation presented in [42] follows Friedmann and Tong's earlier work [40,41] and a modification of Houbolt and Brooks' work [14] to include quadratic non-linear terms. An ordering scheme, also used by other investigators, was employed to neglect several non-linear terms in a systematic way and obtain the equations with quadratic non-linearities. A revised set of non-linear differential equations of motion was later derived by Rosen and Friedmann [48]. The equations developed in [48] for pre-twisted blades with a small pre-cone angle include the effects of offset between the blade's area and mass centroids, shear and aerodynamic centers. As in [32], three axes Euler angles are used to describe the blade's orientation in space.

Some of the work on the flap-lag dynamics of rigid hinged blades and hingeless elastic blades for both cases of hover and forward flight has been examined by Kaza and Kvaternik [50]. For the rigid blade case, two types of hinges were considered, one for which the lead/lag hinge flaps with the blade, and the other for which the flap hinge also executes a lead/lag motion. Stability boundaries in the flap-lead/lag frequencies space were obtained, indicating that such boundaries depend on the physical hinge sequence. This work was also reported in [30]. The same authors also examined in [51] the flap-lag-torsional kinematics of pre-twisted blades using lag-flap-pitch and flap-lag-pitch three-axis Euler angle sequences to describe the orientation of the blade's principal axes. Here, however, a source of disagreement with the work of [32,33] arises when the third Euler angle -- a final rotation about the blade's elastic axis --is equated in [51] to the twist angle of the blade rather than determining

the blade's twist from the component along the elastic axis of the curvature vector (which is obtained as a function of the pre-twist, the three Euler angles and their spatial derivatives), as done in [32,33]. The kinematic expressions obtained in [51] are used in [52] to develop a set of non-linear differential equations of motion with quadratic non-linearities for the forward flight condition using two-dimensional quasi-steady blade element theory. The differential equations for the two Euler angle sequence mentioned above are shown explicitly. Although not mentioned in [52], these equations, which represent the same physical problem, should be equivalent to each other. In [50,52] the authors make explicit use, in their derivation, of the blade's kinematic axial deflection, or foreshortening.

The source of disagreements previously mentioned in regard to the formulation of the flap-lag-torsional dynamics of rotating blades was recently clarified by Hodges, Ormiston and Peters [55]. They considered the kinematics of rotating extensional beams and developed expressions for the curvature vector and the Euler-angle transformation matrix for all possible six sequences of the three-axes Euler angles. By means of an appropriate change of variable, they showed that these sequences are equivalent to each other. In addition, they showed that the angle of twist, obtained by integration of the twisting curvature, does contain a higher-order integral term, and that such an angle is a quasi-coordinate much like the axial deflection used in [50,52] with the foreshortening considered a priori. This angle of twist, first obtained by Peters and Ormiston in [33], is discussed by Nordgren in [61] and was independently obtained by Crespo da Silva and Glynn [62-71] in connection with the nonlinear flexural-flexural-torsional motions of inextensional beams.

The non-linear differential equations of motion for inextensional beams were derived in [62,68] by making use of Kirchhoff's kinetic analogy [72], three-axis Euler angles to describe the orientation of the beam's cross sectional principal axes, and Hamilton's principle with the Lagrangian adjoined to the inextensional

constraint by a Lagrangian multiplier. Physically, the Lagrange multiplier represents a force along the beam's neutral axis necessary to maintain inextensionality. This force is dependent on the deflections in a non-linear manner. The equations derived in [62,68] were then expanded into cubic non-linearities and converted to a set of integro partial differential equations [63,68]. These were subsequently used to investigate the non-linear response, and its stability, exhibited by the beam under several conditions [63-67, 69-71] including support asymmetry [64]. The systematic formulation of [62, 63, 68] may be extended to address the rotor blade dynamics problem by incorporation of effects such as blade rotation, pre-twist, pre-cone angle, and aerodynamic forces.

An investigation of the response, and its stability, of a dynamical system described by a set of nonlinear differential equations with quadratic and cubic nonlinearities acting simultaneously has been presented by Nayfeh and Kamel, and by Nayfeh [73,74]. Although the problem considered in [73,74] is also a complex one, the rotor blade aeroelastic problem is substantially more involved. To this author's knowledge, a systematic investigation of the influence of the extensional and inextensional assumptions, and of the next order -- cubic -- non-linearities on the rotor blade aeroelastic response is missing in the literature.

In this report the work presented in [62,68,32] is extended with the objective of deriving the differential equations of motion for both extensional and inextensional rotor blades with pre-cone and a pitch angle $\theta(x,t)$ -- which incorporates a pre-twist $\theta_{pt}(x)$ -- including third-order (ε^3) non-linearities. The pitch angle θ may be included in several ways. One way, for example, is to start with an unpitched blade and to impose the known pitch angle θ about its longitudinal axis to obtain the directions of the principal axes of the undeformed but "pre-pitched" blade. Then a sequence of Euler angles can be used to bring these axes to their new orientation obtained after the blade's elastic deformations. Another way,

introduced in [32,33], is to start with a set (ξ_3, η_3, ζ_3) of non-principal axes of the pitched blade with, say, ξ_3 along the blade's elastic axis, and (η_3, ζ_3) being two non-principal cross sectional axes at an angle $-\theta$ with the "pre-pitched" (but undeformed) blade's principal axes along its cross section. A sequence of three Euler angles is then employed to bring (ξ_3, η_3, ζ_3) to their corresponding orientation after the blade has undergone the elastic deformations. Since the blade's cross section is now, after the elastic deformations, in its "final" orientation in space, a further simple rotation $\theta(x,t)$ about the "final" position of the $\xi_3 = \xi$ axis will then bring the triad (ξ_3, η_3, ζ_3) into coincidence with the deformed blade's principal axes (ξ, η, ζ). With this approach, the pitch angle $\theta(x,t)$ appears in the differential equations simply as an additive term to the third Euler angle, as long as this angle is a rotation about the blade's elastic axis [32,33,55]. Because of its inherent simplicity and compactness, this is the approach that will be used here.

CHAPTER II - THE DYNAMIC SYSTEM AND BASIC ASSUMPTIONS

Consider an initially straight, but pre-twisted, rotor blade of length R when not elastically deformed, mass m per unit length, and of closed cross section. Its maximum cross section dimension is assumed to be much smaller than its length so that the blade may be approximated as a beam. A blade segment, both in its undeformed and elastically deformed states, is shown in Fig. 2.1. The (ξ, η, ζ) axes -- with unit vectors $(\hat{\xi}, \hat{\eta}, \hat{\zeta} = \hat{\xi} \times \hat{\eta})$ -- are the principal axes of the cross section at the shear center C_e^* of the deformed blade; it is assumed that the cross section is symmetric about the η axis. The ξ axis is tangent at all times to the elastic axis of the blade.

The (x,y,z) axes, with unit vectors $(\hat{x}, \hat{y}, \hat{z} = \hat{x} \times \hat{y})$ shown in Fig. 2.1, are a set of rotating reference axes; the x axis is coincident with the elastic axis of the undeformed blade. These axes rotate in space with constant angular velocity Ω about a direction perpendicular to the rotor hub. This is shown in detail in Fig. 2.2 together with their spatial orientation relative to a set of inertial (X, Y, Z) directions. As seen in Fig. 2.2, the orientation of these axes may be described by first aligning (x, y, z) with (X, Y, Z) and then performing two successive rotations. The first rotation $\tau = \Omega t$ (where t is time) about Z brings the (x, y, z) triad to its new orientation $(X_1, Y_1, Z_1 = Z)$, while a second rotation β about the negative Y_1 direction (i.e., a clockwise rotation) brings (X_1, Y_1, Z_1) to its final orientation (x, y, z). The angle β, which is the angle the undeformed blade x axis makes with the plane of rotation, XY, is the blade's pre-cone angle. This angle is taken to be a constant. The "blade root offset" e_1 shown in Fig. 2.2 will be taken to be zero for simplicity.

The principal axes (η, ζ) of the blade's cross section, centered at C_e^* as shown in Fig. 2.1, make an angle $\theta(x, \tau)$ with two cross section non-principal axes

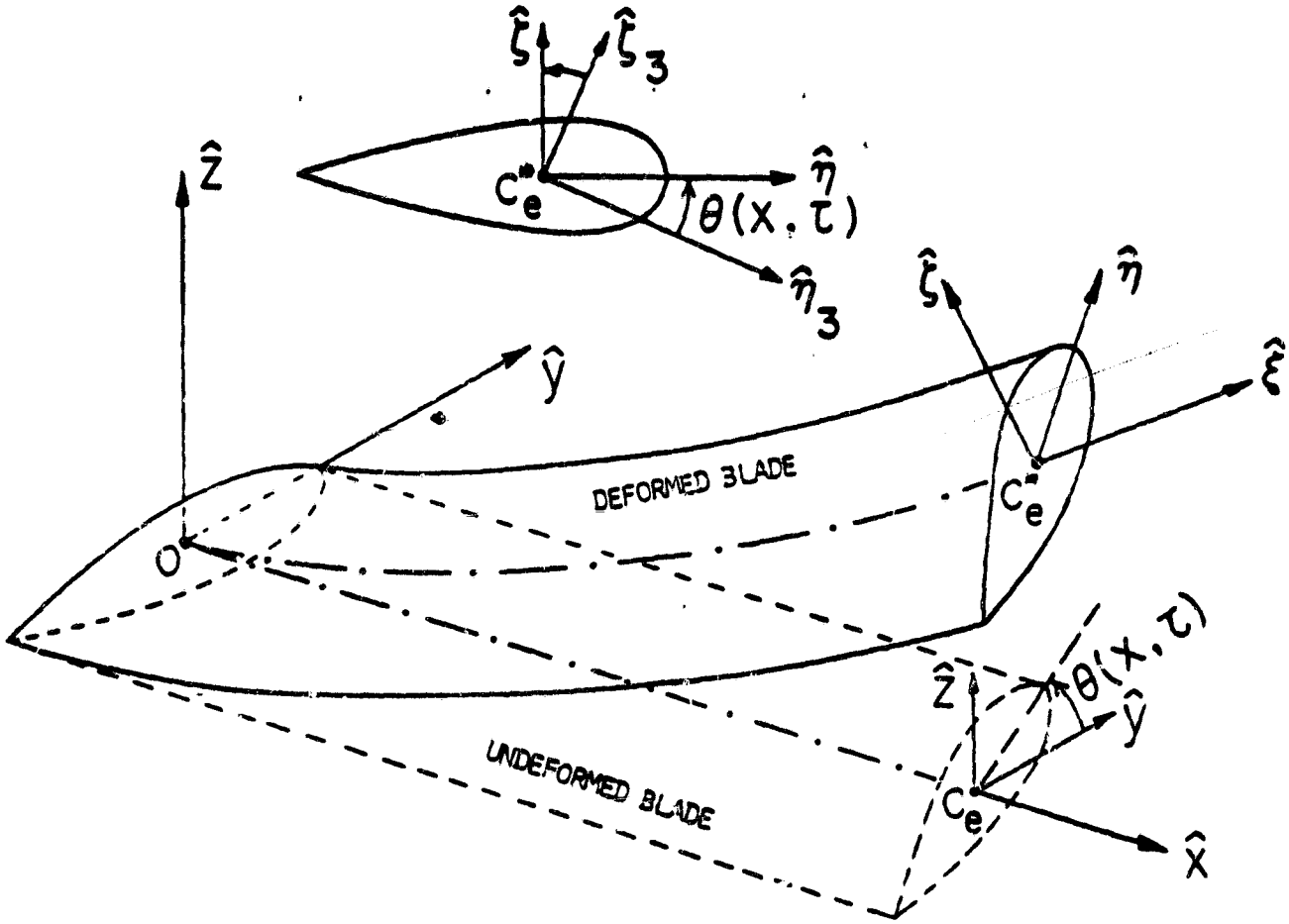

FIG. 2.1 UNDEFORMED AND DEFORMED BLADE CONFIGURATIONS, AND THE ROTATING UNIT VECTOR TRIADS.

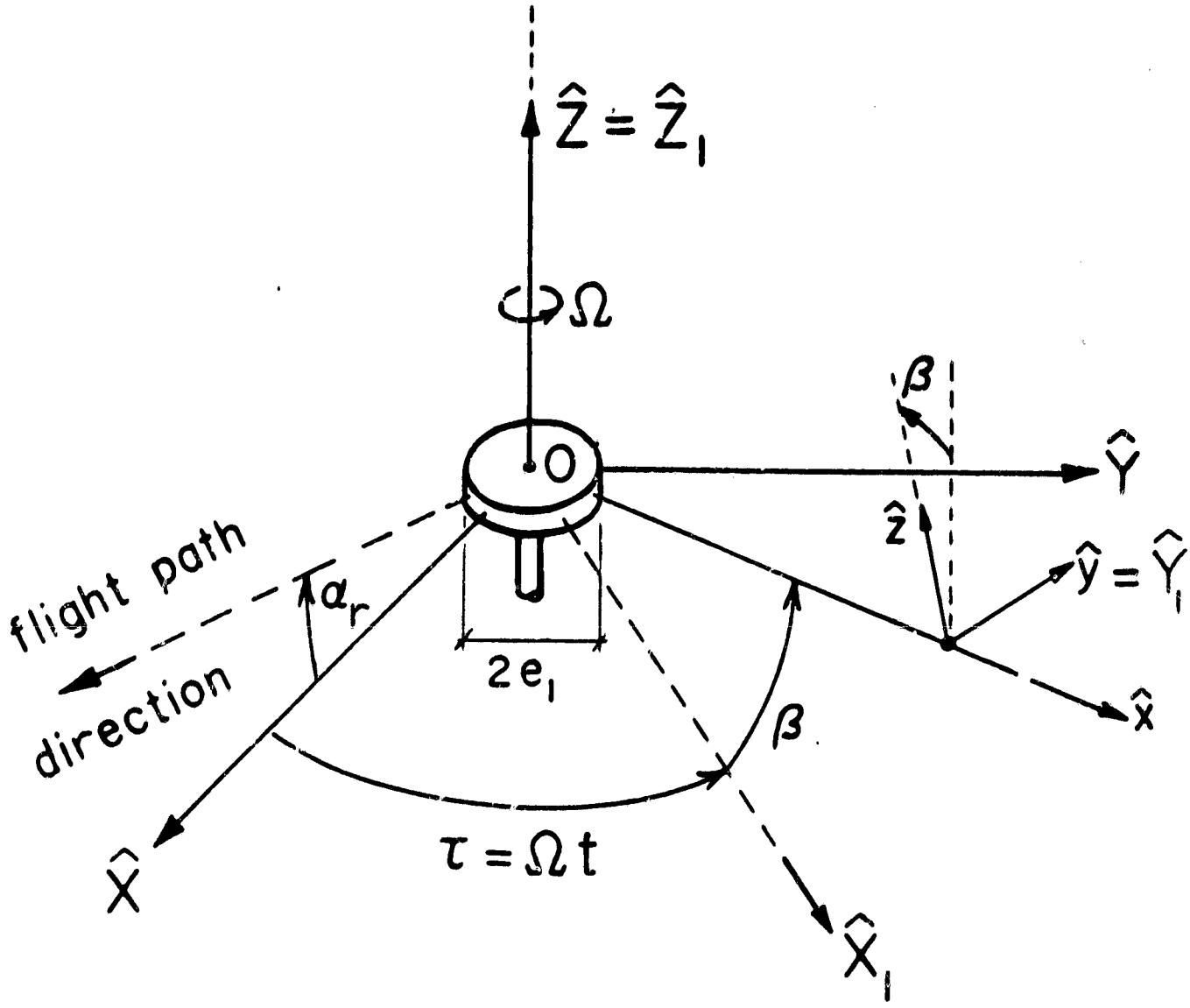

FIG. 2.2 SPATIAL ORIENTATION OF THE ROTOR BLADE REFERENCE UNIT VECTOR TRIAD ($\hat{X}, \hat{Y}, \hat{Z}$).

(η_3, ζ_3) also shown in that figure. It is assumed that the angle $\theta(x,\tau)$ -- the geometric pitch angle of the blade -- is given as

$$\theta(x,\tau) = \theta_c + \theta_{pt}(x) + \sum_{i=1}^{n} \left(\theta_{ic} \cos i\,\tau + \theta_{is} \sin i\,\tau \right) \tag{2.1}$$

where θ_c is the collective pitch angle -- a constant --, $\theta_{pt}(x)$ is a pre-twist angle that may be incorporated to the blade, and θ_{ic} and θ_{is}, $i = 1,2,\ldots,n$, are the harmonic pitch components that may be introduced by a control system. When the blade is elastically undeformed the non-principal η_3 and ζ_3 cross section axes are parallel to the rotating y and z axes, respectively.

During the elastic deformations, point C_e -- the elastic center of the undeformed (but pitched) blade's cross section at location x=x shown in Fig. 2.1 -- moves from its location (x=x, y=0, z=0) to its new location C_e^* whose coordinates relative to the (x,y,z) rotating axes are written respectively, as $Rx + Ru(x,\tau)$, $Rv(x,\tau)$ and $Rw(x,\tau)$. Here u, v and w are the $(\hat{x}, \hat{y}, \hat{z})$ components, non-dimensionalized by the undeformed blade's length R, of the elastic displacement vector of the blade's cross section elastic center. Clearly, $0 \leq x \leq 1$ is a non-dimensional quantity.

In general, each cross section of the blade experiences the elastic displacements $Ru(x,\tau)$, $Rv(x,\tau)$ and $Rw(x,\tau)$ of its elastic center C_e^* and a rotation about C_e^*. The orientation of the principal axes (ξ,η,ζ), through C_e^*, of the blade's cross section at C_e^* may be described by three successive Euler angle rotations. A set of three-axes Euler angles is used here for this purpose. We begin this process by aligning $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ with $(\hat{x}, \hat{y}, \hat{z})$ and then performing the three successive rotations shown in Fig. 2.3*. The first rotation $\theta_z(x,\tau)$ about $\hat{z}$ brings

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*Due to warping of the cross section, the coordinate system (x, η, ζ) is in reality a non-orthogonal system if $\partial\theta/\partial x \neq 0$ since, in this case, $\hat{\xi}$ is not coincident with $\hat{x}$. However, this effect is quite small [75, 76] and, for this reason, it will be neglected here.

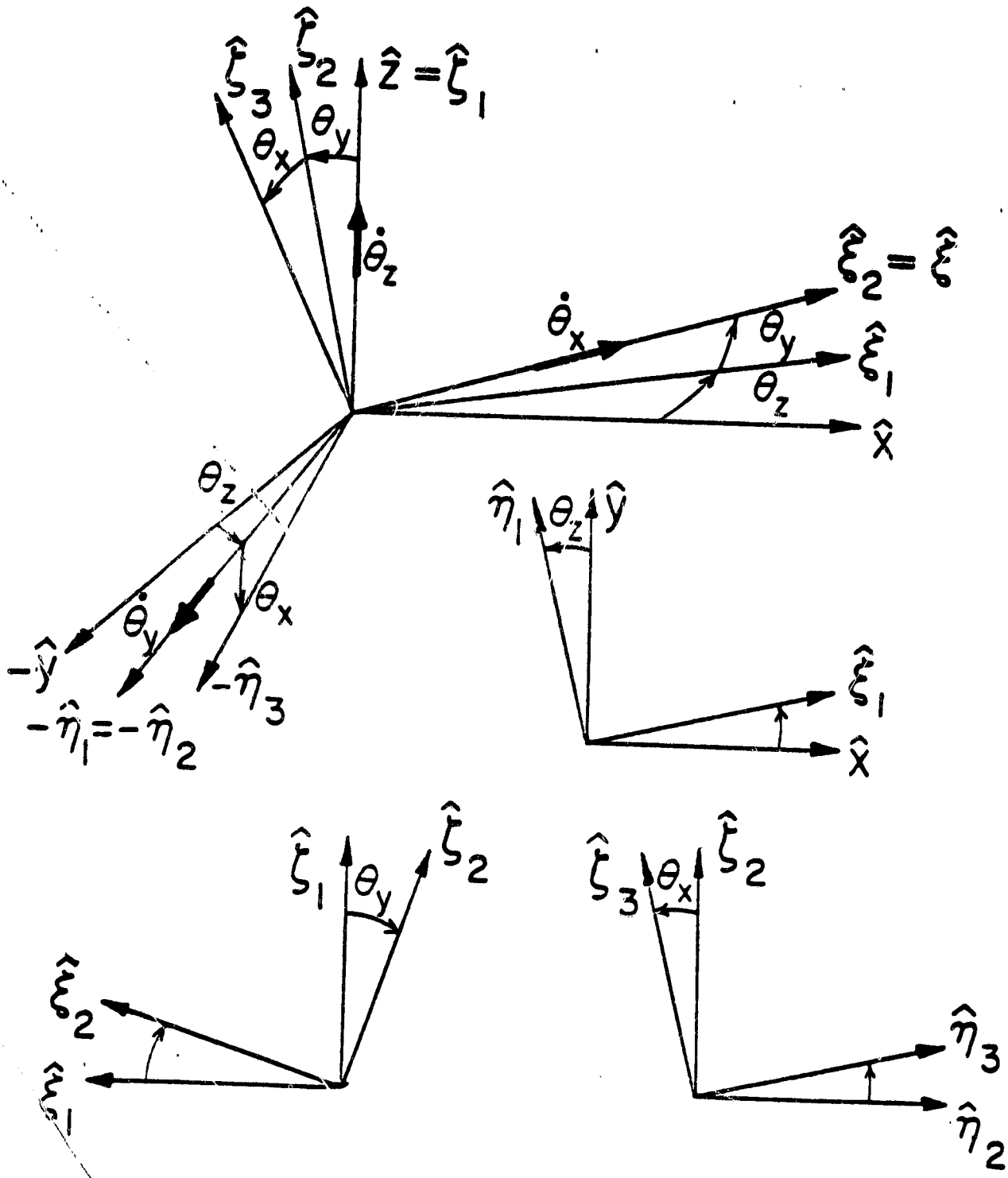

FIG. 2.3 THREE-AXES EULER ANGLE SEQUENCE DESCRIBING THE ORIENTATION OF THE BLADE CROSS SECTION THROUGH C_E^*.

that unit vector triad to $(\hat{\xi}_1, \hat{\eta}_1, \hat{\zeta}_1 = \hat{z})$. The second rotation $\theta_y(x,\tau)$ about the negative direction (a clockwise rotation) of the new position $\hat{\eta}_1$ of the $\hat{\eta}$ unit vector brings $(\hat{\xi}_1, \hat{\eta}_1, \hat{\zeta}_1)$ to $(\hat{\xi}_2, \hat{\eta}_2 = \hat{\eta}_1, \hat{\zeta}_2)$. A third rotation $\theta_x(x,\tau)$ about $\hat{\xi}_2 = \hat{\xi}$ brings this triad to the orientation $(\hat{\xi}, \hat{\eta}_3, \hat{\zeta}_3)$. Thus, the sequence $(\theta_z, \theta_y, \theta_x)$ takes a non-principal triad from its initial orientation associated with the undeformed blade, and aligned with the rotating (x,y,z) reference axes, to its "final" orientation associated with the elastically deformed blade. As indicated in Fig.2.1, an additional rotation of the $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ triad, which is still aligned with $(\hat{\xi}, \hat{\eta}_3, \hat{\zeta}_3)$ after the third Euler angle rotation $\theta_x(x,\tau)$, by an amount equal to the pitch angle $\theta(x,\tau)$ about $\hat{\xi}$ will bring the blade's cross section principal axes to their "final" orientation in space.

The transformation matrix [T] between $(\hat{x},\hat{y},\hat{z})$ and $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$, defined as,

$$\begin{bmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{bmatrix} = [T] \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{2.2}$$

may be readily obtained with the aid of Fig. 2.3. As found in [32, 33, 55] its components t_{ij} are given as

$$[T] = \begin{bmatrix} c\theta_y\, c\theta_z & c\theta_y\, s\theta_z & s\theta_y \\ -c\theta_1\, s\theta_z - s\theta_1\, s\theta_y\, c\theta_z & c\theta_1\, c\theta_z - s\theta_1\, s\theta_y\, s\theta_z & s\theta_1\, c\theta_y \\ s\theta_1\, s\theta_z - c\theta_1\, s\theta_y\, c\theta_z & -s\theta_1\, c\theta_z - c\theta_1\, s\theta_y\, s\theta_z & c\theta_1\, c\theta_y \end{bmatrix} \tag{2.3}$$

where $s\theta_k$ and $c\theta_k$ denote, respectively, $\sin\theta_k$ and $\cos\theta_k$ $(k=1,y,z)$, and θ_1 is given as

$$\theta_1 = \theta(x,\tau) + \theta_x(x,\tau) \tag{2.4}$$

The advantage of incorporating the pitch angle $\theta(x,\tau)$ in the way that is done here is that $\theta(x,\tau)$ simply shows up in the equations as an additive term to the third Euler angle $\theta_x(x,\tau)$, regardless of the fact that the blade is given a known "pre-twist" $\theta_{pt}(x)$, or a general known pitch angle $\theta(x,\tau)$. This approach was first introduced in [33] for pre-twisted blades.

With the position vector of the elastic center C_e^*, relative to the hub center O, given as

$$\bar{r}_e = R\,[(x + u)\,\hat{x} + v\,\hat{y} + w\,\hat{z}] \tag{2.5}$$

it follows that

$$\hat{\xi} = \frac{\partial \bar{r}_e}{\partial (Rr)} = (x + u)^+\,\hat{x} + v^+\,\hat{y} + w^+\,\hat{z} \tag{2.6}$$

where* $(\ )^+ \triangleq \partial(\ )/\partial r$.

Equations (2.2) and (2.6) imply that

$$t_{11} = c\theta_y\, c\theta_z = (x + u)^+ = (1 + u')x^+ \tag{2.7a}$$

$$t_{12} = c\theta_y\, s\theta_z = v^+ = v'\, x^+ \tag{2.7b}$$

---

* The symbol $\triangleq$ is used here to denote "equal by definition". The scalar quantity r denotes arc length, non-dimensionalized by the blade's undeformed length R, measured along the elastic axis.

$$t_{13} = s\theta_y = w^+ = w' x^+ \tag{2.7c}$$

where $(\ )' \triangleq \partial(\ )/\partial x$. Equations (2.7a-c) may be combined to yield the following expression for x^+ in terms of u', v' and w',

$$x^+ = [(1 + u')^2 + v'^2 + w'^2]^{-1/2} \tag{2.8}$$

Equation (2.8) will be used in the sequel to relate the partial derivatives $(\ )^+$ to $(\ )'$ and u', v', w'.

By letting dots denote partial derivative with respect to the non-dimensional time $\tau = \Omega t$, i.e. $(\ )^{\cdot} = \partial(\ )/\partial\tau$, the angular velocity $\bar{\omega}(x,\tau)$ of the principal axis system (ξ,η,ζ) with respect to the inertial frame (X, Y, Z) is obtained directly from Figs. 2.1, 2.2 and 2.3 as

$$\begin{aligned}
\bar{\omega} &= \Omega\,[\dot{\theta}_z\,\hat{z} - \dot{\theta}_y\,\hat{\eta}_1 + \dot{\theta}_1\hat{\xi}] + \Omega\,\hat{Z} \\
&= \Omega\,[\dot{\theta}_1 + (\dot{\theta}_z + c\beta)s\theta_y + s\beta\,c\theta_y\,c\theta_z]\hat{\xi} \\
&\quad + \Omega\,[(\dot{\theta}_z + c\beta)s\theta_1 c\theta_y - \dot{\theta}_y c\theta_1 - (c\theta_1\,s\theta_z + s\theta_1\,s\theta_y\,c\theta_z)s\beta]\hat{\eta} \\
&\quad + \Omega\,[(\dot{\theta}_z + c\beta)c\theta_1\,c\theta_y + \dot{\theta}_y\,s\theta_1 + (s\theta_1\,s\theta_z - c\theta_1\,s\theta_y\,c\theta_z)s\beta]\hat{\zeta} \\
&\triangleq \Omega\,[\omega_\xi\,\hat{\xi} + \omega_\eta\,\hat{\eta} + \omega_\zeta\,\hat{\zeta}]
\end{aligned} \tag{2.9}$$

By making use of Kirchhoff's kinetic analogy [72, Chap. 19], the components

C_ξ , C_η and C_ζ of the elastic axis curvature vector $\bar{C}(x,\tau)$ can be readily obtained from equation (2.9) as

$$R\bar{C} = (\theta_z^+ \hat{z} - \theta_y^+ \hat{\eta}_1 + \theta_1^+ \hat{\xi})$$

$$= [\theta_1^+ + \theta_z^+ s\theta_y]\hat{\xi} + [\theta_z^+ s\theta_1 c\theta_y - \theta_y^+ c\theta_1]\hat{\eta}$$

$$+ [\theta_z^+ c\theta_1 c\theta_y + \theta_y^+ s\theta_1]\hat{\zeta} \triangleq x^+[\rho_\xi \hat{\xi} + \rho_\eta \hat{\eta} + \rho_\zeta \hat{\zeta}] \tag{2.10}$$

Although a total of six dependent variables (u, v, w) and (θ_z, θ_y, θ_x) have been introduced, only four are needed. As seen from equations (2.7a-c), the angles θ_y and θ_z are related to the spatial derivatives u, v and w as

$$\tan \theta_z = v'/(1 + u') \tag{2.11a}$$

$$\sin \theta_y = w'x^+ = w'[(1 + u')^2 + v'^2 + w'^2]^{-1/2} \tag{2.11b}$$

The differential equations of motion to be developed here will involve the elastic displacements $u(x,\tau)$, $v(x,\tau)$ and $w(x,\tau)$, and the third Euler angle $\theta_x(x,\tau)$. To obtain such equations, the extended form of Hamilton's principle [77,78] will be used. For this, expressions for the variations of the kinetic and strain energies of the motion are needed. Expressions for these energies are developed next.

CHAPTER III - KINETIC AND STRAIN ENERGIES OF THE MOTION

The position of an arbitrary point P on the cross section through C_e^*, which is located by the vector $\bar{r}_e$ given by equation (2.5), is shown in Fig. 3.1. Due to warping, P experiences a small axial displacement given approximately as $-(R\psi)(\rho_\xi x^+)\hat{\xi}$, where $\psi(\eta, \zeta)$ is the warp function -- normalized by R^2 -- obtained by solving Laplace's equation for the cross section [79, 80]. In terms of non-dimensional coordinates η and ζ (along the η and ζ axes, respectively), the position vector of P relative to the hub center O is then written as

$$\bar{r}_p = \bar{r}_e + R(\eta\,\hat{\eta} + \zeta\,\hat{\zeta}) - R\psi\rho_\xi x^+\hat{\xi}$$

$$= R[(x + u + \eta t_{21} + \zeta t_{31} - \psi\rho_\xi x^+ t_{11})\hat{x}$$

$$+ (v + \eta t_{22} + \zeta t_{32} - \psi\rho_\xi x^+ t_{12})\hat{y} + (w + \eta t_{23} + \zeta t_{33} - \psi\rho_\xi x^+ t_{13})\hat{z}] \qquad (3.1)$$

The velocity of P relative to O is then

$$\bar{v}_p = d\bar{r}_p/dt = \Omega[(d\bar{r}_p/d\tau)_{\hat{x},\hat{y},\hat{z}} + \hat{Z} \times \bar{r}_p] \qquad (3.2a)$$

where $(d\bar{r}_p/d\tau)_{\hat{x},\hat{y},\hat{z}}$ is the velocity of P as seen by an observer fixed to the rotating reference frame $(\hat{x},\hat{y},\hat{z})$.

Equation (3.2a) yields

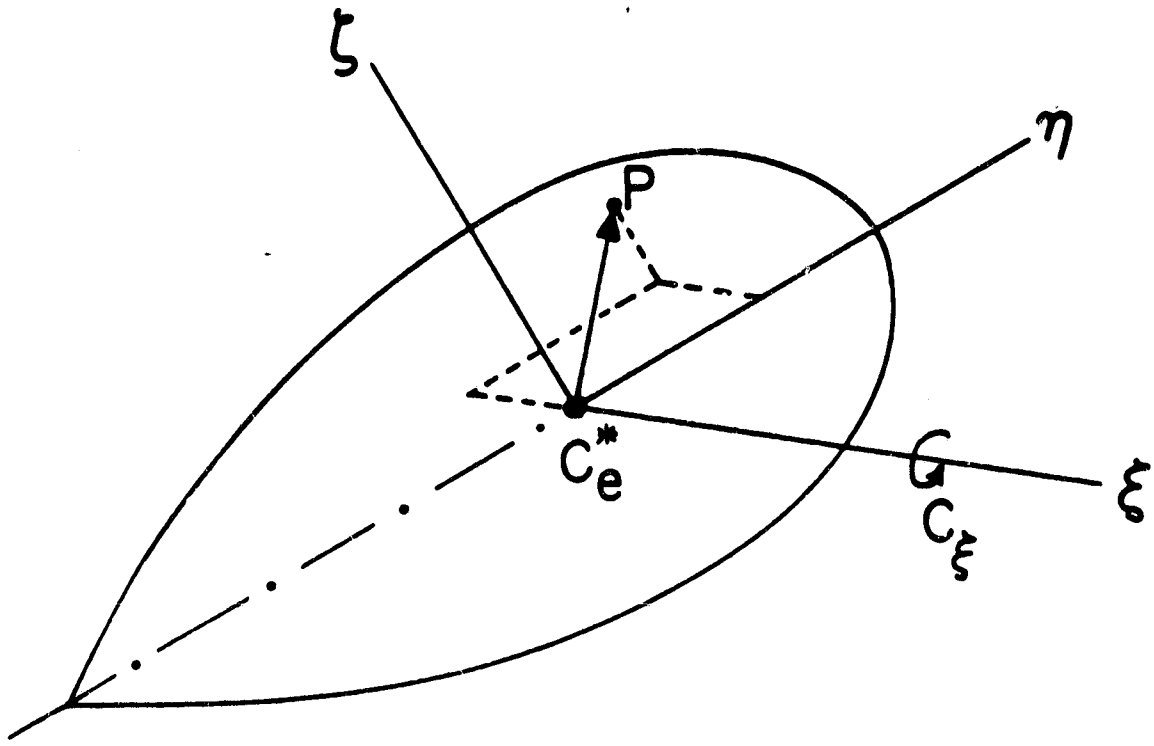

FIG. 3.1 POSITION OF AN ARBITRARY POINT P ON THE BLADE'S CROSS SECTION THROUGH C_E^*, INCLUDING THE EFFECT OF WARPING.

$$\bar{v}_p/(\Omega R) = [\dot{u} - vc\beta + \eta(\dot{t}_{21} - t_{22}c\beta) + \zeta(\dot{t}_{31} - t_{32}c\beta) + \psi\rho_\xi x^+ t_{12}c\beta - (\psi\rho_\xi x^+ t_{11})^\cdot]\hat{x}$$

$$+[\dot{v} + (x + u)c\beta - ws\beta + \eta(\dot{t}_{22} + t_{21}c\beta - t_{23}s\beta)$$

$$+\zeta(\dot{t}_{32} + t_{31}c\beta - t_{33}s\beta) + \psi\rho_\xi x^+(t_{13}s\beta - t_{11}c\beta) - (\psi\rho_\xi x^+ t_{12})^\cdot]\hat{y}$$

$$+[\dot{w} + vs\beta + \eta(\dot{t}_{23} + t_{22}s\beta) + \zeta(\dot{t}_{33} + t_{32}s\beta) - \psi\rho_\xi x^+ t_{12}s\beta - (\psi\rho_\xi x^+ t_{13})^\cdot]\hat{z} \tag{3.2b}$$

Assuming that the velocity of the hub center 0 is constant, which is true for both cases of hover and of non-accelerated forward flight, the kinetic energy of the motion of the blade is then

$$T = \frac{1}{2} R^3 \int_{x=0}^{1} \iint_A \rho\, \bar{v}_p \cdot \bar{v}_p \, d\eta\, d\zeta\, dx \tag{3.3}$$

where ρ is the blade's material density at point P and A is the undeformed blade's cross section area, normalized by R^2. The mass per unit length of the blade, m, is defined as the following area integral

$$m \triangleq R^2 \iint_A \rho\, d\eta\, d\zeta \tag{3.4a}$$

and will be assumed to be a constant. This implies that ρ is not a function of x, that is, $\rho = \rho(\eta, \zeta)$. It will also be assumed here that the blade's cross section has material symmetry about the η axis, so that

$$\iint_A \rho\, \zeta\, d\eta\, d\zeta = \iint_A \rho\, \eta\, \zeta\, d\eta\, d\zeta = 0 \tag{3.4b}$$

and that the warp function $\psi(\eta,\zeta)$ satisfies the following relations

$$\iint_A \rho\psi\, d\eta\, d\zeta = \iint_A \rho\psi\eta\, d\eta\, d\zeta = 0 \tag{3.4c}$$

With the aforementioned assumptions, and defining the following cross section integrals

$$m\, e \triangleq R^2 \iint_A \rho\eta\, d\eta\, d\zeta \tag{3.5a}$$

$$m j_\eta \triangleq R^2 \iint_A \rho\, \zeta^2\, d\eta\, d\zeta \quad ; \quad m j_\zeta \triangleq R^2 \iint_A \rho\, \eta^2\, d\eta\, d\zeta \tag{3.5b}$$

$$j_\xi \triangleq j_\eta + j_\zeta \tag{3.5c}$$

the kinetic energy given by equation (3.3) may be written, after making use of the expressions for the elements t_{ij} of the transformation matrix [T] given by equation (2.3), as

$$\begin{aligned} T/(m\Omega^2R^3) = {} & 1/2 \int_0^1 \{(\dot{u}-vc\beta)^2 + [\dot{v} + (x+u)c\beta - ws\beta]^2 + (\dot{w} + vs\beta)^2\}dx \\ & + 1/2 \int_0^1 (j_\xi\, \omega_\xi^2 + j_\eta\, \omega_\eta^2 + j_\zeta\omega_\zeta^2)\, dx \\ & + \int_0^1 e\{(\dot{u}-vc\beta)\,(\dot{t}_{21} - t_{22}c\beta) + (\dot{w} + vs\beta)\,(\dot{t}_{23} + t_{22}s\beta) \\ & \qquad + [\dot{v} + (x+u)c\beta - ws\beta]\,(\dot{t}_{22} + t_{21}c\beta - t_{23}s\beta)\}dx \\ & + T^*/(m\Omega^2R^3) \end{aligned} \tag{3.6}$$

The terms not shown explicitly in equation (3.6) are included in T^*. The expression for T^* is given in the Appendix. The quantities e and j_α ($\alpha = \xi, \eta, \zeta$) defined by equations (3.5a-c) are the blade's mass center offset from the elastic axis, normalized by R, and its distributed mass moments of inertia, normalized by mR^2.

To obtain the expression for the strain energy of the deformed blade, a strain tensor, represented here by a matrix $[\varepsilon]$ with elements ε_{ij}, is needed. As a measure of the deformation of the blade, the square of the distances between two infinitesimally close points on the blade, before and after the deformation, is used [80,81]. The nonlinear strain-displacement relations for the deformed blade are sought for the case of large displacements, and for this the strain components ε_{ij} are formulated in terms of the increments dr, $d\eta$ and $d\zeta$ for the deformed configuration. The strain components are still assumed to be small enough, however, so that Hooke's linear law relating the stresses at any point P of the deformed blade's material to the strains at P is still applicable. The matrix $[\varepsilon]$ of strain components ε_{ij} is thus determined as

$$d\bar{r}_P \cdot d\bar{r}_P - d\bar{r}_{P0} \cdot d\bar{r}_{P0} = 2R^2 [dr \; d\eta \; d\zeta] \; [\varepsilon] \begin{bmatrix} dr \\ d\eta \\ d\zeta \end{bmatrix} \tag{3.7a}$$

where, as obtained from equations (3.1) and (2.10),

$$\bar{r}_{P0} = \bar{r}_P \Big|_{u=v=w=\theta_x=0} = R[(x - \psi_0\theta')\hat{x} + (\eta_0 c\theta - \zeta_0 s\theta)\hat{y} + (\eta_0 s\theta + \zeta_0 c\theta)\hat{z}] \tag{3.7b}$$

In equation (3.7b), ψ_0, η_0 and ζ_0 denote, respectively, ψ, η and ζ when $u = v = w = \theta_x = 0$. As discussed in [32], the difference between the quantities α_0 and α ($\alpha = \psi, \eta, \zeta$) for the blade's cross section is very small. In-plane cross section distortion will be neglected here, and the approximations $\eta_0 \approx \eta$, $\zeta_0 \approx \zeta$ and $\psi_0 \approx \psi$ are adopted.

From equations (3.1) and (3.7b) it follows that

$$d\bar{r}_{P0}/R = \{(x^+ - \psi\theta'' x^+)dr - \theta'[(\partial\psi/\partial\eta)d\eta + (\partial\psi/\partial\zeta)d\zeta]\}\hat{x}$$

$$+[-\theta'(\eta s\theta + \zeta c\theta)dr + (c\theta)d\eta - (s\theta)d\zeta]\hat{y}$$

$$+[\theta'(\eta c\theta - \zeta s\theta)dr + (s\theta)d\eta + (c\theta)d\zeta]\hat{z} \qquad (3.8)$$

$$d\bar{r}_P/R = [(x + u)^+ \hat{x} + v^+\hat{y} + w^+\hat{z} - \psi(\rho_\xi x^+)^+\hat{\xi}]dr$$

$$+(\hat{\eta} - \hat{\xi}\,\rho_\xi x^+\,\partial\psi/\partial\eta)d\eta + (\hat{\zeta} - \hat{\xi}\,\rho_\xi x^+\,\partial\psi/\partial\zeta)d\zeta$$

$$+ x^+(\rho_\xi\hat{\xi} + \rho_\eta\hat{\eta} + \rho_\zeta\hat{\zeta}) \times (\eta\,\hat{\eta} + \zeta\,\hat{\zeta} - \psi\rho_\xi\, x^+\hat{\xi})dr \qquad (3.9a)$$

Making use of equations (2.1), (2.2) and (2.7a-c), and defining, for the sake of compacteness, $\hat{x}_1 = \hat{x}$, $\hat{x}_2 = \hat{y}$ and $\hat{x}_3 = \hat{z}$, equation (3.9a) may be written, after performing the cross product indicated and noticing that $(\rho_\xi x^+)^+ = x^+(\rho_\xi x^+)'$, as

$$d\bar{r}_p/R = \sum_{i=1}^{3} \{[t_{1i} + (\zeta\rho_\eta - \eta\rho_\zeta)t_{1i}x^+ - \psi(\rho_\xi x^+)' \; t_{1i}x^+$$

$$-(\zeta + \psi\rho_\zeta x^+)\rho_\xi x^+ t_{2i} + (\eta + \psi\rho_\eta x^+)\rho_\xi x^+ t_{3i}]dr$$

$$+(t_{2i} - t_{1i}\,\rho_\xi x^+\,\partial\psi/\partial\eta)d\eta + (t_{3i} - t_{1i}\,\rho_\xi x^+\,\partial\psi/\partial\zeta)d\zeta\}\hat{x}_i \tag{3.9b}$$

Equations (3.7a), (3.8) and (3.9b) immediately yield the following expressions for the strain components ε_{ij},

$$2\varepsilon_{11} = [1 + (\zeta\rho_\eta - \eta\rho_\zeta)x^+ - \psi x^+(\rho_\xi x^+)']^2 + [(\zeta + \psi\rho_\zeta x^+)^2 + (\eta + \psi\rho_\eta x^+)^2](\rho_\xi x^+)^2$$

$$- [(1 - \psi\theta'')^2 + (\eta^2 + \zeta^2)\theta'^2]x^{+2} \tag{3.10a}$$

$$2\varepsilon_{12} = -(\zeta + \partial\psi/\partial\eta)\;(\rho_\xi - \theta')x^+ + [(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\eta - \psi\rho_\zeta]\rho_\xi x^{+2}$$

$$+ [\rho_\xi x^+(\rho_\xi x^+)' - \theta'\;\theta'']x^+\psi\;\partial\psi/\partial\eta \tag{3.10b}$$

$$2\varepsilon_{13} = (\eta - \partial\psi/\partial\zeta)\;(\rho_\xi - \theta')x^+ + [(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\zeta + \psi\rho_\eta]\rho_\xi x^{+2}$$

$$+ [\rho_\xi x^+(\rho_\xi x^+)' - \theta'\;\theta'']x^+\psi\;\partial\psi/\partial\zeta \tag{3.10c}$$

$$2\varepsilon_{22} = [(\rho_\xi x^+)^2 - \theta'^2]\;(\partial\psi/\partial\eta)^2 \tag{3.10d}$$

$$2\varepsilon_{23} = [(\rho_\xi x^+)^2 - \theta'^2]\;(\partial\psi/\partial\eta)(\partial\psi/\partial\zeta) \tag{3.10e}$$

$$2\varepsilon_{33} = [(\rho_\xi x^+)^2 - \theta'^2]\;(\partial\psi/\partial\zeta)^2 \tag{3.10f}$$

The strain energy, U, is given by

$$U = \frac{R^3}{2} \iiint_{(\text{blade})} [\sigma_{11}\,\varepsilon_{11} + \sigma_{22}\,\varepsilon_{22} + \sigma_{33}\,\varepsilon_{33} + 2(\sigma_{12}\,\varepsilon_{12} + \sigma_{13}\,\varepsilon_{13} + \sigma_{23}\,\varepsilon_{23})]\,d\eta\,d\zeta\,dx \tag{3.11}$$

where σ_{ij} are the stresses acting on the blade. For a linear and isotropic elastic material, the stresses and the components ε_{ij} of the strain tensor at any point P of the blade are related by Hooke's law as

$$\varepsilon_{11} = [\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})]/E_y \tag{3.12a}$$

$$\varepsilon_{22} = [\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})]/E_y \tag{3.12b}$$

$$\varepsilon_{33} = [\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})]/E_y \tag{3.12c}$$

$$\varepsilon_{12} = (1 + \nu)\,\sigma_{12}/E_y = \sigma_{12}/(2G_s) \tag{3.12d}$$

$$\varepsilon_{13} = (1 + \nu)\,\sigma_{13}/E_y = \sigma_{13}/(2G_s) \tag{3.12e}$$

$$\varepsilon_{23} = (1 + \nu)\,\sigma_{23}/E_y = \sigma_{23}/(2G_s) \tag{3.12f}$$

where $E_y(x)$, ν and $G_s(x) = E_y/[2(1 + \nu)]$ are, respectively, the Young modulus, Poisson ratio and the shear modulus for the blade's material.

With the assumption that the blade's maximum cross section dimension is much smaller than its length, the blade is approximated as a beam, and the normal

stresses σ_{22} and σ_{33} are neglected. Under this assumption, equations (3.12a-c) reduce to

$$\varepsilon_{11} \simeq \sigma_{11}/E_y \quad ; \quad \varepsilon_{22} \simeq \varepsilon_{33} \simeq -\nu\varepsilon_{11} \tag{3.13}$$

and the strain energy U, calculated from equations (3.11), (3.12d-f) and (3.13), becomes

$$\begin{aligned}
U = \frac{R^3}{8} \int_{x=0}^{1} \iint_A E_y \Big\{ & 1 - x^{+2} + 2(\zeta\rho_\eta - \eta\rho_\zeta)x^{+} + (\zeta\rho_\eta - \eta\rho_\zeta)^2 x^{+2} + (\eta^2 + \zeta^2)\,(\rho_\xi^2 - \theta'^2)x^{+2} \\
& + \psi^2 \Big[(\rho_\xi x^{+})'^2 + (\rho_\eta^2 + \rho_\zeta^2)\rho_\xi^2 x^{+2} - \theta''^2\Big] x^{+2} + 2\psi \Big[\theta'' x^{+2} - (\rho_\xi x^{+})' x^{+} \\
& + (\zeta\rho_\zeta + \eta\rho_\eta)\rho_\xi^2 x^{+3} - (\zeta\rho_\eta - \eta\rho_\zeta)x^{+2}\,(\rho_\xi x^{+})'\Big]\Big\}^2 d\eta\, d\zeta\, dx \\
& + \frac{R^3}{2} \int_{x=0}^{1} \iint_A G_s \Big\{ (\rho_\xi - \theta')^2 \Big[(\zeta + \partial\psi/\partial\eta)^2 + (\eta - \partial\psi/\partial\zeta)^2\Big] x^{+2} \\
& + 2\rho_\xi(\rho_\xi - \theta') \Big\{ (\eta - \partial\psi/\partial\zeta) \Big[(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\zeta + \psi\rho_\eta\Big] x^{+3} \\
& - (\zeta + \partial\psi/\partial\eta) \Big[(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\eta - \psi\rho_\zeta\Big] x^{+3} \Big\}\Big\}\, d\eta\, d\zeta\, dx + U_s^{*}
\end{aligned} \tag{3.14}$$

where the term U_s^{*} is given in the Appendix.

Assuming that the blade cross section is symmetric about the η axis, and that the warping function ψ is antisymmetric, it follows that

$$\iint_A \zeta \, d\eta \, d\zeta = \iint_A \zeta\eta \, d\eta \, d\zeta = \iint_A \zeta\eta^2 \, d\eta \, d\zeta = \iint_A \zeta^3 \, d\eta \, d\zeta = 0 \tag{3.15a}$$

$$\iint_A \psi \, d\eta \, d\zeta = \iint_A \eta\psi \, d\eta \, d\zeta = \iint_A \psi(\eta^2 + \zeta^2) d\eta \, d\zeta = 0 \tag{3.15b}$$

Non-dimensionalizing the Young and the shear moduli as

$$E \triangleq E_y/(m\Omega^2) \quad ; \quad G \triangleq G_s/(m\Omega^2) \tag{3.15c}$$

and defining the following non-dimensional cross section integrals,

$$D_\xi \triangleq G \iint_A [(\zeta + \partial\psi/\partial\eta)^2 + (\eta - \partial\psi/\partial\zeta)^2] d\eta \, d\zeta \tag{3.16a}$$

$$D_\eta \triangleq E \iint_A \zeta^2 \, d\eta \, d\zeta \quad ; \quad D_\zeta \triangleq E \iint_A \eta^2 \, d\eta \, d\zeta \tag{3.16b}$$

$$I_\xi \triangleq \iint_A (\eta^2 + \zeta^2) d\eta \, d\zeta \quad ; \quad A \triangleq \iint_A d\eta \, d\zeta \quad ; \quad Ae_A \triangleq \iint_A \eta \, d\eta \, d\zeta \tag{3.16c}$$

$$B_1 \triangleq \iint_A (\eta^2 + \zeta^2)^2 \, d\eta \, d\zeta \quad ; \quad B_2 \triangleq \iint_A \eta(\eta^2 + \zeta^2) d\eta \, d\zeta \tag{3.16d}$$

$$B_3 \triangleq \iint_A \zeta^2\eta \, d\eta \, d\zeta \quad ; \quad B_4 \triangleq B_2 - B_3 \tag{3.16e}$$

$$C_1 \triangleq \iint_A \psi^2 \, d\eta \, d\zeta \quad ; \quad C_1^* \triangleq \iint_A \zeta \, \psi \, d\eta \, d\zeta \tag{3.16f}$$

$$J_\eta \triangleq \iint_A [(\eta - \partial\psi/\partial\zeta) \, (\psi - \zeta \, \partial\psi/\partial\zeta) + (\zeta + \partial\psi/\partial\eta)\zeta \, \partial\psi/\partial\eta] d\eta \, d\zeta \tag{3.16g}$$

$$J_\zeta \triangleq \iint_A [(\eta - \partial\psi/\partial\zeta)\eta\, \partial\psi/\partial\zeta + (\zeta + \partial\psi/\partial\eta)\,(\psi - \eta\, \partial\psi/\partial\eta)]\, d\eta\, d\zeta \tag{3.16h}$$

the strain energy given by equation (3.14) may be written as

$$U/(m\Omega^2R^3) = \frac{1}{2}\int_0^1 [D_\xi(\rho_\xi - \theta')^2 + D_\eta\, \rho_\eta^2 + D_\zeta\, \rho_\zeta^2]x^{+2}\, dx + \frac{1}{8}\int_0^1 EA(1-x^{+2})^2\, dx$$

$$+ \frac{1}{4}\int_0^1 (1 - x^{+2})\, \{[EI_\xi(\rho_\xi^2 - \theta'^2) + D_\eta\rho_\eta^2 + D_\zeta\rho_\zeta^2]x^{+2} - 2EAe_A\rho_\zeta x^+\}dx$$

$$+ \frac{1}{2}\int_0^1 \{\frac{1}{4}\, EB_1(\rho_\xi^2 - \theta'^2)^2x^{+4} - EB_2\rho_\zeta(\rho_\xi^2 - \theta'^2)x^{+3}$$

$$-E(3B_3\, \rho_\eta^2\, \rho_\zeta + B_4\rho_\zeta^3)x^{+3} + EC_1\theta_x''^2 - 2EC_1^*\rho_\eta(\rho_\xi' - \theta'')\}\, dx$$

$$+ \int_0^1 G(J_\eta\rho_\eta + J_\zeta\rho_\zeta)\, (\rho_\xi - \theta')\rho_\xi x^{+3}\, dx + U^*/(m\Omega^2R^3) \tag{3.17}$$

The term U^*, which includes most of the contribution due to warping, is given in the Appendix .

The quantities D_ξ, D_η and D_ζ defined by equations (3.16a,b) are, respectively, the torsional and flexural stiffnesses of the blade, normalized by $m\Omega^2R^4$, while I_ξ -- defined by the first of equations (3.16c) -- is the polar moment of inertia for the cross section (referred to its shear center C_e^*) normalized by R^4; A is the blade's cross section area normalized by R^2, and e_A is the cross section area centroid offset from the shear center, normalized by R. It is worth noticing that the curvature components $\rho_\xi x^+$, $\rho_\eta x^+$ and $\rho_\zeta x^+$ appear directly in the expression for the strain energy U. This is a direct consequence of defining U via the strain components formulated in terms of the increments dr, $d\eta$ and $d\zeta$ for the deformed configuration.

CHAPTER IV - HAMILTON'S PRINCIPLE AND THE DIFFERENTIAL EQUATIONS OF MOTION, AND BOUNDARY CONDITIONS

In this report the extended form of Hamilton's principle [77,78] is used to obtain the differential equations of motion for the blade, and their boundary conditions. This principle may be expressed as

$$\int_{\tau_1}^{\tau_2} [\delta(T) - \delta(U) + \delta W]d\tau \quad \underline{\Delta}(m\Omega^2R^3)\delta I = 0 \tag{4.1}$$

where δW -- which is not necessarily the variation of a function W, and hence the the distinction from the notation used for $\delta(U)$ for example -- designates the virtual work done by the external and damping forces. Here, δW is expressed in terms of the virtual displacements of the non-dimensional generalized coordinates u,v,w and θ_x as,

$$\delta W = m\Omega^2R^3 \{ \delta W_B \Big|_{x=0}^{1} + \int_0^1 (Q_u \, \delta u + Q_v \, \delta v + Q_w \, \delta w + Q_{\theta_x} \, \delta\theta_x)dx \} \tag{4.2}$$

The quantities Q_u, Q_v and Q_w are the distributed forces associated with the virtual displacements $R\delta u$, $R\delta v$ and $R\delta w$, respectively, normalized by $m\Omega^2R$, while Q_{θ_x} is the distributed moment associated with the virtual rotation $\delta\theta_x$, normalized by $m\Omega^2R^2$. The term δW_B is included in equation 4.2 to account for the cases where the virtual work done by the distributed forces acting on the system is also dependent on the variation of the spatial derivative of any of the variables u,v,w and θ_x. In such cases, an integration by parts in the expression for δW immediately yields a δW_B term, as in equations (6.15a,b) in Chapter VI.

From equations (2.3) and (2.8) to (2.10) the following quantities, which appear in the expressions for the variations of the kinetic and strain energies, are obtained

$$\delta\omega_\xi = [(\dot{\theta}_z + c\beta)c\theta_y - s\beta\, s\theta_y\, c\theta_z]\delta\theta_y - s\beta\, c\theta_y\, s\theta_z\, \delta\theta_z + \delta\dot{\theta}_x + s\theta_y\, \delta\dot{\theta}_z \tag{4.3a}$$

$$\delta\omega_\eta = \omega_\zeta\delta\theta_x - [(\dot{\theta}_z + c\beta)s\theta_y + t_{11}s\beta]s\theta_1\, \delta\theta_y - t_{22}s\beta\, \delta\theta_z - c\theta_1\delta\dot{\theta}_y + t_{23}\, \delta\dot{\theta}_z \tag{4.3b}$$

$$\delta\omega_\zeta = -\,\omega_\eta\, \delta\theta_x - [(\dot{\theta}_z + c\beta)s\theta_y + t_{11}s\beta]c\theta_1\, \delta\theta_y - t_{32}s\beta\, \delta\theta_z + s\theta_1\, \delta\dot{\theta}_y + t_{33}\, \delta\dot{\theta}_z \tag{4.3c}$$

$$\delta\rho_\xi = \theta_z'\, c\theta_y\, \delta\theta_y + \delta\theta_x' + s\theta_y\, \delta\theta_z' \tag{4.3d}$$

$$\delta\rho_\eta = \rho_\zeta\, \delta\theta_x - \theta_z'\, s\theta_y\, s\theta_1\, \delta\theta_y - c\theta_1\, \delta\theta_y' + c\theta_y\, s\theta_1\, \delta\theta_z' \tag{4.3e}$$

$$\delta\rho_\zeta = -\,\rho_\eta\, \delta\theta_x - \theta_z'\, s\theta_y\, c\theta_1\, \delta\theta_y + s\theta_1\, \delta\theta_y' + c\theta_y\, c\theta_1\, \delta\theta_z' \tag{4.3f}$$

$$\delta t_{21} = t_{31}\, \delta\theta_x - t_{23}\, c\theta_z\, \delta\theta_y - t_{22}\, \delta\theta_z \tag{4.3g}$$

$$\delta t_{22} = t_{32}\, \delta\theta_x - t_{23}\, s\theta_z\, \delta\theta_y + t_{21}\, \delta\theta_z \tag{4.3h}$$

$$\delta t_{23} = t_{33}\, \delta\theta_x - s\theta_1\, s\theta_y\, \delta\theta_y \tag{4.3i}$$

$$x^+\delta x^+ = -\,[(1 + u')\delta u' + v'\delta v' + w'\delta w']/[(1 + u'^2) + v'^2 + w'^2]^2 \tag{4.3j}$$

The virtual variations $\delta\theta_y$ and $\delta\theta_z$ may be found directly from equations (2.11a,b) as

$$\delta\theta_y = \{\frac{-w'\,[(1+u')\delta u' + v'\delta v']}{[(1+u')^2 + v'^2]^{1/2}} + [(1+u')^2 + v'^2]^{1/2}\,\delta w'\}/[(1+u')^2 + v'^2 + w'^2]$$

$$= (\partial\theta_y/\partial u')\delta u' + (\partial\theta_y/\partial v')\delta v' + (\partial\theta_y/\partial w')\delta w' \tag{4.4a}$$

$$\delta\theta_z = [-v'\,\delta u' + (1+u')\delta v']/[1+u')^2 + v'^2] \tag{4.4b}$$

$$= (\partial\theta_z/\partial u')\delta u' + (\partial\theta_z/\partial v')\delta v' \tag{4.4b}$$

From equations (4.1) to (4.4b), (3.6) and (3.17) the following differential equations of motion are obtained after performing a few integrations by parts in equation (4.1)

$$\{(T_{\theta_y} + U_{\theta_y})\partial\theta_y/\partial u' + (T_{\theta_z} + U_{\theta_z})\partial\theta_z/\partial u' + T_e(1+u') + h_u\}' \triangleq G_u'(x,\tau)$$

$$= (u + et_{21})^{\cdot\cdot} - 2(v + et_{22})^{\cdot}c\beta + (w + et_{23})s\beta\, c\beta - (x + u + et_{21})c^2\beta - Q_u \tag{4.5a}$$

$$\{(T_{\theta_y} + U_{\theta_y})\partial\theta_y/\partial v' + (T_{\theta_z} + U_{\theta_z})\partial\theta_z/\partial v' + T_e v' + h_v\}' \triangleq G_v'(x,\tau)$$

$$= (v + et_{22})^{\cdot\cdot} + 2(u + et_{21})^{\cdot}\, c\beta - 2(w + et_{23})^{\cdot}\, s\beta - (v + et_{22}) - Q_v \tag{4.5b}$$

$$\{(T_{\theta_y} + U_{\theta_y})\partial\theta_y/\partial w' + T_e w' + h_w\}' \triangleq G_w'(x,\tau)$$

$$= (w + et_{23})^{\cdot\cdot} + 2(v + et_{22})^{\cdot}\, s\beta + (x + u + et_{21})s\beta\, c\beta - (w + et_{23})s^2\beta - Q_w \tag{4.5c}$$

$$T_{\theta_x} - U_{\theta_x} + h_{\theta_x} + Q_{\theta_x} = 0 \tag{4.5d}$$

where

$$T_e = -x^{+4}\Big\{D_\xi(\rho_\xi - \theta')^2 + (3 - 2x^{+2})\ (D_\eta\rho_\eta^2 + D_\zeta\rho_\zeta^2)/2 + EI_\xi(1 - 2x^{+2})\ (\rho_\xi^2 - \theta'^2)/2$$

$$- EAe_A\rho_\zeta(1 - 3x^{+2})/\ (2x^+) + E(\rho_\xi^2 - \theta'^2)x^+[B_1(\rho_\xi^2 - \theta'^2)x^+ - 3B_2\rho_\zeta]/2 - EA(1 - x^{+2})/2$$

$$- 3E(3B_3\rho_\eta^2\ \rho_\zeta + B_4\rho_\zeta^3)x^+/2 + 3G(J_\eta\rho_\eta + J_\zeta\rho_\zeta)\ (\rho_\xi - \theta')\rho_\xi x^+\Big\} \tag{4.6a}$$

$$T_{\theta_y} = (j_\zeta\omega_\zeta s\theta_1 - j_\eta\omega_\eta c\theta_1)^\bullet - j_\xi\omega_\xi[(\dot\theta_z + c\beta)c\theta_y - s\beta\ s\theta_y c\theta_z]$$

$$+(j_\eta\omega_\eta\ s\theta_1 + j_\zeta\omega_\zeta c\theta_1)\ [(\dot\theta_z + c\beta)s\theta_y + s\beta\ c\theta_y\ c\theta_z]$$

$$-e\Big\{\Big[\ddot u - \Big(2\dot v - ws\beta + (x + u)c\beta\Big)c\beta\Big]c\theta_z c\theta_y + \Big[\ddot v + 2\dot u\ c\beta - 2\dot w\ s\beta - v\Big]s\theta_z c\theta_y$$

$$+ \Big[\ddot w + \Big(2\dot v - w\ s\beta + (x + u)\ c\beta\Big)s\beta\Big]s\theta_y\Big\}s\theta_1 \tag{4.6b}$$

$$T_{\theta_z} = [j_\xi\omega_\xi\ s\theta_y + (j_\eta\omega_\eta s\theta_1 + j_\zeta\omega_\zeta c\theta_1)c\theta_y]^\bullet + [j_\xi\omega_\xi c\theta_y s\theta_z + j_\eta\omega_\eta(c\theta_1 c\theta_z - s\theta_1 s\theta_y s\theta_z)$$

$$- j_\zeta\omega_\zeta(s\theta_1 c\theta_z + c\theta_1 s\theta_y s\theta_z)]s\beta - e\Big\{\Big[\ddot u - \Big(2\dot v - w\ s\beta + (x + u)c\beta\Big)c\beta\Big]t_{22}$$

$$- \Big[\ddot v + 2\dot u\ c\beta - 2\dot w\ s\beta - v\Big]t_{21}\Big\} \tag{4.6c}$$

$$T_{\theta_x} = (J_\eta - J_\zeta)\omega_\eta\omega_\zeta - J_\xi\dot{\omega}_\xi - e\Big\{\Big[\ddot{u} - \Big(2\dot{v} - w\, s\beta + (x + u)c\beta\Big)\, c\beta\Big]t_{31}$$

$$+ \Big[\ddot{v} + 2\dot{u}\, c\beta - 2\dot{w}\, s\beta - v\Big]t_{32} + \Big[\ddot{w} + \Big(2\dot{v} - w\, s\beta + (x + u)c\beta\Big)s\beta\Big]c\theta_1\, c\theta_y\Big\} \qquad (4.6d)$$

$$U_{\theta_y} = \Big\{D_\xi(\rho_\xi - \theta') + \rho_\xi\Big[(1 - x^{+2})EI_\xi/2 + EB_1(\rho_\xi^2 - \theta'^2)x^{+2}/2 - EB_2\rho_\zeta x^+\Big] + (EC_1^*\rho_\eta)'/x^{+2}$$

$$+ G(J_\eta\rho_\eta + J_\zeta\rho_\zeta)\,(2\rho_\xi - \theta')x^+\Big\}x^{+2}\theta_z'\, c\theta_y - \tfrac{1}{2}(3 - x^{+2})\,(D_\eta\rho_\eta s\theta_1 + D_\zeta\rho_\zeta c\theta_1)x^{+2}\theta_z' s\theta_y$$

$$+ \Big\{\tfrac{1}{2}\Big[EA(1 - x^{+2})e_A\, x^+ + EB_2(\rho_\xi^2 - \theta'^2)x^{+3} + 3E(B_3\rho_\eta^2 + B_4\rho_\zeta^2)x^{+3}\Big]c\theta_1$$

$$+ \Big[3EB_3\rho_\eta\rho_\zeta\, x^{+3} + EC_1^*(\rho_\xi' - \theta'')\Big]s\theta_1 - G(J_\eta\, s\theta_1 + J_\zeta\, c\theta_1)\rho_\xi(\rho_\xi - \theta')x^{+3}\Big\}\theta_z'\, s\theta_y$$

$$+ \Big\{\tfrac{1}{2}\Big[(3-x^{+2})\,(D_\eta\rho_\eta c\theta_1 - D_\zeta\rho_\zeta s\theta_1) + EB_2(\rho_\xi^2 - \theta'^2)x^+\, s\theta_1 + 3E(B_3\rho_\eta^2 + B_4\rho_\zeta^2)x^+ s\theta_1\Big]x^{+2}$$

$$+ \tfrac{1}{2}\, EA(1 - x^{+2})e_A x^+\, s\theta_1 - 3EB_3\,\rho_\eta\rho_\zeta x^{+3}\, c\theta_1 - EC_1^*(\rho_\xi' - \theta'')c\theta_1$$

$$+ G(J_\eta\, c\theta_1 - J_\zeta\, s\theta_1)\rho_\xi(\rho_\xi - \theta')\, x^{+3}\Big\}' \qquad (4.6e)$$

$$U_{\theta_z} = -\Big\{\Big[D_\xi(\rho_\xi - \theta') + \frac{E}{2} I_\xi(1 - x^{+2})\rho_\xi + \frac{E}{2} B_1(\rho_\xi^2 - \theta'^2)\rho_\xi x^{+2} - EB_2\rho_\zeta\rho_\xi x^{+} + (EC_1^*\rho_\eta)'/x^{+2}$$

$$+ G(J_\eta\rho_\eta + J_\zeta\rho_\zeta)\ (2\rho_\xi - \theta')x^{+}\Big]x^{+2}\ s\theta_y + \frac{1}{2}(3-x^{+2})\ (D_\eta\rho_\eta\ s\theta_1 + D_\zeta\rho_\zeta c\theta_1)x^{+2}c\theta_y$$

$$- \frac{1}{2}\Big[EA(1-x^{+2})e_A x^{+} + EB_2(\rho_\xi^2 - \theta'^2)x^{+3} + 3E(B_3\rho_\eta^2 + B_4\rho_\zeta^2)x^{+3}\Big]c\theta_1\ c\theta_y$$

$$- \Big[3EB_3\ \rho_\eta\rho_\zeta x^{+3} + EC_1^*(\rho_\xi' - \theta'')\Big]s\theta_1\ c\theta_y + G(J_\eta\ s\theta_1 + J_\zeta\ c\theta_1)\rho_\xi(\rho_\xi - \theta')x^{+3}c\theta_y\Big\}'$$

$$\triangleq - H_z' \tag{4.6f}$$

$$U_{\theta_x} = -\Big\{\Big[D_\xi(\rho_\xi - \theta') + \frac{E}{2} I_\xi\ (1 - x^{+2})\rho_\xi + \frac{E}{2} B_1(\rho_\xi^2 - \theta'^2)\rho_\xi x^{+2} - EB_2\rho_\zeta\rho_\xi x^{+} + (EC_1^*\rho_\eta)'/x^{+2}$$

$$+ G(J_\eta\rho_\eta + J_\zeta\rho_\zeta)\ (2\rho_\xi - \theta')x^{+}\Big]x^{+2} - (EC_1\theta_x'')'\Big\}' + \frac{1}{2}(3-x^{+2})\ (D_\eta - D_\zeta)\rho_\eta\rho_\zeta x^{+2}$$

$$+ \frac{1}{2}\Big[EA(1-x^{+2})e_A x^{+} + EB_2(\rho_\xi^2 - \theta'^2)x^{+3} + 3E(B_3\rho_\eta^2 + B_4\rho_\zeta^2)x^{+3}\Big]\rho_\eta$$

$$- \Big[3EB_3\ \rho_\eta\rho_\zeta x^{+3} + EC_1^*(\rho_\xi' - \theta'')\Big]\rho_\zeta + G(J_\eta\rho_\zeta - J_\zeta\rho_\eta)\rho_\xi(\rho_\xi - \theta')x^{+3} \tag{4.6g}$$

The second term, $EI_\xi(1 - x^{+2})\ \rho_\xi/2$, that appears in the expression for U_{θ_x} is a tension-torsion coupling term that reflects the increase in the effective torsional stiffness of the blade due to axial tension, e.g. [45,75]. This is discussed further in Chapter VII.

The term T_e/x^{+4}, given by equation (4.6a), is the coefficient of $-x^+\delta x^+$ that appear in the variation of $(U-U^*)/(m\Omega^2R^3)$ given by equation (3.17). The terms h_u, h_v, h_w and h_{θ_x} that appear in the left hand side of equations (4.5 a-d) denote the small contributions due to $(T^*-U^*)/(m\Omega^2R^3)$. The terms in equation (4.1) that were integrated by parts yield the following boundary condition equation

$$\Big\{G_u\,\delta u + G_v\,\delta v + G_w\,\delta w + H_x\,\delta\theta_x + E(C_1\theta''_x - C_1^*\rho_\eta)\delta\theta'_x - EC_1^*\rho_\eta(s\theta_y)\delta\theta'_z + H_z[(\partial\theta_z/\partial u')\delta u'$$

$$+ (\partial\theta_z/\partial v')\delta v'] - H_y\big[(\partial\theta_y/\partial u')\delta u' + (\partial\theta_y/\partial v')\delta v' + (\partial\theta_y/\partial w')\delta w'\big] - \delta W_B\Big\}_{x=0}^{1} = 0 \tag{4.7}$$

where,

$$H_x = \Big[D_\xi(\rho_\xi - \theta') + \tfrac{1}{2}\,EI_\xi(1 - x^{+2})\rho_\xi + \tfrac{1}{2}\,EB_1(\rho_\xi^2 - \theta'^2)\rho_\xi x^{+2} - EB_2\rho_\zeta\rho_\xi x^+$$

$$+ G(J_\eta\rho_\eta + J_\zeta\rho_\zeta)\,(2\rho_\xi - \theta')x^+\Big]x^{+2} + (EC_1^*\,\rho_\eta - EC_1\theta''_x)' \tag{4.8a}$$

and

$$H_y = \Big[\tfrac{1}{2}\,(3 - x^{+2})\,(D_\eta\rho_\eta\,c\theta_1 - D_\zeta\rho_\zeta\,s\theta_1) + \tfrac{1}{2}\,EB_2(\rho_\xi^2 - \theta'^2)x^+ s\theta_1 + \tfrac{3}{2}\,E(B_3\rho_\eta^2 + B_4\rho_\zeta^2)x^+ s\theta_1$$

$$- 3EB_3\rho_\eta\rho_\zeta\,x^+\,c\theta_1 + G(J_\eta\,c\theta_1 - J_\zeta\,s\theta_1)\rho_\xi\,(\rho_\xi - \theta')x^+\Big]x^{+2}$$

$$+ \tfrac{1}{2}\,EA(1 - x^{+2})e_A x^+\,s\theta_1 - EC_1^*[(\rho'_\xi - \theta'')c\theta_1 - \rho_\eta\theta'_z\,c\theta_y] \tag{4.8b}$$

CHAPTER V - EQUATIONS OF MOTION WITH $O(\varepsilon^3)$ NON-LINEARITIES

Equations (4.5a-d) are a set of four non-linear coupled partial differential equations satisfying the boundary conditions extracted from equation (4.6). In general, they admit non-zero particular solutions which are either an equilibrium state α = constant $\triangleq \alpha_e$ ($\alpha = u,v,w,\theta_x$) for the case of hover, or an steady state solution for the case of forward flight. Of special interest is the determination of these particular solutions, and the analysis of the stability of the perturbed motion about them. Due to the complexity of equations (4.5a-d), they will now be restricted to "moderately large" deflections. Toward this end, the non-linearities in those equations are expanded in Taylor series and the results truncated to a certain degree in order to obtain a set of approximate differential equations that are more amenable to analysis. It is well known that the resulting equations, with polynomial non-linearities, still retain relevant information about the motion. Here the expansions will be carried to include terms up to third degree.

To expand equations (4.5a-d) into third degree polynomial non-linearities a small parameter ε, of the order of the bending deflections, is introduced and an order of magnitude $\varepsilon^n (n = 0,1,2,...)$ is assigned to the variables and parameters of the system, e.g. [32,42,52,62]. For example, the blade's geometric pitch angle is taken to be of $O(1)$, but its space and time varying components [see equation (2.1)] are assumed to be small so that $\theta_{pt}(x) = O(\varepsilon)$, $\theta_{ic} = O(\varepsilon)$ and $\theta_{is} = O(\varepsilon)$. The assumed order of magnitude of the quantities that appear in equations (4.5a-d) are as follows*

Order ε^0:

x, θ_c, D_η, D_ξ; D_ζ, EI_ξ

* The order of magnitude of the quantities indicated after a semi-colon follow as a consequence of the assumption for the order of magnitude of the appropriate $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^2)$ quantities indicated before a semi-colon.

Order ε:

$$v,\ w,\ \theta_x,\ \theta_{pt},\ \theta_{ic},\ \theta_{is},\ \sin\beta,\ \eta,\ \zeta,\ e;\ e_A,\ EB_2,\ EB_3,\ EB_4,\ EC_1^*,\ GJ_\eta,\ GJ_\zeta$$

Order ε^2:

$$u,\ \psi;\ j_\xi,\ j_\eta,\ j_\zeta,\ EB_1,\ EC_1$$

In addition, $EA = O(\varepsilon^{-2})$

From equations (2.11a,b) and (4.4a,b) the following expansions are obtained for θ_y, θ_z and their partial derivatives with respect to the elastic displacements,

$$\theta_y = w'(1 - u' - v'^2/2) - w'^3/3 + O(\varepsilon^5) \tag{5.1a}$$

$$\theta_z = v'(1 - u') - v'^3/3 + O(\varepsilon^5) \tag{5.1b}$$

$$\partial\theta_y/\partial u' = - w' + O(\varepsilon^3) \tag{5.1c}$$

$$\partial\theta_y/\partial v' = - v'w' + O(\varepsilon^4) \tag{5.1d}$$

$$\partial\theta_y/\partial w' = 1 - u' - v'^2/2 - w'^2 + O(\varepsilon^4) \tag{5.1e}$$

$$\partial\theta_z/\partial u' = - v' + O(\varepsilon^3) \tag{5.1f}$$

$$\partial\theta_z/\partial v' = 1 - u' - v'^2 + O(\varepsilon^4) \tag{5.1g}$$

Also, from equations (2.8), (4.6a) and (2.1) it follows that

$$x^{+} = 1 - u' - (v'^2 + w'^2)/2 + O(\varepsilon^4) \tag{5.1h}$$

$$\frac{E}{2} A\,(1 - x^{+2}) = EA\{u' + \frac{v'^2}{2} + \frac{w'^2}{2} + \frac{u'^2}{2} - 2[u' + \frac{v'^2}{2} + \frac{w'^2}{2}]^2\} + O(\varepsilon^4) \tag{5.1i}$$

$$\begin{aligned} T_e = EA\Big\{ & u' + \frac{v'^2}{2} + \frac{w'^2}{2} + \frac{u'^2}{2} - 6(u' + \frac{v'^2}{2} + \frac{w'^2}{2})^2 - e_A\Big\{\Big[v''(1 - u' - v'^2 \\ & - \frac{w'^2}{2}) - v'\,u''\Big]c\theta_1 + \Big[w''(1 - u' - \frac{v'^2}{2} - w'^2) - w'(u'' + v'\,v'')\Big]s\theta_1 \\ & - 6(u' + \frac{v'^2}{2} + \frac{w'^2}{2})\,(v''\,c\theta_c + w''\,s\theta_c)\Big\}\Big\} - D_\xi\theta_x'^2 + \frac{E}{2} I_\xi\theta_x'(\theta_x' + 2\theta') \\ & - \frac{1}{2}\Big[D_\eta(v''\,s\theta_c - w''\,c\theta_c)^2 + D_\zeta(v''\,c\theta_c + w''\,s\theta_c)^2\Big] + O(\varepsilon^3) \end{aligned} \tag{5.1j}$$

After expanding the remaining terms in equations (4.6 a-g), and noticing that the terms h_u, h_v, h_w and h_{θ_x} in equations (4.5 a-d) due to the T^* and U^* contributions to the kinetic and strain energies are of higher order -- as can be readily verified from equations A.1, A.2 and A.3 in the Appendix--the following differential equations with polynomial non-linearities are obtained*

$$\begin{aligned} G_u' = \Big\{ & v'\Big[D_\eta(v''\,s\theta_c - w''\,c\theta_c)s\theta_c + D_\zeta\,(v''\,c\theta_c + w''\,s\theta_c)c\theta_c - EAe_A(u' + \frac{v'^2}{2} + \frac{w'^2}{2})c\theta_c\Big]' \\ & - w'\Big[D_\eta(v''\,s\theta_c - w''\,c\theta_c)c\theta_c - D_\zeta(v''\,c\theta_c + w''\,s\theta_c)s\theta_c + EAe_A(u' + \frac{v'^2}{2} + \frac{w'^2}{2})s\theta_c\Big]' \\ & - xe(v'\,c\theta_c + w'\,s\theta_c)c^2\beta + T_e(1 + u')\Big\}' = \ddot{u} - e(\ddot{v}\,c\theta_c + \ddot{w}\,s\theta_c)' \\ & - 2(\dot{v} - e\dot{\theta}_1\,s\theta_c)c\beta + (w + e\,s\theta_c)s\beta\,c\beta - \Big[x + u - e(v'\,c\theta_c + w'\,s\theta_c)\Big]c^2\beta \\ & - Q_u + O(\varepsilon^3) \end{aligned} \tag{5.2a}$$

---

* Although $\sin\beta$ and $\cos\beta$ are related as $\cos^2\beta + \sin^2\beta = 1$, the O(1) quantity $\cos\beta$ is left in the equations as $c\beta$ for simplicity.

$$
\begin{aligned}
G_v' = \Bigg\{ & - v'w' \Big\{ 2\, x\, e\, s\theta_c\, c^2\beta + \Big[D_\eta (v''\, s\theta_c - w''\, c\theta_c) c\theta_c - D_\zeta (v''\, c\theta_c + w''\, s\theta_c) s\theta_c + EAe_A (u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2}) s\theta_c \Big]' \Big\} \\
& + J_\eta \Big[\ddot{v}'\, s\theta_c - (\ddot{w}' - 2\dot{\theta}_1\, c\beta) c\theta_c \Big] s\theta_c + J_\zeta \Big[\ddot{v}'\, c\theta_c + (\ddot{w}' - 2\dot{\theta}_1\, c\beta) s\theta_c \Big] c\theta_c + (J_\eta - J_\zeta) s\beta\, c\beta\, s\theta_c\, c\theta_c \\
& - e \Big\{ \Big[\ddot{u} - u\, c^2\beta + w\, s\beta\, c\beta + x(u' + \tfrac{3}{2} v'^2) c^2\beta \Big] c\theta_c - (2\dot{v} + x\, c\beta) c\theta_1\, c\beta + (\ddot{v} - v)\, (v'\, c\theta_c + w'\, s\theta_c) \Big\} \\
& - \Big\{ w' D_\xi (\theta_x' + v''\, w') + w' [EC_1^* (v''\, s\theta_c - w''\, c\theta_c)]' + (D_\eta\, s^2\theta_1 + D_\zeta\, c^2\theta_1) v'' + (D_\zeta - D_\eta) w''\, s\theta_1\, c\theta_1 \\
& - (D_\eta\, s^2\theta_c + D_\zeta\, c^2\theta_c) \Big[v''(2u' + \tfrac{3}{2} v'^2 + \tfrac{3}{2} w'^2) + v' u'' \Big] - (D_\zeta - D_\eta) \Big[(2u' + v'^2 + 2w'^2) w'' + w'(u'' + v' v'') \Big] s\theta_c\, c\theta_c \\
& - \tfrac{E}{2} B_2\, \theta_x' (\theta_x' + 2\theta') c\theta_c - \tfrac{3}{2} E \Big[B_3 (v''\, s\theta_c - w''\, c\theta_c)^2 + B_4 (v''\, c\theta_c + w''\, s\theta_c)^2 \Big] c\theta_c \\
& - 3EB_3 (v''\, s\theta_c - w''\, c\theta_c)\, (v''\, c\theta_c + w''\, s\theta_c) s\theta_c - EC_1^* (\theta_x' + v''\, w')'\, s\theta_1 + G(J_\eta\, s\theta_c + J_\zeta\, c\theta_c)\, \theta_x' (\theta_x' + \theta') \Big\}' \Big]' \\
& + \Big\{ EAe_A \Big\{ (u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2}) c\theta_1 - \Big[(u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2})\, (3u' + \tfrac{3}{2} v'^2 + 2w'^2) - \tfrac{u'^2}{2} \Big] c\theta_c \Big\} \Big\}'' \\
& + \Big\{ (u' + v'^2) \Big[v''(D_\eta\, s^2\theta_c + D_\zeta\, c^2\theta_c) + (D_\zeta - D_\eta) w''\, s\theta_c\, c\theta_c - EAe_A (u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2}) c\theta_c \Big]' + T_e v' \Big\}' \\
= & \Big[v + e(c\theta_1 - \tfrac{v'^2}{2} c\theta_c - v'\, w'\, s\theta_c) \Big]^{\cdot\cdot} - v - e\, c\theta_1 + e\, (\tfrac{v'^2}{2} c\theta_c + v'\, w'\, s\theta_c) - 2\dot{w}\, s\beta \\
& + 2 \Big[\dot{u} - e(\dot{v}'\, c\theta_1 + \dot{w}'\, s\theta_1) + e\dot{\theta}_1 (v'\, s\theta_c - w'\, c\theta_c) \Big] c\beta - 2e\dot{\theta}_1\, s\beta\, c\theta_c - Q_v + O(\varepsilon^4)
\end{aligned}
\tag{5.2b}
$$

$$
\begin{aligned}
G'_w = \Bigg\{ & j_\zeta\Big[(\ddot{v}'\, c\theta_c + \ddot{w}'\, s\theta_c)s\theta_c + \dot{\theta}_1\, c2\theta_c\, c\beta\Big] - j_\eta\Big[(\ddot{v}'\, s\theta_c - \ddot{w}'\, c\theta_c)c\theta_c + \dot{\theta}_1\, c2\theta_c\, c\beta\Big] - j_\xi(\dot{\theta}_1 + s\beta + w'\, c\beta)c\beta \\
& + (j_\eta\, s^2\theta_c + j_\zeta\, c^2\theta_c)\,(w'\, c\beta + s\beta)\, c\beta \quad - e\Big\{\Big[\ddot{u} - (u\, c\beta - w\, s\beta)c\beta + x(u' + v'^2 + \tfrac{3}{2}w'^2)c^2\beta + v'(\ddot{v} - v) \\
& + w'(\ddot{w} + x\, s\beta\, c\beta)\Big]s\theta_c \quad - (2\dot{v} + x\, c\beta)s\theta_1\, c\beta\Big\} + v''\Big\{D_\xi(\theta'_x + v''\, w') + [EC_1^*(v''\, s\theta_c - w''\, c\theta_c)]' \\
& - w'\Big[(D_\eta\, s^2\theta_c + D_\zeta\, c^2\theta_c)v'' + (D_\zeta - D_\eta)w''\, s\theta_c\, c\theta_c \quad - EAe_A(u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2})c\theta_c\Big]\Big\} \\
& + \Big\{(D_\eta - D_\zeta)v''\, s\theta_1\, c\theta_1 - (D_\eta - D_\zeta)\Big[v''(2u' + \tfrac{3}{2}v'^2 + w'^2) + v'u''\Big]s\theta_c\, c\theta_c \quad - (D_\eta\, c^2\theta_1 + D_\zeta\, s^2\theta_1)w'' \\
& + (D_\eta\, c^2\theta_c + D_\zeta\, s^2\theta_c)\Big[w''(2u' + v'^2 + \tfrac{3}{2}w'^2) + w'(u'' + v'v'')\Big] + \tfrac{E}{2}B_2\,\theta'_x(\theta'_x + 2\theta')s\theta_c + \tfrac{3}{2}E\Big[B_3(v''\, s\theta_c - w''\, c\theta_c)^2 \\
& + B_4(v''\, c\theta_c + w''\, s\theta_c)^2\Big]s\theta_c - 3EB_3(v''\, s\theta_c - w''\, c\theta_c)\,(v''\, c\theta_c + w''\, s\theta_c)c\theta_c\Big\}'\Big]' \quad + \Big\{\Big[EAe_A\Big\{(u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2})s\theta_1 \\
& + \Big[\tfrac{u'^2}{2} - 3(u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2})^2\Big]s\theta_c\Big\} - EC_1^*(\theta'_x + v''\, w')'\, c\theta_1 \quad + G(J_\eta\, c\theta_c - J_\zeta\, s\theta_c)\,\theta'_x\,(\theta'_x + \theta')\Big\}' \\
& - (u' + \tfrac{v'^2}{2} + w'^2)\Big[(D_\eta - D_\zeta)v''\, s\theta_c\, c\theta_c - (D_\eta\, c^2\theta_c + D_\zeta\, s^2\theta_c)w'' \quad + EAe_A(u' + \tfrac{v'^2}{2} + \tfrac{w'^2}{2})s\theta_c\Big]' + T_e\, w'\Big]' \\
= & \Big[w + e(s\theta_1 - \tfrac{w'^2}{2}\, s\theta_c)\Big]^{\cdot\cdot} + 2\,(\dot{v} - e\,\dot{\theta}_1\, s\theta_c)s\beta \quad - (w + e\, s\theta_c)\, s^2\beta + \Big[x + u - e(v'\, c\theta_c + w'\, s\theta_c)\Big]s\beta\, c\beta - Q_w + O(\varepsilon^4)
\end{aligned}
\tag{5.2c}
$$

$$
\begin{aligned}
& j_\xi(\ddot{\theta}_1 + \dot{w}'\ c\beta) + (j_\zeta - j_\eta)\left[(2\dot{v}' + c\beta)s\theta_1\ c\theta_1 - \dot{w}'\ c2\theta_c\right]c\beta - e\Big\{(v's\theta_1 - w'c\theta_1)(2\dot{v} + x\ c\beta)c\beta \\
& -(\ddot{w} + 2\dot{v}\ s\beta + x\ s\beta\ c\beta)c\theta_1 + (\ddot{v} - 2\dot{w}\ s\beta - v + 2\dot{u}\ c\beta)s\theta_1\Big\} - \Big\{D_\xi\Big[\theta_x' + v''\ w' \\
& -\theta_x'(2u' + v'^2 + w'^2)\Big] + EI_\xi\theta_1'\ (u' + \frac{v'^2}{2} + \frac{w'^2}{2}) - EB_2\theta_1'\ (v''\ c\theta_c + w''\ s\theta_c)\Big\}' \\
& +(v''\ s\theta_1 - w''\ c\theta_1)\left[(D_\eta - D_\zeta)\ (v''\ c\theta_1 + w''\ s\theta_1) + EAe_A(u' + \frac{v'^2}{2} + \frac{w'^2}{2})\right] \\
& + (EC_1\theta_x'')'' - [EC_1^*(v''s\theta_1 - w''\ c\theta_1)]'' - EC_1^*\ \theta_x''(v''c\theta_c - w''\ s\theta_c) \\
& - \Big\{(\theta_x' + \theta_1')\left[GJ_\eta(v''\ s\theta_c - w''\ c\theta_c) + GJ_\zeta(v''\ c\theta_c + w''\ s\theta_c)\right]\Big\}' = Q_{\theta_x} + 0\ (\varepsilon^4)
\end{aligned}
\tag{5.2d}
$$

In equations (5.1j) and 5.2 b-d) the expansions for $\sin\theta_1$ and $\cos\theta_1$, truncated to the appropriate order, should be used. With θ_1 expressed as

$$
\theta_1(x,\tau) \equiv \theta_c + \theta_x(x,\tau) + \theta_{pt}(x) + \sum_{i=1}^{n} (\theta_{ic}\ ci\tau + \theta_{is}\ si\tau) \triangleq \theta_c + \theta_{11}(x,\tau) \tag{5.3}
$$

these expansions are

$$
s\theta_1 = s\theta_c + \theta_{11}\ c\theta_c - \frac{1}{2}\theta_{11}^2\ s\theta_c - \frac{1}{6}\theta_{11}^3\ c\theta_c + 0(\varepsilon^4) \tag{5.4a}
$$

$$
c\theta_1 = c\theta_c - \theta_{11}\ s\theta_c - \frac{1}{2}\theta_{11}^2\ c\theta_c + 0(\varepsilon^3) \tag{5.4b}
$$

Equations (5.2 a-d) may be reduced to a set of three integro partial differential equations in v, w and θ_x, with cubic non-linearities in these variables. For a rotor blade with the end x = 1 free, equation (5.2 a) may be integrated with $G_u(x = 1, \tau=\tau) = 0$ to yield an expression for u' in terms of the remaining variables. To obtain such expression, the integrated form of equation (5.2 a) is first written as

$$
\begin{aligned}
T_e = {} & [D_\eta(v'' s\theta_c - w'' c\theta_c)]' (w' c\theta_c - v' s\theta_c) + x(v' c\theta_c + w' s\theta_c)e\, c^2\beta \\
& + (v' c\theta_c + w' s\theta_c)\left[EAe_A(u' + \frac{v'^2}{2} + \frac{w'^2}{2}) - D_\zeta(v'' c\theta_c + w'' s\theta_c)\right]' \\
& + \int_1^x \Big\{ \ddot{u} - \left[2\dot{v} - w\, s\beta + (x + u)c\beta\right]c\beta \\
& \qquad - e\left[(\ddot{v}' - v' c^2\beta)c\theta_c + (\ddot{w}' - 2\dot{\theta}_1 c\beta - s\beta\, c\beta - w' c^2\beta)s\theta_c\right]\Big\} dx \\
& -(1 - u') \int_1^x Q_u\, dx - \frac{u'}{2}(1 - x^2)c^2\beta + O(\varepsilon^3)
\end{aligned}
\tag{5.5}
$$

Equating the right hand side of equations (5.1j) and (5.5) to each other, and multiplying the resulting equation by $[1 + 6(u' + v'^2/2 + w'^2/2) - u'/2 + (v'^2 + w'^2)/4]$, the following expression for $EA(u' + v'^2/2 + w'^2/2)$, which is needed to $O(\varepsilon^2)$ in equations (5.2 b,c), results when $Q_u = o(\varepsilon)$

$$EA(u' + \frac{v'^2}{2} + \frac{w'^2}{2}) = [D_\eta(v'' \; s\theta_c - w'' \; c\theta_c)]'(w' \; c\theta_c - v' \; s\theta_c)$$

$$-[D_\zeta(v'' \; c\theta_c + w'' \; s\theta_c)]'(v' \; c\theta_c + w' \; s\theta_c) + xe(v' \; c\theta_c + w' \; s\theta_c)c^2\beta$$

$$+ D_\xi\theta_x'^2 - \frac{1}{2} EI_\xi\theta_x'(\theta_x' + 2\theta') + \frac{1}{2}\left[D_\eta(v'' \; s\theta_c - w'' \; c\theta_c)^2 + D_\zeta(v'' \; c\theta_c + w'' \; s\theta_c)^2\right]$$

$$+ \int_1^x \left\{ \ddot{u} - \left[2\dot{v} - w \; s\beta + (x+u)c\beta\right]c\beta - e\left[(\ddot{v}' - v' \; c^2\beta)c\theta_c\right.\right.$$

$$\left.\left. + (\ddot{w}' - 2\dot{\theta}_1 \; c\beta - s\beta \; c\beta - w' \; c^2\beta)s\theta_c\right]\right\}dx \quad - EAe_A\left\{\left[\frac{3}{2} u' - \frac{1}{4}(v'^2 - w'^2)\right]v'' \; c\theta_c\right.$$

$$\left. + \left[\frac{3}{2} u' + \frac{1}{4}(v'^2 - w'^2)\right]w'' \; s\theta_c - v' \; w' \; w'' \; c\theta_c - v'' \; c\theta_1 - w'' \; s\theta_1\right\}$$

$$- \frac{1}{8} EA(v'^2 + w'^2)^2 - \int_1^x Q_u \, dx + (EAe_A)' \; (v' \; c\theta_c + w' \; s\theta_c)(u' + \frac{v'^2}{2} + \frac{w'^2}{2})$$

$$+ \frac{1}{2}(1 - x^2)\left[6(u' + \frac{v'^2}{2} + \frac{w'^2}{2}) - \frac{3}{2} u' + \frac{1}{4}(v'^2 + w'^2)\right]c^2\beta + o(\varepsilon^3) \qquad (5.6)$$

The remaining u-terms in equations (5.2 b,c) are needed to $O(\varepsilon^3)$ at most, while only an $O(\varepsilon^2)$ expansion for u is needed to eliminate u and its derivatives in the right hand side of equation (5.6). To $O(\varepsilon^3)$, equation (5.6) may be solved for u' as

$$u' = e_A(v'' \; c\theta_1 + w'' \; s\theta_1) - \frac{1}{2}(v'^2 + w'^2) + \frac{1}{EA}\left[\frac{1}{2}(1 - x^2)c^2\beta - \int_1^x (2\dot{v} \; c\beta + Q_u)dx\right] + O(\varepsilon^4) \qquad (5.7)$$

Or, with the boundary condition $u(x = 0, \tau = \tau) = 0$, and making use of equations (5,4a,b),

$$u = \int_0^x [e_A(v'' c\theta_c + w'' s\theta_c) - e_A\theta_{11}(v'' s\theta_c - w'' c\theta_c) - (v'^2 + w'^2)/2]dx$$

$$+ \int_0^x \frac{1}{EA} [\tfrac{1}{2}(1 - x^2)c^2\beta - \int_1^x (2\dot{v}\ c\beta + Q_u)dx]dx + O(\varepsilon^4) \tag{5.8}$$

When equations (5.7) and (5.8) are used in the right hand side of equation (5.6), the expression for $EA(u' + v'^2/2 + w'^2/2)$ becomes

$$EA(u' + \frac{v'^2}{2} + \frac{w'^2}{2}) = \left[D_\eta(v'' s\theta_c - w'' c\theta_c)\right]'(w' c\theta_c - v' s\theta_c)$$

$$- \left[D_\zeta(v'' c\theta_c + w'' s\theta_c)\right]'(v' c\theta_c + w' s\theta_c) + xe(v' c\theta_c + w' s\theta_c)c^2\beta + D_\xi\theta_x'^2$$

$$- \frac{1}{2} EI_\xi\theta_x' (\theta_x' + 2\theta') + \frac{1}{2}\left[D_\eta(v'' s\theta_c - w'' c\theta_c)^2 + D_\zeta(v'' c\theta_c + w'' s\theta_c)^2\right]$$

$$- \int_1^x \left\{(2\dot{v} - w s\beta + x c\beta)c\beta + (e - e_A)\left[(\ddot{v}' - v' c^2\beta)c\theta_c + (\ddot{w}' - w' c^2\beta)s\theta_c\right]\right.$$

$$\left. - e(2\dot{\theta}_1 + s\beta)c\beta\ s\theta_c\right\}dx \quad - \int_1^x \int_0^x \left\{(v' \dot{v}' + w' \dot{w}')^{\cdot} + \frac{1}{2}\left[\frac{(1 - x^2)}{EA} c^2\beta - v'^2\right.\right.$$

$$\left.\left. - w'^2\right]c^2\beta\right\}dx\ dx \quad + \frac{1}{2}(1 - x^2)\ (v'^2 + w'^2)c^2\beta \quad - \frac{1}{8}EA\ (v'^2 + w'^2)^2$$

$$+ \frac{3}{4}(3\ c^2\beta - 1)\ (1 - x^2)e_A\ (v'' c\theta_c + w'' s\theta_c) + \frac{9}{8EA}\ (1 - x^2)^2\ c^4\beta$$

$$+ EAe_A\left\{\left[v''\ (v'^2 + \frac{w'^2}{2}) + v'\ w'\ w''\right]c\theta_c + (w'^2 + \frac{v'^2}{2})w''\ s\theta_c\right\} - \int_1^x Q_u\ dx$$

$$+ EAe_A(v'' c\theta_1 + w'' s\theta_1) - \frac{3}{2} EAe_A^2(v'' c\theta_c + w'' s\theta_c)^2 + o(\varepsilon^3) \tag{5.9}$$

The integro partial differential equations in v, w and θ_x with non-linearities to $O(\varepsilon^3)$ are obtained when u and its derivatives in equations (5.2b,d) are replaced by the expressions given by equations (5.7), (5.8) and (5.9). The resulting equations are written here as equations (5.10) to (5.12) for the special simpler case of constant stiffnesses and $e = e_A = 0$.

$$
\begin{aligned}
G_v' = {} & \Bigg\{ \Big[(D_\eta - D_\zeta)\, \frac{w''}{2}\, s2\theta_1 - (D_\eta\, s^2\theta_1 + D_\zeta\, c^2\theta_1) v'' \Big]' + (D_\eta - D_\zeta) \Big\{ \frac{w'''}{2} (v'^2 + w'^2) \\
& + \frac{c^2\beta}{EA} \Big[w - \frac{3}{2}\, (1 - x^2) w'' \Big]' \Big\} s\theta_c\, c\theta_c - (D_\eta\, s^2\theta_c + D_\zeta\, c^2\theta_c) \Big\{ \frac{v'}{2}\, (v'^2 + w'^2)'' \\
& + \frac{c^2\beta}{EA} \Big[v - \frac{3}{2}\, (1 - x^2) v'' \Big]' \Big\} - \Big\{ w' \Big[D_\xi (\theta_x' + v''\, w') + EC_1^* (v'''\, s\theta_c - w'''\, c\theta_c) \Big] \\
& - \frac{E}{2} \Big[B_2 \theta_x' \, (\theta_x' + 2\theta') + 3B_3\, (v''\, s\theta_c - w''\, c\theta_c)^2 + 3B_4 (v''\, c\theta_c + w''\, s\theta_c)^2 \Big] c\theta_c \\
& - 3EB_3 (v''\, s\theta_c - w''\, c\theta_c)\, (v''\, c\theta_c + w''\, s\theta_c) s\theta_c - EC_1^* (\theta_x' + v''\, w')'\, s\theta_1 \\
& + G(J_\eta\, s\theta_c + J_\zeta\, c\theta_c) \theta_x' \, (\theta_x' + \theta') \Big\}' - v' \Big\{ \int_1^x \Big[(2\dot{v} - w\, s\beta + x\, c\beta) c\beta \\
& + \frac{c^4\beta}{2EA} \Big(x - \frac{x^3}{3} \Big) + Q_u \Big] dx + \int_1^x \int_0^x \Big[(v'\, \dot{v}' + w'\, \dot{w}')^{\cdot} - \frac{1}{2} (v'^2 + w'^2) c^2\beta \Big] dx\, dx \\
& - (1 - x^2)\, \frac{c^2\beta}{4} \Big[v'^2 + w'^2 - (1 - x^2)\, \frac{c^2\beta}{EA} \Big] \Big\} + j_\eta (\ddot{v}\, s\theta_c - \ddot{w}\, c\theta_c)'\, s\theta_c \\
& + j_\zeta (\ddot{v}\, c\theta_c + \ddot{w}\, s\theta_c)'\, c\theta_c + (j_\eta - j_\zeta)\, (2\dot{\theta}_1 + s\beta)\, c\beta\, s\theta_c\, c\theta_c \Big]' \\
= {} & \ddot{v} - v - 2\, \dot{w}\, s\beta - 2(c\beta) \int_0^x \Big[v'\, \dot{v}' + w'\dot{w}' + \frac{1}{EA} \int_1^x (2\, \ddot{v}\, c\beta + \dot{Q}_u) dx \Big] dx - Q_v + O(\varepsilon^4)
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
G'_w = {} & \Bigg\{\Big[(D_\eta - D_\zeta)\,\frac{v''}{2}\,s2\theta_1 - (D_\eta\, c^2\theta_1 + D_\zeta\, s^2\theta_1)w''\Big]' + (D_\eta - D_\zeta)\Big\{v'\,w'\,w''' \\
& + \Big[\frac{v''}{2}(v'^2 + w'^2) + v'\,w'\,w''\Big]' + \frac{c^2\beta}{EA}\Big[v - \frac{3}{2}(1 - x^2)v''\Big]'\Big\}\, s\theta_c\, c\theta_c \\
& - (D_\eta\, s^2\theta_c + D_\zeta\, c^2\theta_c)w'\,(v'v'')' - (D_\eta\, c^2\theta_c + D_\zeta\, s^2\theta_c)\Big\{\frac{c^2\beta}{EA}\Big[w - \frac{3}{2}(1 - x^2)w''\Big]' \\
& + w'(w'\,w'')'\Big\} + v''\Big[D_\xi(\theta'_x + v''\,w') + EC_1^*(v'''\,s\theta_c - w'''\,c\theta_c)\Big] \\
& + \Big\{\frac{E}{2}\Big[B_2\theta'_x\,(\theta'_x + 2\theta') + 3B_3(v''\,s\theta_c - w''\,c\theta_c)^2 + 3B_4(v''\,c\theta_c + w''\,s\theta_c)^2\Big]s\theta_c \\
& - 3EB_3(v''\,s\theta_c - w''\,c\theta_c)\,(v''\,c\theta_c + w''\,s\theta_c)c\theta_c - EC_1^*(\theta'_x + v''\,w')'\,c\theta_1 \\
& + G(J_\eta\, c\theta_c - J_\zeta\, s\theta_c)\theta'_x\,(\theta'_x + \theta')\Big\}' - w'\Big\{\int_1^x \Big[(2\dot{v} - w\,s\beta + x\,c\beta)c\beta \\
& + \frac{c^4\beta}{2EA}\Big(x - \frac{x^3}{3}\Big) + Q_u\Big]dx + \int_1^x\int_0^x \Big[(v'\,\dot{v}' + w'\,\dot{w}')^{\cdot} - \frac{1}{2}(v'^2 + w'^2)c^2\beta\Big]dx\,dx \\
& - (1 - x^2)\,\frac{c^2\beta}{4}\Big[v'^2 + w'^2 - (1 - x^2)\,\frac{c^2\beta}{EA}\Big]\Big\} - j_\eta\,(\ddot{v}\,s\theta_c - \ddot{w}\,c\theta_c)'\,c\theta_c \\
& + j_\zeta(\ddot{v}\,c\theta_c + \ddot{w}\,s\theta_c)'\,s\theta_c - (j_\eta - j_\zeta)\dot{\theta}_1\,c2\theta_c\,c\beta - j_\xi(\dot{\theta}_1 + s\beta + w'\,c\beta)c\beta \\
& + (j_\eta\, s^2\theta_c + j_\zeta\, c^2\theta_c)\,(w'\,c\beta + s\beta)c\beta\Bigg\}' = \ddot{w} + (2\dot{v} - w\,s\beta)s\beta \\
& + \Big[x - \frac{1}{2}\int_0^x (v'^2 + w'^2)dx + \frac{c^2\beta}{2EA}\Big(x - \frac{x^3}{3}\Big)\Big]s\beta\,c\beta - Q_w + O(\varepsilon^4)
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
& j_\xi(\dot{\theta}_1 + w' \, c\beta)^{\cdot} + (j_\zeta - j_\eta)\left[(\dot{v}' + \frac{c\beta}{2}) \, s2\theta_1 - \dot{w}' \, c2\theta_c\right]c\beta \\
& - \left\{D_\xi\left[\theta_x' + v'' \, w' - \theta_x'(1 - x^2)\frac{c^2\beta}{EA}\right] + EI_\xi\theta_1'(1 - x^2)\frac{c^2\beta}{2EA} - EB_2\theta_1'(v'' \, c\theta_c + w'' \, s\theta_c)\right\}' \\
& + (D_\eta - D_\zeta)(v'' \, s\theta_1 - w'' \, c\theta_1)(v'' \, c\theta_1 + w'' \, s\theta_1) + EC_1\theta_x'''' - EC_1^*\left[(v'' \, s\theta_1 - w'' \, c\theta_1)''\right. \\
& \left. + \theta_x''(v'' \, c\theta_c + w'' \, s\theta_c)\right] - \left\{(\theta_x' + \theta_1')\left[GJ_\eta(v'' \, s\theta_c - w'' \, c\theta_c)\right.\right. \\
& \left.\left. + GJ_\zeta(v'' \, c\theta_c + w'' \, s\theta_c)\right]\right\}' = Q_{\theta_x} + O(\epsilon^4)
\end{aligned}
\tag{5.12}
$$

CHAPTER VI - GENERALIZED AERODYNAMIC FORCES EXPANDED TO $O(\epsilon^3)$ NON-LINEARITIES

The velocity $\bar{v}_e$ of a point on the blade's elastic axis, relative to the hub center 0, may be obtained directly from equation (3.2b) in Chapter III as

$$\bar{v}_e = \Omega R\{(\dot{u} - v\, c\beta)\hat{x} + [\dot{v} + (x + u)c\beta - w\, s\beta]\,\hat{y} + (\dot{w} + v\, s\beta)\hat{z}\,\} \tag{6.1}$$

The rotation of the blade, with angular velocity $\Omega\hat{Z}$ (see Fig.2.2 in Chapter II), induces a "downwash" air inflow with velocity $-\Omega R v_i \hat{Z} = -\Omega R v_i(\hat{x}\, s\beta + \hat{z}\, c\beta)$. It should be noted that in the notation used here v_i denotes the magnitude of the induced inflow, normalized by ΩR. If V_f denotes the magnitude of the constant forward velocity of the hub center, 0, normalized by ΩR, the velocity $\bar{v}_{e/a}$ of the blade's elastic axis relative to the air is written in terms of the advance ratio $\mu \triangleq V_f \cos\alpha_r = O(1)$ and inflow ratio $\lambda \triangleq V_f \sin\alpha_r + v_i = O(\epsilon)$ as [6,9,28]

$$\bar{v}_{e/a} = \bar{v}_e + \Omega R V_f(\hat{X}c\alpha_r + \hat{Z}s\alpha_r) + \Omega R v_i \hat{Z}$$

$$= \bar{v}_e + \Omega R\{\mu[(\hat{x}\, c\beta - \hat{z}\, s\beta)c\tau - \hat{y}\, s\tau] + \lambda\,(\hat{x}\, s\beta + \hat{z}\, c\beta)\} \tag{6.2}$$

Making use of equations (2.2) and (2.3) in Chapter II, equation (6.2) may be expressed as

$$\bar{v}_{e/a} = \Omega R(U_R\hat{\xi} + U_T\hat{\eta} + U_P\hat{\zeta}) \tag{6.3.1}$$

where

$$\begin{bmatrix} U_R \\ U_T \\ U_P \end{bmatrix} = [T] \begin{bmatrix} \dot{u} - v\,c\beta + \mu\,c\beta\,c\tau + \lambda\,s\beta \\ \dot{v} + (x + u)\,c\beta - w\,s\beta - \mu\,s\tau \\ \dot{w} + v\,s\beta - \mu\,s\beta\,c\tau + \lambda\,c\beta \end{bmatrix} \tag{6.3.2}$$

The aerodynamic forces and moments will be modelled using Greenberg's extension of Theodorsen's theory in which only the U_T and U_P components of the velocity $\bar{v}_{e/a}$ are assumed to effect the aerodynamic loading [17-19,34,46,52,56]. According to this theory, the lift $\bar{L}$ and aerodynamic moment $\bar{M}_a$ per unit length along the blade, expressed in the $(\hat{\xi},\hat{\eta},\hat{\zeta})$ unit vector triad, may be written for quasi-steady aerodynamics as

$$\bar{L} = (m\Omega^2 R)\ [L_C(\hat{\zeta}\ c\alpha + \hat{\eta}\ s\alpha) + L_{NC}\hat{\zeta}] \tag{6.4}$$

$$\bar{M}_a = (m\Omega^2 R^2) M_A \hat{\xi} \tag{6.5}$$

where the non-dimensional quantities L_C, L_{NC} and M_A are given by* [56]

$$L_C = \frac{\gamma}{6}\ U[-U_P + (\frac{c}{2} - x_A)\omega_\xi] \tag{6.6a}$$

---

* A more general expression for the generalized aerodynamic forces in hover may be obtained by allowing the aerodynamic coefficients to be a function of angle of attack [82,83]. As shown in [83], this procedure yields a refined expression for calculating the steady induced inflow velocity in hover.

$$L_{NC} = \frac{\gamma}{24} c[-\dot{U}_P + (\frac{c}{4} - x_A)\dot{\omega}_\xi] \tag{6.6b}$$

$$M_A = -\frac{\gamma}{6}[x_A UU_P + (\frac{c}{4} - x_A)^2 U\omega_\xi - \frac{c}{4}(\frac{c}{4} - x_A)\dot{U}_P + \frac{c}{8}(\frac{3c^2}{16} - cx_A + 2x_A^2)\dot{\omega}_\xi] \tag{6.6c}$$

As illustrated in Fig. 6.1 the circulatory lift component $m\Omega^2 RL_C$ acts in the direction normal to the velocity ΩRU, where

$$U = (U_T^2 + U_P^2)^{1/2} \tag{6.7}$$

The angle α is the angle of attack for the airfoil, while c and x_A are, respectively, the airfoil chord and its aerodynamic center offset from the shear center C_e^*, both normalized by R. These quantities are shown in Fig. 6.1. The non-dimensional parameter γ that appears in equations (6.6 a-c) is the Lock number for the flow, defined as

$$\gamma = 6\pi\rho_A R^2 c/m \tag{6.8}$$

and ρ_A is the air density. The distributed aerodynamic drag acting on the blade is also shown in Figure 6.1 and is written in terms of the airfoil profile drag coefficient c_{d0} as

$$\bar{D}_a = \frac{1}{2}\rho_A Rcc_{d0}(\Omega RU)^2(\hat{\zeta}\, s\alpha - \hat{\eta}\, c\alpha) = (m\Omega^2 R)\frac{\gamma}{6}\frac{c_{d0}}{2\pi}U^2(\hat{\zeta}\, s\alpha - \hat{\eta}\, c\alpha)$$

$$\triangleq (m\Omega^2 R)\, D\, (\hat{\zeta}\, s\alpha - \hat{\eta}\, c\alpha) \tag{6.9}$$

As can be seen from Fig. 6.1, sin α and cos α may be expressed as

$$s\alpha = -U_P/U \quad ; \quad c\alpha = U_T/U \tag{6.10}$$

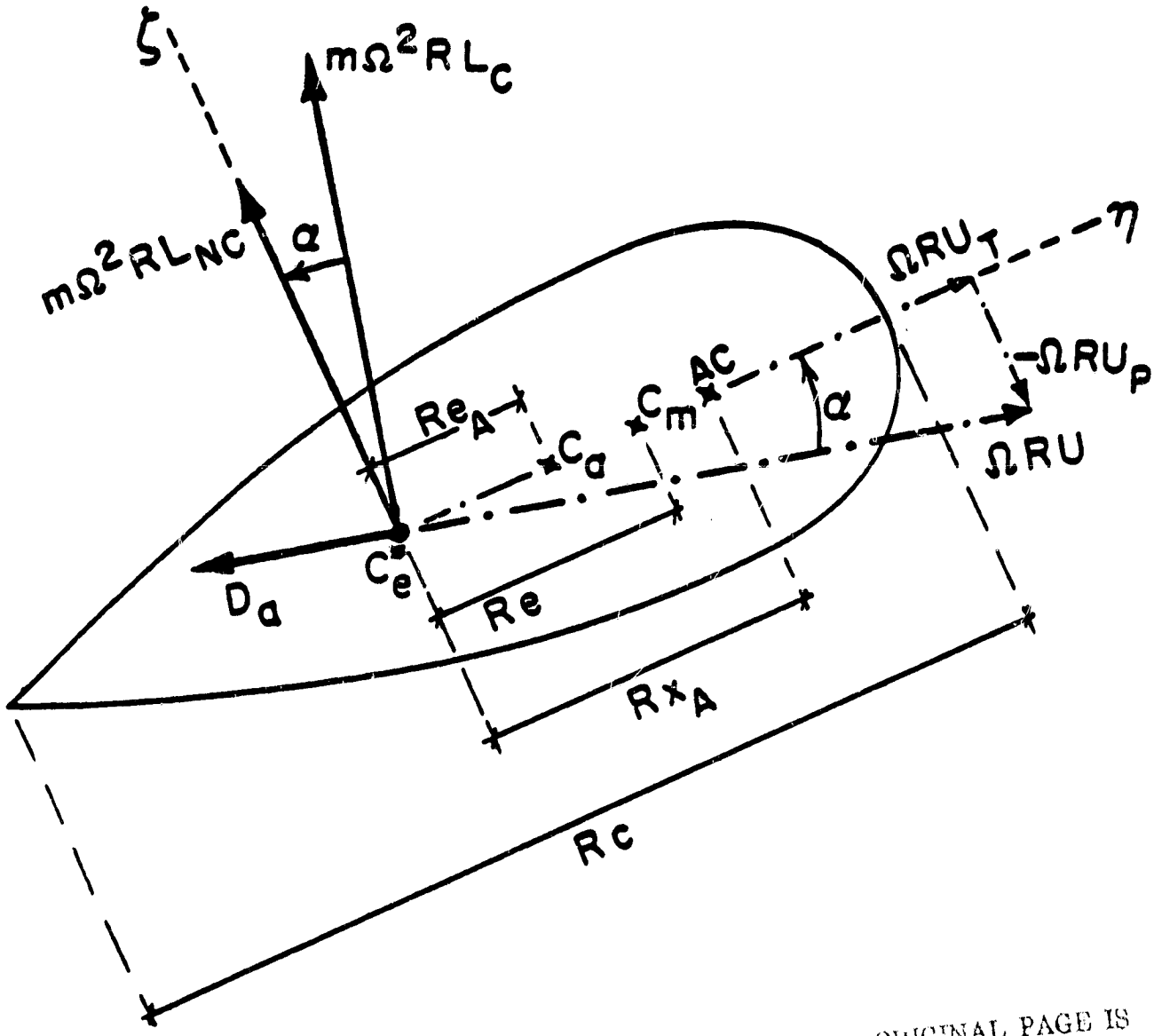

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FIG. 6.1 DIRECTION OF THE AERODYNAMIC FORCES ACTING ON THE AIRFOIL.

Combining equations (6.4), (6.9) and (6.10) the resultant, $\bar{F}_A$, of the distributed aerodynamic forces acting on the blade is determined as

$$\bar{F}_A/(m\Omega^2 R) = (L_C\ s\alpha - D\ c\alpha)\hat{\eta} + (L_{NC} + L_C\ c\alpha + D\ s\alpha)\hat{\zeta}$$

$$\triangleq F_\eta\ \hat{\eta} + F_\zeta\ \hat{\zeta} \tag{6.11}$$

where

$$F_\eta = \frac{\gamma}{6}\ [U_P^2 - (\frac{c}{2} - x_A)U_P\omega_\xi - \frac{c_{d0}}{2\pi}\ UU_T] \tag{6.12a}$$

$$F_\zeta = \frac{\gamma}{6}\ [-\ U_P U_T + (\frac{c}{2} - x_A)U_T\omega_\xi\ - \frac{c}{4}\ \dot{U}_P + \frac{c}{4}(\frac{c}{4} - x_A)\dot{\omega}_\xi - \frac{c_{d0}}{2\pi}\ UU_P] \tag{6.12b}$$

The virtual work done by $\bar{F}_A$ during a virtual displacement $R(\hat{x}\ \delta u + \hat{y}\ \delta v + \hat{z}\ \delta w)$ is then

$$(\delta W)_{\text{due to } \bar{F}_A} = (m\Omega^2 R^3)\ \int_0^1\ (F_\eta\hat{\eta} + F_\zeta\hat{\zeta}) \cdot (\hat{x}\ \delta u + \hat{y}\ \delta v + \hat{z}\ \delta w)dx$$

$$= (m\Omega^2 R^3)\ \int_0^1\ (L_u\delta u + L_v\delta v + L_w\delta w)dx \tag{6.13}$$

where, according to equations (2.2) and (2.3) in Chapter II,

$$L_u = t_{21}\ F_\eta + t_{31}\ F_\zeta \tag{6.14a}$$

$$L_v = t_{22}\ F_\eta + t_{32}\ F_\zeta \tag{6.14b}$$

$$L_w = t_{23}\ F_\eta + t_{33}\ F_\zeta \tag{6.14c}$$

With the virtual rotation about the ξ - axis of the airfoil obtained directly from either equation (2.9) or (2.10) as $[\delta\theta_x + (s\theta_y)\,\delta\theta_z]\hat{\xi}$, the virtual work associated with the aerodynamic moment $\bar{M}_a$ is

$$(\delta W)_{\text{due to } \bar{M}_a} = (m\Omega^2R^3)\int_0^1 M_A\{\delta\theta_x + [(\partial\theta_z/\partial u')\delta u' + (\partial\theta_z/\partial v')\delta v']s\theta_y\}dx$$

$$= (m\Omega^2R^3)\left\{\delta W_B\Big|_0^1 + \int_0^1 \{M_A\delta\theta_x - [M_A(\partial\theta_z/\partial u')s\theta_y]'\delta u - [M_A(\partial\theta_z/\partial v')s\theta_y]'\delta v\}dx\right\} \tag{6.15a}$$

where, upon substitution of $s\theta_y$ by $w'\,x^+$,

$$\delta W_B = M_A\,[(\partial\theta_z/\partial u')\delta u + (\partial\theta_z/\partial v')\delta v]w'x^+ \tag{6.15b}$$

The δW_B term given by equation (6.15b) is included in the boundary condition equation for the system [equation (4.7) in Chapter IV].

Equations (6.13) and (6.15a) yield the following non-dimensional generalized forces due to the aerodynamic loading

$$(Q_u)_{aero} = L_u - [M_A(\partial\theta_z/\partial u')w'x^+]' \tag{6.16a}$$

$$(Q_v)_{aero} = L_v - [M_A(\partial\theta_z/\partial v')w'x^+]' \tag{6.16b}$$

$$(Q_w)_{aero} = L_w \tag{6.16c}$$

$$(Q_{\theta_x})_{aero} = M_A \tag{6.16d}$$

If there are no other forces acting on the blade, such as damping for example, $Q_K = (Q_K)_{aero}$ $(K = u,v,w,\theta_x)$.

To expand the generalized aerodynamic forces into polynomial non-linearities, the angle of attack, α, of the airfoil is assumed to be small so that $U_P = O(\varepsilon)$. With $U_T = O(1)$, equations (6.12 a,b), (6.14 a-c) and (2.3) indicate that the expansion for U_P is needed to $O(\varepsilon^3)$ and the expansion for U_T to $O(\varepsilon^2)$. We then express U_P and U_T as

$$U_P = U_{P1} + U_{P2} + U_{P3} + O(\varepsilon^4) \tag{6.17a}$$

$$U_T = U_{T0} + U_{T1} + U_{T2} + O(\varepsilon^3) \tag{6.17b}$$

where U_{Pi} and U_{Ti} denote the order ε^i $(i = 0,1,2...)$ term in the expansions for U_P and U_T, respectively. With

$$\gamma/6 = O(1)$$

$$c = O(\varepsilon) \quad ; \quad x_A = O(\varepsilon)$$

$$c_{d0}/(2\pi) = O(\varepsilon^2)$$

and $\omega_\xi = \dot{\theta}_1 + w' c\beta + s\beta + w'\dot{v}' + O(\varepsilon^3)$, as given by equations (2.9) and (5.1a,b), the following expansions are obtained from equations (6.12 a,b) for F_η and F_ζ

$$F_\eta = F_{\eta 2} + F_{\eta 3} + O(\varepsilon^4) \tag{6.18a}$$

$$F_\zeta = F_{\zeta 1} + F_{\zeta 2} + F_{\zeta 3} + O(\varepsilon^4) \tag{6.18b}$$

where

$$F_{\eta 2} = (\gamma/6)[U_{P1}^2 - \frac{c_{d0}}{2\pi} U_{T0}^2] = O(\varepsilon^2) \tag{6.18c}$$

$$F_{\eta 3} = (\gamma/6)\ [2U_{P1}\ U_{P2} - (\frac{c}{2} - x_A)\ U_{P1}\ (\dot{\theta}_1 + w'\ c\beta + s\beta) - \frac{c_{d0}}{\pi} U_{T0}U_{T1}] = O(\varepsilon^3) \tag{6.18d}$$

$$F_{\zeta 1} = -(\gamma/6)\ U_{P1}\ U_{T0} = O(\varepsilon) \tag{6.18e}$$

$$F_{\zeta 2} = -(\gamma/6)\ [U_{P1}\ U_{T1} + U_{P2}\ U_{T0} - (\frac{c}{2} - x_A)\ (\dot{\theta}_1 + w'\ c\beta + s\beta)U_{T0} + \frac{c}{4}\ \dot{U}_{P1}] = O(\varepsilon^2) \tag{6.18f}$$

$$F_{\zeta 3} = -(\gamma/6)\ \{U_{P1}\ U_{T2} + U_{P2}\ U_{T1} + U_{P3}\ U_{T0} - (\frac{c}{2} - x_A)\ [(\dot{\theta}_1 + w'\ c\beta + s\beta)U_{T1} + w'\dot{v}'U_{T0}]$$

$$+ \frac{c}{4}\ \dot{U}_{P2} - \frac{c}{4}(\frac{c}{4} - x_A)(\ddot{\theta}_1 + \dot{w}'\ c\beta) + \frac{c_{d0}}{2\pi} U_{T0}\ U_{P1}\} = O(\varepsilon^3) \tag{6.18g}$$

Making use of the expansions for $\sin\theta_1$ and $\cos\theta_1$ given by equations (5.4 a,b), the following expanded forms for the generalized aerodynamic forces are then obtained from equations (6.16 a-d), (6.14 a-c) and (6.6 c)

$$(Q_u)_{aero} = (v'\ s\theta_c - w'\ c\theta_c)\ F_{\zeta 1} + O(\varepsilon^3) \tag{6.19a}$$

$$\begin{aligned}(Q_v)_{aero} = &- F_{\zeta 1}\ s\theta_c + \{(F_{\eta 2} - \theta_{11}\ F_{\zeta 1})c\theta_c - F_{\zeta 2}\ s\theta_c\} \\ &+ \{(F_{\eta_3} - \theta_{11}\ F_{\zeta 2} - v'\ w'\ F_{\zeta 1})c\theta_c \\ &- (\theta_{11}\ F_{\eta 2} - \frac{1}{2}\ \theta_{11}^2\ F_{\zeta 1} + F_{\zeta 3} - \frac{v'^2}{2}\ F_{\zeta 1})s\theta_c\} + O(\varepsilon^4) \\ &+ (\gamma/6)x_A[w'U_{T0}U_{P1}]' + O(\varepsilon^4)\end{aligned} \tag{6.19b}$$

$$(Q_w)_{aero} = F_{\zeta 1}\ c\theta_c + \{(F_{\eta 2} - \theta_{11}\ F_{\zeta 1})s\theta_c + F_{\zeta 2}\ c\theta_c\}$$

$$+ \{(F_{\eta 3} - \theta_{11}\ F_{\zeta 2})s\theta_c + (\theta_{11}\ F_{\eta 2} - \frac{1}{2}\ \theta_{11}^2\ F_{\zeta 1} + F_{\zeta 3} - \frac{w'^2}{2}\ F_{\zeta 1})c\theta_c\}$$

$$+ O(\epsilon^4) \tag{6.19c}$$

$$(Q_{\theta_x})_{aero} = -\ \frac{\gamma}{6}\ \{x_A(U_{T0}U_{P1} + U_{T0}U_{P2} + U_{T1}U_{P1}) + (\frac{c}{4} - x_A)^2(\dot{\theta}_1 + w'\ c\beta + s\beta)U_{T0}$$

$$- \frac{c}{4}(\frac{c}{4} - x_A)\dot{U}_{P1}\} + O(\epsilon^4) \tag{6.19d}$$

To complete the formulation of the generalized aerodynamic forces, only the closed form expressions for the terms U_{Pi} and U_{Ti} ($i = 0,1,2,\ldots$), as defined by equations (6.17 a,b), remain to be determined. Making use of equations (2.3), (5.1 a,b) and (5.3), the following approximate expressions are obtained for U_T and U_p

$$U_T \approx (v'\ c\theta_1 + w'\ s\theta_1)\ (v - \mu\ c\tau)c\beta + (\dot{v} + u\ c\beta - w\ s\beta)c\theta_1$$

$$+ (x\ c\beta - \mu\ s\tau)\ [(1 - \frac{v'^2}{2})c\theta_1 - v'\ w'\ s\theta_1] + [\dot{w} + (v - \mu\ c\tau)s\beta + \lambda\ c\beta]s\theta_1$$

$$\tag{6.20a}$$

$$U_p \approx (\dot{u} + \lambda\ s\beta - v\ c\beta)\ (v'\ s\theta_1 - w'\ c\theta_1) + \mu(c\tau\ c\beta)\ [v'(1 - u' - \frac{v'^2}{2})s\theta_1$$

$$- w'(1 - u' - v'^2 - \frac{w'^2}{2})c\theta_1] - (u\ c\beta - w\ s\beta)s\theta_1$$

$$- (\dot{v} - \mu\ s\tau + x\ c\beta)\ [(1 - \frac{v'^2}{2})s\theta_1 + v'\ w'\ c\theta_1] + v\ s\beta\ c\theta_1$$

$$+ (\dot{w} - \mu\ c\tau\ s\beta + \lambda\ c\beta)\ (1 - \frac{w'^2}{2})c\theta_1 \tag{6.20b}$$

With $s\theta_1$ and $c\theta_1$ expressed as in equations (5.4 a,b), and with U_P when $u = v = w = \theta_{11} = 0$ being an $O(\varepsilon)$ quantity, the following expressions for U_{Ti} and U_{Pi} $(i = 0,1,\ldots)$ are obtained from equations (6.20 a,b)

$$U_{T0} = (x\, c\beta - \mu\, s\tau)c\theta_c \tag{6.21a}$$

$$U_{T1} = -\ (v'\, c\theta_c + w'\, s\theta_c)\mu\, c\tau\, c\beta + \dot{v}\, c\theta_c - (x\, c\beta - \mu\, s\tau)\theta_{11}\, s\theta_c$$

$$+ (\dot{w} - \mu\, c\tau\, s\beta + \lambda\, c\beta)s\theta_c \tag{6.21b}$$

$$U_{T2} = (v'\, c\theta_c + w'\, s\theta_c)v\, c\beta + (v'\, s\theta_c - w'\, c\theta_c)\theta_{11}\, \mu\, c\tau\, c\beta + (u\, c\beta - w\, s\beta)c\theta_c$$

$$+ (v\, s\beta - \dot{v}\, \theta_{11})s\theta_c - (x\, c\beta - \mu\, s\tau)\left(\frac{v'^2}{2}\, c\theta_c + v'\, w'\, s\theta_c + \frac{1}{2}\theta_{11}^2\, c\theta_c\right)$$

$$+ (\dot{w} - \mu\, c\tau\, s\beta + \lambda\, c\beta)\theta_{11}\, c\theta_c \tag{6.21c}$$

$$U_{P1} = [(\lambda\, c\beta - \mu\, c\tau\, s\beta)c\theta_c - (x\, c\beta - \mu\, s\tau)s\theta_c] - (x\, c\beta - \mu\, s\tau)\theta_{11}\, c\theta_c$$

$$+ (v'\, s\theta_c - w'\, c\theta_c)\mu\, c\tau\, c\beta - \dot{v}\, s\theta_c + \dot{w}\, c\theta_c \tag{6.21d}$$

$$U_{P2} = (w'\, c\theta_c - v'\, s\theta_c)v\, c\beta + \theta_{11}\, (v'\, c\theta_c + w'\, s\theta_c)\mu\, c\tau\, c\beta - (u\, c\beta - w\, s\beta)s\theta_c$$

$$- \dot{v}\, \theta_{11}\, c\theta_c + (x\, c\beta - \mu\, s\tau)\left[\frac{v'^2}{2}\, s\theta_c - v'\, w'\, c\theta_c + \frac{1}{2}\theta_{11}^2\, s\theta_c\right] + v\, s\beta\, c\theta_c$$

$$- (\dot{w} - \mu\, c\tau\, s\beta + \lambda\, c\beta)\theta_{11}\, s\theta_c \tag{6.21e}$$

$$
\begin{aligned}
U_{P3} = {} & (\dot{u} + \lambda\, s\beta)\,(v'\, s\theta_c - w'\, c\theta_c) - \theta_{11}\,(v'\, c\theta_c + w'\, s\theta_c)\, v\, c\beta \\
& - \mu(c\tau\, c\beta)\,[\tfrac{1}{2}\,\theta_{11}^2\,(v'\, s\theta_c - w'\, c\theta_c) + v'(u' + \tfrac{v'^2}{2})s\theta_c - w'\,(u' + v'^2 + \tfrac{w'^2}{2})c\theta_c] \\
& - (u\, c\beta - w\, s\beta)\theta_{11}\, c\theta_c + \dot{v}[\tfrac{1}{2}(\theta_{11}^2 + v'^2)s\theta_c - v'w'\, c\theta_c] \\
& + (x\, c\beta - \mu\, s\tau)\,[(\tfrac{1}{6}\,\theta_{11}^2 + \tfrac{v'^2}{2})c\theta_c + v'\, w'\, s\theta_c]\dot{\theta}_{11} - \theta_{11}\, v\, s\beta\, s\theta_c \\
& - \tfrac{1}{2}\,(\dot{w} - \mu\, c\tau\, s\beta + \lambda\, c\beta)\,(\theta_{11}^2 + w'^2)c\theta_c
\end{aligned}
\tag{6.21f}
$$

CHAPTER VII - CONCLUDING REMARKS

A case of particular interest, for which the equations of motion reduce to a simpler form, is that of an inextensional blade. For this case no EA terms appear in the differential equations of motion as the blade behaves as infinitelly stiff to axial extension. The equations of motion for this case may be obtained by taking the limit $EA \to \infty$ in equations (5.2b - d), with u obtained from equation (5.8) in order to recover the $O(\varepsilon^3)$ inextensionality constraint [55]. It should be noted that the spatial independent variable for this case is now the arc-length $r(0 \leq r \leq 1)$ along the blade's neutral axis.

A question that naturally arises is "what line along the blade's span is the inextensional line." It can be shown with the aid of equations (5.7) and (3.9b) that in the limit as $EA \to \infty$ the line that connects the area centroid of each cross section (point $C_a(\xi = 0, \eta = e_A, \zeta = 0)$ shown in Fig. 6.1) along the blade's span is the inextensional line. According to equations (3.9b) and (2.7a - c), if the line of area centroids is inextensional, the following condition has to be satisfied when warping is neglected, for simplicity,

$$\left|\frac{\partial}{\partial x}\bar{r}_{C_a}\right|^2 = [(1/x^+ - e_A\rho_\zeta)t_{11} + e_A\rho_\xi t_{31}]^2 + [(1/x^+ - e_A\rho_\zeta)\, t_{12} + e_A\rho_\xi t_{32}]^2$$

$$+ [(1/x^+ - e_A\rho_\zeta)t_{13} + e_A\rho_\xi t_{33}]^2 \equiv (1/x^+ - e_A\rho_\zeta)^2 + (e_A\rho_\xi)^2 = 1 \qquad (7.1a)$$

or

$$\{[(1 + u')^2 + v'^2 + w'^2]^{1/2} - e_A\rho_\zeta\}^2 + (e_A\rho_\xi)^2 - 1 = 0 \qquad (7.1b)$$

It can now be readily verified that, in the limit $EA \to \infty$, equation (5.7) is simply the $O(\varepsilon^3)$ expansion to equation (7.1b).

An alternate way to obtain the differential equations of motion governing the flexural - flexural - torsional motion of an inextensional blade via Hamilton's principle is to adjoin the inextensionality constraint to the Lagrangian of the motion by a Lagrange multiplier as in [62, 68]. The advantage of using the limiting process previously mentioned is that both the extensional and inextensional cases may be investigated via the same equations of motion.

The differential equations of motion for an extensional rotor blade were derived in this report in terms of the three elastic displacements $u(x, \tau)$, $v(x, \tau)$ and $w(x, \tau)$ for any point on the blade's elastic axis, and an Euler angle $\theta_x(x, \tau)$. The total angle of twist of the blade, $\phi_t(x, \tau)$, may be obtained by first expressing the twisting curvature $C_\xi = \rho_\xi x^+$ defined by equation (2.10) as [32, 33, 55, 61, 63]

$$\phi_t^+ \equiv \phi_t' x^+ = x^+(\theta' + \theta_x' + \theta_z' s\theta_y) \triangleq x^+ \rho_\xi \tag{7.2}$$

and then integrating equation (7.2) over the domain $0 \leq x \leq 1$. Since the total angle of twist, ϕ_t, includes the non-elastic pitch angle $\theta(x, \tau)$, the elastic angle of twist of the blade, $\phi(x, \tau)$, is then

$$\phi(x, \tau) = \int_0^x (\phi_t' - \theta')dx + \phi_t(x=0, \tau=\tau) - \theta(x=0, \tau=\tau)$$

$$= \theta_x(x, \tau) + \int_0^x \theta_z' s\theta_y \, dx = \theta_x + \int_0^x v'' w' \, dx + O(\varepsilon^4) \tag{7.3}$$

Equation (7.3), relating θ_x to ϕ, could be used in the outset to derive the differential equations of motion for the blade in terms of u, v, w and ϕ. When this approach is taken, the expression for $\delta\rho_\xi$ for example, given by equation

(4.3 d), simply becomes $\delta\phi'$. In such case, the term $D_\xi(\rho_\xi - \theta')\delta\rho_\xi = D_\xi\phi'\delta\phi'$, obtained when the variation of the strain energy U given by equation (3.17) is taken, gives rise to a single term $D_\xi\phi''$ in the $\delta\phi$ equation. In contrast, if $\delta\theta_x$ is used instead, the term $D_\xi\phi'\ \delta\rho_\xi$ gives rise to the term $D_\xi\phi''$ in the $\delta\theta_x$ equation but also to the $O(\varepsilon^2)$ terms $(D_\xi\phi'w')''$ and $(D_\xi\phi'v'')'$ in the δv and δw equations respectively [equations (5.2 b, c)]. However, the differential equations derived via δu, δv, δw and $\delta\theta_x$ as done here and in [62, 63], and those derived via δu, δv, δw and $\delta\phi$, as done in [32] to $O(\varepsilon^2)$, are equivalent to each other since one set of equations may be converted to the other by the integral transformation given by equation (7.3).

As mentioned in Chapter IV, the term $EI_\xi(1 - x^{+2})\rho_\xi/2$ that appears in equation (4.6 g) is a tension - torsion coupling term that reflects the increase in the effective torsion stiffness of the blade due to axial tension. To show this, the case of a non-rotating beam undergoing only static axial extension and torsion, as in [75], is now briefly addressed. For simplicity, it will be assumed that no external torsional moment, Q_{θ_x}, is applied to the beam and that the beam's cross section is symmetric about both the η and ζ axes. From the expression for the variation δU of the strain energy, the following differential equation is obtained when the coefficient of $\delta\theta_x'$ is equated to zero*

$$D_\xi\theta_x' + \frac{E}{2} I_\xi(1 - x^{+2})(\theta_x' + \theta') + \frac{E}{2} B_1(\theta_x'^2 + 2\theta_x'\theta')(\theta_x' + \theta')x^{+2}$$

$$- (EC_1\theta_x'')'/x^{+2} = 0 \qquad (7.4a)$$

---

* Here D_ξ, for example, denotes the torsional stiffness of the beam, as defined in Chapter III, but without carrying out the normalization process implied by equations (3.15 c) and (3.17). A similar observation applies to the other quantities in equations (7.4a,b).

With $x^+ = 1/(1 + u') \simeq 1 - u'$, and $u' = -(\int_1^x Q_u dx)/(EA) = T_e/(EA)$ as given by equations (5.7) and (5.5), equation (7.4a) may be approximated as

$$(D_\xi + T_e I_\xi/A)\theta'_x + T_e I_\xi\, \theta'/A - (EC_1\theta''_x)' + O(\varepsilon^4) = 0 \tag{7.4b}$$

The coefficient of θ'_x in equation (7.4b) is the effective torsional stiffness of the beam [14, 75, 84]. For slender beams, the first two terms in that equation are the dominant ones and equation (7.4b) reduces to equation (18) in [14]. A more accurate formulation for the torsion - tension coupling problem that takes into account the non-orthogonality of the coordinate system (x, η, ζ) for the elastically undeformed pre-twisted beam was recently presented in [75]. The work in [75] disclosed that an extra term proportional to θ' should be present in equation (7.4b) and that for values of G/E that are typical of isotropic materials the influence of this extra term is quite small and may be neglected.

APPENDIX - KINETIC AND STRAIN ENERGY CONTRIBUTIONS T^*, U_s^* and U^*

The warping terms not shown explicitly in equation (3.16) are included in T^* which is given as

$$
\begin{aligned}
T^*/(m\Omega^2R^3) = \int_0^1 C_{T1}\Big\{ & \Big[t_{12}\rho_\xi x^+ c\beta - (t_{11}\rho_\xi x^+)^\cdot\Big](\dot{t}_{31} - t_{32}c\beta) \\
& + \Big[(t_{13}s\beta - t_{11}c\beta)\rho_\xi x^+ - (t_{12}\rho_\xi x^+)^\cdot\Big](\dot{t}_{32} + t_{31}c\beta - t_{33}s\beta) \\
& - \Big[t_{12}\,\rho_\xi x^+ s\beta + (t_{13}\rho_\xi x^+)^\cdot\Big](\dot{t}_{33} + t_{32}s\beta)\Big\}dx \\
& + \frac{1}{2}\int_0^1 C_{T2}\Big\{\Big[t_{12}\rho_\xi x^+ c\beta - (t_{11}\rho_\xi x^+)^\cdot\Big]^2 + \Big[t_{12}\rho_\xi x^+ s\beta + (t_{13}\rho_\xi x^+)^\cdot\Big]^2 \\
& + \Big[(t_{13}s\beta - t_{11}c\beta)\rho_\xi x^+ - (t_{12}\rho_\xi x^+)^\cdot\Big]^2\Big\}dx
\end{aligned}
\tag{A.1}
$$

where

$$
mC_{T1} = R^2 \iint_A \rho\psi\zeta \, d\eta \, d\zeta \quad ; \quad mC_{T2} = R^2 \iint_A \rho\psi^2 \, d\eta \, d\zeta \tag{A.2}
$$

The expression for the term U_s^* appearing in equation (3.14) is

$$
\begin{aligned}
U_s^* = \frac{R^3}{2}\int_0^1 \iint_A G_s\Big\{ & \Big[(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\eta - \psi\rho_\zeta\Big]^2 \rho_\xi^2 x^{+4} + \Big[(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\zeta + \psi\rho_\eta\Big]\rho_\xi^2 x^{+4} \\
& + \psi^2\Big[(\partial\psi/\partial\eta)^2 + (\partial\psi/\partial\zeta)^2\Big]\Big[\rho_\xi(\rho_\xi x^+)' x^+ - \theta'\theta''\Big]^2 x^{+2} \\
& + 2\psi(\partial\psi/\partial\zeta)\Big[\rho_\xi(\rho_\xi x^+)' x^+ - \theta'\theta''\Big]\Big\{(\eta - \partial\psi/\partial\zeta)\,(\rho_\xi - \theta') + \Big[(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\zeta + \psi\rho_\eta\Big]\rho_\xi x^+\Big\}x^{+2}
\end{aligned}
$$

$$- 2\psi(\partial\psi/\partial\eta)\left[\rho_\xi(\rho_\xi x^+)' x^+ - \theta'\theta''\right]\left\{(\zeta + \partial\psi/\partial\eta)(\rho_\xi - \theta') + \left[(\eta\rho_\zeta - \zeta\rho_\eta)\partial\psi/\partial\eta - \psi\rho_\zeta\right]\rho_\xi x^+\right\}x^{+2}$$

$$+ \left[(\rho_\xi^2 x^{+2} - \theta'^2)\ (\partial\psi/\partial\eta)\ (\partial\psi/\partial\zeta)\right]^2\Bigg\} d\eta\ d\zeta dx \tag{A.2}$$

The strain energy contribution U^* appearing in equation (3.17) is obtained directly from equation (3.14) as

$$U^* = U_s^* + \Bigg\{\frac{1}{8}\int_0^1 \iint_A E\ \Big\{\psi^4 H_2^2 + 4\psi^3 H_2\ (L_2 + H_3) + 4\psi^2\ (H_3^2 + 2L_2H_3)$$

$$+ 2\psi^2 H_2\ (H_1 + L_1) + 4\psi\left[(L_2 + H_3)H_1 + L_1H_3\right]\Big\} d\eta\ d\zeta\ dx$$

$$+ \frac{1}{8}\int_0^1 \iint_A E\ \left[(\zeta\rho_\eta - \eta\rho_\zeta)^4 + 2(\eta^2 + \zeta^2)\ (\zeta\rho_\eta - \eta\rho_\zeta)^2\ (\rho_\xi^2 - \theta'^2)\right]x^{+4}\ d\eta\ d\zeta\ dx$$

$$+ \frac{1}{2}\int_0^1 EC_1\left\{x^{+2}\left[(\rho_\xi x^+)' - \theta'' x^+\right]^2 - \theta_x''^2\right\}dx$$

$$- \int_0^1 EC_1^*\rho_\eta\left\{\left[(\rho_\xi x^+)' - \theta'' x^+\right]x^{+2} - (\rho_\xi' - \theta'')\right\}dx\Bigg\}\ m\Omega^2 R^3 \tag{A.3}$$

where,

$$H_1 = 1 - x^{+2} - 2\eta\rho_\zeta x^+ + (\zeta\rho_\eta - \eta\rho_\zeta)^2\ x^{+2} + (\eta^2 + \zeta^2)\ (\rho_\xi^2 - \theta'^2)x^{+2} \tag{A.4}$$

$$L_1 = 2\zeta\ \rho_\eta x^+ \tag{A.5}$$

$$H_2 = [(\rho_\xi x^+)'^2 + (\rho_\eta^2 + \rho_\zeta^2)\rho_\xi^2 x^{+2} - \theta''^2]x^{+2} \tag{A.6}$$

$$L_2 = \theta'' x^{+2} - (\rho_\xi x^+)' x^+ \tag{A.7}$$

$$H_3 = (\zeta\rho_\zeta + \eta\rho_\eta)\rho_\xi^2 x^{+3} - (\zeta\rho_\eta - \eta\rho_\zeta)\ (\rho_\xi x^+)' x^{+2} \tag{A.8}$$

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