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KINEMATIC PRECISION OF GEAR TRAINS

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ABSTRACT

Kinematic precision is affected by errors which are the result of either intentional adjustments or accidental defects in manufacturing and assembly of gear trains. This paper explains a general method for the determination of kinematic precision of gear trains. The general method is based on the exact kinematic relations for the contact point motions of the gear tooth surfaces under the influence of errors. An approximate method is also explained.

Example applications of the general and approximate methods are demonstrated for gear trains consisting of involute (spur and helical) gears, circular-arc (Wildhaber-Novikov) gears, and spiral-bevel gears. Gear noise measurements from a helicopter transmission are presented and discussed with relation to the kinematic precision theory.

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Transmission error is a measure of the kinematic precision of gear trains. Transmission error is defined as the departure of the meshed gear pair (or entire gear train) from a constant ratio of angular motions as defined by the ratio of tooth numbers. It is true that in a gear pair each gear has a whole number of teeth and this defines the nominal ratio of angular positions between the two. But the instantaneous ratio during a meshing cycle can vary slightly from the nominal ratio. Transmission error is the measure of instantaneous variation from the ideal nominal value. The precision of gears was investigated by Litvin [1]*, Litvin and Gutman [2], and Michalec [3].

When the mating teeth in a gear train have profiles that transmit motion having no error, they are said to be conjugate pairs. In theory, it is possible to select an arbitrary shape for a driving tooth and then to find a profile for the driven tooth which will give conjugate action. Involute gear teeth happen to have the same form for driving and driven member teeth. A benefit of the involute form is that small errors in center distance between gears will not produce transmission errors. This is not true for other tooth profile forms [4, 5].

In general, there are many causes for transmission error, and they cannot be avoided in practice. Such things as shaft misalignment, profile error, tooth deflections under load, mounting location errors, and gear support deflections may combine to cause transmission error. The effects of transmission error are most often harmful. These are high vibration and noise, pitting and scoring of gear teeth, and reduced reliability of the gear train. Sometimes it is beneficial to introduce small intentional errors into the gear

*Numbers in square brackets denote references.
tooth profile to compensate for the probable occurrence of accidental and unavo-
nable errors in the assembled and operating gear train. Tip relief to re-
duce dynamic loading and combined mismatch in spiral-bevel gears to reduce mis-
alignment sensitivity are two examples of intentional errors which are ben-
ficial.

Baxter has studied the effect of various types of misalignment on tooth
contact in bevel and hypoid gears [6]. Townsend, Coy, and Hatvani have
examined gear train noise as a test of its precision during an intentional
loss of lubricant destruction test [7].

Of course, the effect of all errors (intentional or otherwise) on
transmission error must be predicted analytically if the gear design process
is to remain rational and not collapse into a confusing heap of empiricism.

Errors of manufacturing and assemblage of gears induce kinematic errors
in gear-drives which may be presented by the following function:

\[ \Delta \theta_2(\varphi_1, \Delta Q) \]  

(1)

Here \( \varphi_1 \) is the angle of rotation of the driving gear, \( 1 \),

\[ \Delta Q = (\Delta q_1, \Delta q_2, \ldots) \]  

(2)

is the vector of errors, and

\[ \Delta \theta_2 = \varphi_2^0 - \varphi_2 \]  

(3)

is the kinematic error of the gear drive, represented as the difference between the
theoretical and actual angles of rotation of driven gear 2.

In this paper two methods to determine Function (1) are presented: (a) a
numerical method for computer solution and (b) an approximate geometric solution
which leads to simple, accurate results in an analytical form.
NOMENCLATURE

A shortest distance between gear axes of rotation
\( \Delta e_i \) eccentricity vector of gear i
\( H_1, H_2 \) axial settings of gears (see Fig. 2)
\( M_{ij} \) angular velocity ratio, gear i, gear j
\( N_i \) number of teeth on gear i
\( n_f(i) \) vector function representing unit normals of surface of gear i in fixed coordinate system (f)
\( \Delta n(i) \) change in unit normal vector due to errors in gear i
\( \Delta n_r(i) \) change in unit normal vector due to point motion relative to gear i
\( \Delta n_{tr}(i) \) change in unit normal vector due to transfer point motion with gear i
\( n_{abs}(i) \) absolute velocity of tip of unit normal vector of surface i
\( n_r(i) \) similar to \( \Delta n_r(i) \) and \( \Delta n_{tr}(i) \) but velocities rather than displacements
\( O_i \) center of rotation of gear i
\( O(i) \) geometrical center of gear i
\( P \) pitch point
\( \Delta Q_i \) vector of errors
\( \Delta q_i \) components of vector of errors
\( R \) vector from origin to axis of gear rotation
\( r_b(i) \) base circle radius of gear i
\( r_f(i) \) vector function representing surface of gear i in fixed coordinate system (f)
velocity of contact point
coordinate system i
coordinate system rigidly connected to frame
absolute displacement of contact point of gear i
displacement of contact point due to errors in gear i
displacement of contact point relative to gear i
displacement of contact point in transfer motion with gear i
tangent plane
surface coordinate of gear i surface
similar to \( ds_{\text{abs}}^i \), \( ds_r^i \), \( ds_{\text{tr}}^i \), but velocities rather than displacements
angular position of eccentricity vector of gear i
initial angular position of eccentricity vector of gear i
rotation vector representing position change of gear axis of rotation
surface coordinate of gear surface i
change in kinematic error function as measured on shaft of gear i
proportionality factor
radius vector
surface i
angle of rotation of gear 1
actual angle of rotation of gear 2
theoretical angle of rotation of gear 2
The theory and exact solution method for kinematic precision

In the process of motion the tooth surfaces of two gears, $i_1$ and $i_2$ (Fig. 1), are in tangency if the following equations are satisfied:

$$ r_f^1(u_1, \phi_1, \varphi_1) = r_f^2(u_2, \phi_2, \varphi_2) $$

$$ n_f^1(u_1, \phi_1, \varphi_1) = n_f^2(u_2, \phi_2, \varphi_2) $$

Here $r_f^i$ is the position vector of the contact point on gear $i$; $n_f^i$ is the surface unit normal vector at the contact point $M$; $u_i$, and $\phi_i$ are the surface coordinates of the gear surfaces; and $\varphi_i$ is the angle of rotation of gear $i$. Subscript $f$ denotes a coordinate system which is rigidly connected to the frame.

For a gearset with kinematic errors, represented by $\Delta Q_1$ and $\Delta Q_2$, conditions for tangency may be expressed as

$$ r_f^1(u_1, \phi_1, \varphi_1, \Delta Q_1) = r_f^2(u_2, \phi_2, \varphi_2, \Delta Q_2) $$

$$ n_f^1(u_1, \phi_1, \varphi_1, \Delta Q_1) = n_f^2(u_2, \phi_2, \varphi_2, \Delta Q_2) $$

Equations (6) and (7) yield the functions

$$ \varphi_2(\phi_1, \Delta Q_1, \Delta Q_2) = \varphi_2^0(\phi_1) + \Delta \varphi_2(\phi_1, \Delta Q_1, \Delta Q_2) $$

$$ u_1(\phi_1, \Delta Q_1, \Delta Q_2) ; \phi_1(\phi_1, \Delta Q_1, \Delta Q_2) $$
The functions

\[ r_f(i)(u_1, \theta_1) ; u_1(\varphi_1, \Delta Q_1, \Delta Q_2), \theta_1(\varphi_1, \Delta Q_1, \Delta Q_2) \quad (i = 1, 2) \]  

(10)

represent the path of the contact point on gear surface \( \Sigma_i \) corresponding to the meshing of gears with errors of manufacturing and assembly. Functions

\[ r_f(i)(u_1^0(\varphi_1), \theta_1^0(\varphi_1) ; u_1^0(\varphi_1), \theta_1^0(\varphi_1) \quad (i = 1, 2) \]  

(11)

represent the path of the contact point on gear surface \( \Sigma_i \) corresponding to meshing without errors. Comparison of functions (10) and (11) yields the change of the contact point path induced by errors.

Consider the solution of equations (4) and (5) and (6) and (7). Vector equations (4) and (5) yield only five independent scalar equations since

\[ |n(1)| = |n(2)| = 1. \]  

These equations may be presented as

\[ f_j(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2) = 0 \quad (j = 1, 2, \ldots, 5) \]  

(12)

It is assumed that

\[ \{f_1, f_2, f_3, f_4, f_5\} \in C^1 \]

The symbol \( C^1 \) denotes that functions \( f_j \) have continuous partial derivatives of the first order (at least) by all arguments.

It is assumed that equation system (12) is satisfied by a set of parameters

\[ p(1) = (u_1(1), \theta_1(1), \varphi_1(1), u_2(1), \theta_2(1), \varphi_2(1)) \]  

(13)

and that surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are in tangency at point \( M_0 \). Surfaces \( \Sigma_1 \) and \( \Sigma_2 \) will be in point contact in the neighborhood of \( M_0 \) if by the set of parameters \( p(1) \) the following Jacobian differs from zero:
If inequality (14) is satisfied, equation (12) may be solved in the neighborhood of $P^{(1)}$ with the functions

$$\{u_1(\theta_1), \phi_1(\xi_1), u_2(\theta_2), \phi_2(\xi_2), \phi_0(\xi_1)\} \in C^1$$

(15)

The function $\phi_0(\xi_1)$ represents the ideal law of motion. In most cases (for conjugate tooth action) function $\phi_0(\xi_1)$ is linear.

Equations (6) and (7) also yield a system of five independent equations in six unknowns $(u_1, \theta_1, u_2, \theta_2, \phi_1, \phi_2)$

$$g_j(u_1, \theta_1, u_2, \theta_2, \phi_1, \phi_2, \Delta Q) = 0 \quad (j = 1, 2, \ldots, 5)$$

(16)

It is assumed that this system is satisfied by a set of parameters

$$p^{(2)} = (u_1^{(2)}, \theta_1^{(2)}, u_2^{(2)}, \theta_2^{(2)}, \phi_1^{(2)}, \phi_2^{(2)})$$

(17)

with the same value of $\phi_1^{(1)}$ as in the set $P^{(1)}$. If in the neighborhood of $P^{(2)}$ the Jacobian

$$\frac{D(g_1, g_2, g_3, g_4, g_5)}{D(u_1, \theta_1, u_2, \theta_2, \phi_2)} \neq 0$$

(18)

then system (16) may be solved with the functions

$$\{u_1(\theta_1, \Delta Q), \phi_1(\xi_1, \Delta Q), u_2(\theta_1, \Delta Q), \phi_2(\xi_1, \Delta Q), \phi_0(\xi_1, \Delta Q)\} \in C^1$$

(19)
Function \( \psi_2(\varphi_1, \Delta Q) \) represents the actual law of motion transformation - the law of transformation of motion which corresponds to errors of manufacturing and assembly. Kinematic errors of the gear drive are represented by the function

\[
\Delta \psi_2 = \psi_2(\varphi_1, \Delta Q) - \psi_2(\varphi_1)
\]  

This method of solution can provide, not only the kinematic errors of a gear-set, but also the new path of the contact point (see functions (10)).

In general, the numerical solution of a system of five nonlinear equations is a difficult problem which requires many iterations. To save computer time an effective method of solution was recently proposed by Litvin and Gutman [2]. The principle of this method follows:

The system of equation (16) may be represented as

\[
f_1(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2, A, H_1, H_2, \Delta Q) = 0
\]  

\[
f_2(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2, A, H_1, H_2, \Delta Q) = 0
\]  

\[
f_3(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2, A, H_1, H_2, \Delta Q) = 0
\]  

\[
f_4(u_1, \theta_1, u_2, \theta_2, \varphi_2, \Delta Q) = 0
\]  

\[
f_5(u_1, \theta_1, u_2, \theta_2, \varphi_2, \Delta Q) = 0
\]

Equations (21) to (23) are determined from vector equation (6), and equations (24) to (25) from vector equation (7). Here, \( A \) represents the shortest distance between the axes of rotation of the two gears and \( H_1 \) and \( H_2 \) represent the axial settings of the gears (Fig. 2).
Systems $S_1(X_1,Y_1,Z_1)$ and $S_2(X_2,Y_2,Z_2)$ shown in figure 2 are rigidly connected to the driving and driven gears, respectively. Now suppose that some points $M_1(u_1,e_1)$ and $M_2(u_2,e_2)$ on surfaces $\Sigma_1$ and $\Sigma_2$ are chosen. With a set of known parameters $(u_1,e_1,u_2,e_2)$, equations (24) and (25) become a system of two equations in two unknowns which may be expressed as

$$F_1(\Psi_1,\Psi_2) = 0$$

$$F_2(\Psi_1,\Psi_2) = 0$$

Upon solving for $\Psi_1$ and $\Psi_2$, one checks that the following equations are satisfied:

$$A - K_1(u_1,e_1,u_2,e_2,\Psi_2,4Q) = 0$$

$$H_1 - K_2(u_1,e_1,u_2,e_2,\Psi_2,4Q) = 0$$

$$H_2 - K_3(u_1,e_1,u_2,e_2,\Psi_2,4Q) = 0$$

where $A$, $H_1$, $H_2$, and $4Q$ are given values.

In general, the solution of the above two systems of equations ((26) and (27) and (28) and (30)) requires an iterative procedure. In practice, one of the four variable parameters $(u_1,e_1,u_2,e_2)$ is fixed, and the other three are changed such that the two equation systems are satisfied. The advantage of the above method lies in the ability to divide the system of five equations ((25) to (30)) into two subsystems of two and three equations, and to solve them separately.

This method was applied to investigate the sensitivity of Wildhaber-Novikov gears to errors of center distance mounting [1]. These gears are generated by two rack cutters which have normal sections as shown in Fig. 3.
Surfaces of the rack cutters are in tangency along a straight line $M - M'$ which is parallel to axis $Z_a$ and passes through point $M$ of the normal section. In the normal section, the shape of each rack cutter is a circular arc of radius $\rho_i$ ($i = I, II$). The location of point $M$ is defined by the parameter $\theta = \theta_0$ (Fig. 3). The line $M - M'$ generates a helix on the gear tooth surface which is the path of contact points. Although the procedure described above is primarily for numerical solution, in this case analytical results were obtained.

The investigation showed that the error of center distance $\Delta A$ resulted in the change of location of the contact point path. The new location of the path is represented by the equation

$$\sin \theta = \frac{\Delta A - b_{II}}{\rho_I - \rho_{II}}$$

$$R_1 = \sqrt{(\rho_I \sin \theta + r_1)^2 + (\rho_I \cos \theta \sin \lambda)^2}$$ (31)

Where $R_1$ is the radius of gear $1$ cylinder on which lies the helix of the new contact point path. Parameters $\theta$, $\rho_I$, $\rho_{II}$, and $b_{II}$ are shown in Fig. 3; $r_1$ is the pitch cylinder radius.

With $\Delta A = 0$,

$$\sin \theta_0 = \frac{b_{II}}{\rho_{II} - \rho_I}$$ (32)

where $\theta_0$ is the parameter corresponding to the desired location of the contact point path.
The following relations are the basis of the second method for the determination of kinematic errors in gear trains. As stated above, the tangency of gear tooth surfaces is represented by equations (4) and (5). Because of the continuity of tangency of these surfaces, it is required that

\[ \dot{r}_f^1(u_1, \Theta_1, \Phi_1) = \dot{r}_f^2(u_2, \Theta_2, \Phi_2) \]  

\[ \dot{n}_f^1(u_1, \Theta_1, \Phi_1) = \dot{n}_f^2(u_2, \Theta_2, \Phi_2) \]  

Here \( \dot{r}_f^i \) (\( i = 1, 2 \)) is the velocity of the surface contact point in absolute motion (with respect to the frame); \( \dot{n}_f^i \) is the velocity of the tip of the surface unit normal in absolute motion (also with respect to the frame). Henceforth, \( \dot{r}_f^i \) is designated as \( v^i \) and \( \dot{n}_f^i \) as \( n^i \).

The velocity of absolute motion may be represented as the sum of two components: (a) the velocity of transfer motion (together with the surface) and (b) the velocity of relative motion (with respect to the surface). Consequently,

\[ v^1 = v^{(1)} + v^{(1)} \]  
\[ v^2 = v^{(2)} + v^{(2)} \]  
\[ n^1 = n^{(1)} + n^{(1)} \]  
\[ n^2 = n^{(2)} + n^{(2)} \]

For a surface represented by a vector-function

\[ r^i(u_i, \Theta_i, \Phi_i) \quad (i = 1, 2) \]
and a surface unit normal
\[ n(i)(u_i, \theta_i, \psi_i) \quad (i = 1, 2) \] (38)

the following comes from definitions (a) and (b) above:
\[ v_{tr} = \frac{ar(i)}{\omega_i} \frac{d \omega_i}{dt}, \quad v_r = \frac{ar(i)}{au_i} \frac{du_i}{dt} + \frac{ar(i)}{\omega_i} \frac{d \omega_i}{dt} \] (39)
\[ n_{tr} = \frac{an(i)}{\omega_i} \frac{d \omega_i}{dt}, \quad n_r = \frac{an(i)}{au_i} \frac{du_i}{dt} + \frac{an(i)}{\omega_i} \frac{d \omega_i}{dt} \] (40)

Transfer velocity may also be determined in a kinematical way by supposing that a gear with surface \( z_i \) rotates about an axis that does not pass through the origin \( O_i \) of coordinate system \( S_f \), which is rigidly connected to the frame. Vector \( \omega(i) \) is the vector of angular velocity of the gear's rotation. Then,
\[ v_{tr} = \omega(i) \times r(i) + R(i) \times \omega(i) \] (41)

where \( r(i) \) is the position vector drawn from origin \( O_i \) to the contact point on the tooth surface and \( R(i) \) is a vector drawn from \( O_i \) to an arbitrary point on the gear's axis of rotation.

The transfer velocity \( n_{tr} \) is represented by the equation
\[ n_{tr} = \omega(i) \times n(i) \] (42)

Equations (33) to (36) yield the following kinematic relations for two tooth surfaces which are in continuous tangency:
\[ v_{tr}^{(1)} + v_r^{(1)} = v_{tr}^{(2)} + v_r^{(2)} \] (43)
\[ \mathbf{n}^{(1)}_{tr} + \mathbf{n}^{(1)}_{r} = \mathbf{n}^{(2)}_{tr} + \mathbf{n}^{(2)}_{r} \] (44)

Equations (43) and (44) were first proposed by Litvin [1]. On the basis of these equations, important problems in the theory of gearing were solved, such as avoiding tooth undercutting, deriving the relations between curvatures of two gear tooth surfaces in mesh, and determining the kinematic errors of gear drives which are caused by errors of manufacturing and assembly.

**APPROXIMATE METHOD OF CALCULATION OF GEAR DRIVE KINEMATIC ERRORS**

As a general rule, kinematic errors of a gear drive determined by the exact method must be obtained in a numerical way using a computer. This is a disadvantage of the exact method. Therefore, an approximate method with the opportunity to obtain accurate results analytically is now presented.

Figure 4 shows two gear surfaces \( \Sigma_1 \) and \( \Sigma_2 \) which are not in tangency due to errors of manufacturing and assembly. Points \( M^{(1)} \) and \( M^{(2)} \) do not coincide, position vectors \( r^{(1)} \) and \( r^{(2)} \) are not equal, and surface unit normal vectors \( n^{(1)}_f \) and \( n^{(2)}_f \) do not coincide. To bring the two surfaces into contact it is sufficient to hold one gear fixed and rotate the other gear by an additional small angle. Since the gear with surface \( \Sigma_1 \) is the driving gear, it is preferable to fix the position of surface \( \Sigma_1 \) and rotate surface \( \Sigma_2 \) to bring it back into contact with \( \Sigma_1 \). The additional angle of rotation \( \phi_2 \) represents the change of the theoretical angle of rotation \( \phi_2^0 \) which is exerted by errors of the manufacturing of manufacturing and assembly. The \( \phi_2 \) is as yet an unknown function of the vector of errors \( \Delta Q \) and varies in the process of motion. Thus

\[ \Delta \phi_2 = f(\phi_2^0, \Delta Q) \] (45)
The determination of the function (45) is based on the following kinematic relations, which are analogous to (43) and (44):

\[
\begin{align*}
    ds(1)_r + ds(1)_q + ds(1)_r &= ds(2)_r + ds(2)_r + ds(2)_q \\
    dn(1)_r + dn(1)_q + dn(1)_r &= dn(2)_r + dn(2)_r + dn(2)_q
\end{align*}
\]  (46)  (47)

where \( ds(i)_q \) is the displacement of contact point and \( dn(i)_q \) \( (i = 1, 2) \) is the change in direction of the surface unit normal due to errors of manufacturing and assembly. To bring the surfaces into contact, it is sufficient to rotate only gear 2, holding gear 1 at rest. Therefore, \( ds(1)_r \text{ and } dn(1)_r \) are zero, and

\[
\begin{align*}
    ds(1)_r + ds(1)_q &= ds(2)_r + ds(2)_r + ds(2)_q \\
    dn(1)_r + dn(1)_q &= dn(2)_r + dn(2)_r + dn(2)_q
\end{align*}
\]  (48)  (49)

To determine relations between \( ds(2)_r \text{, } ds(1)_q \text{, and } ds(2)_q \) take the following scalar products:

\[
n \cdot (ds(1)_r + ds(1)_q) = n \cdot (ds(2)_r + ds(2)_r + ds(2)_q)
\]  (50)

Since vectors \( ds(1)_r \) and \( ds(2)_r \) must lie in the common tangent plane \( T \), equation (50) is reduced to the following:

\[
n \cdot ds(2)_r = n \cdot (ds(1)_q - ds(2)_q)
\]  (51)

The vector \( ds(2)_r \) may be represented by the following cross product:
where $\phi^{(2)}$ is the incremental angle of rotation of gear 2 and $\mathbf{r}^{(M)}_2$ is the position vector drawn from an arbitrary point of the axis of rotation to the contact point M.

Equations (51) and (52) yield

$$
\begin{bmatrix}
\phi^{(2)}_2 \\
\mathbf{r}^{(M)}_2 n
\end{bmatrix} = 
\begin{bmatrix}
d_{s1}^{(1)} \\
-d_{s1}^{(2)}
\end{bmatrix} n
$$

Equation (53) is the basic equation for the determination of kinematic errors of gear drives. Its application will be demonstrated in the following sections.

Analogous scalar products can be composed on the basis of equation (49).

It can be proven that these scalar products are zero because the vectors in equation (49) all belong to the tangent plane. Hereinafter, the following notations will apply:

$$
\Sigma \Delta q^{(1)}_i = d_{s1}^{(1)}, \quad \Sigma \Delta q^{(2)}_j = d_{s1}^{(2)},
$$

where $\Sigma \Delta q^{(1)}_i$ and $\Sigma \Delta q^{(2)}_j$ represent the sum of linear-error vectors due to manufacture and assembly of gears 1 and 2, respectively.

In many cases, however, errors in gear trains do not result from linear displacements, but rather from angular displacements. For instance, kinematic errors may result from the misalignment of gear shafts.

Figure 5 shows the axis of gear 2 rotation a-a in its ideal position. Suppose that, due to an error of assembly, axis a-a is rotated about a nonintersecting axis B-B. Such an error of assembly may be represented by the vector $\Delta \phi$, which is directed along axis B-B, where the direction of $\Delta \phi$ corresponds to the direction of rotation by the right-hand rule.
With the given vector $\Delta \delta$ the displacement $\Delta q(2)$ of contact point $M$ may be determined as follows:

(a) Vector $\Delta \delta$, directed along the axis $B-B$, is replaced by an equal vector $\Delta \delta$, which passes through the origin $O_2$ and the vector-moment $R \times \Delta \delta$. Here $R$ is a position vector drawn from $O_2$ to an arbitrary point on the line of action of vector $\Delta \delta$ (Fig. 5).

(b) The displacement $\Delta q(2)$ corresponding to $\Delta \delta$ may be represented by

$$\Delta q(2) = \Delta \delta \times r_2^{(M)} + R \times \Delta \delta = \Delta \delta \times (r_2^{(M)} - R)$$

A similar equation may be developed to determine the displacement of the contact point $M$ exerted by an angular error corresponding to gear 1.

With notations (54) the equation (53) for the determination of kinematic errors may be represented as follows:

$$\left(\Delta q(2) \times r_2^{(M)} + \Sigma \Delta q\right) \cdot n^{(M)} = 0$$

(56)

where $\Sigma \Delta q = \Delta q_{i}^{(2)} - \Delta q_{i}^{(1)}$ and $n^{(M)}$ is the surface unit normal at the contact point $M$.

The location of the contact point $M$ and the direction of the unit normal $n^{(M)}$ change in the process of motion. A further simplification of equation (56) results by assuming that in all positions the contacting tooth surfaces have a common normal which passes through the contact point $M$ and the pitch point $P$. This is the fundamental law for uniform motion transmission. For planar gears the pitch point $P$ coincides with the point of tangency of the pitch circles (gear centrodies). The pitch point for bevel gears is located on the line of tangency of pitch cones. In both cases the surface unit normal $n$
is collinear to \( \overline{PM} \) and
\[
\overline{r}_2(M) = \overline{r}_2(P) + \overline{PM} = \overline{r}_2(P) + \lambda n(M)
\]  
(57)

Equations (56) and (57) yield
\[
(\Delta \varphi (2) x \overline{r}_2(P) + \Sigma \Delta q) \cdot n(M) = 0
\]
(58)

because
\[
(\Delta \varphi (2) x \overline{r}_2(M)) \cdot n(M) = \left[ \Delta \varphi (2) x (\overline{r}_2(P) + \lambda n(M)) \right] \cdot n(M)
\]
\[
= (\Delta \varphi (2) x \overline{r}_2(P)) \cdot n(M) + \left[ \Delta \varphi (2) \lambda n(M) n(M) \right]
\]
\[
= (\Delta \varphi (2) x \overline{r}_2(P)) \cdot n(M)
\]
(59)

Application of equation (58) in place of equation (56) has the advantage that the location of the pitch point may be considered as a constant \( \overline{r}_2(P) = \text{const} \). However, the direction of the surface unit normal is a function of \( \varphi_1 \). Three types of gears - involute (spur and helical) and Wildhaber-Novikov - are exceptions to this statement. For these gears the unit normal of the gear surfaces at their contact point does not change its direction.

Because of kinematic errors, the angular velocity ratio fluctuates as the gear teeth pass through mesh. Figure 6 shows functions for two types of kinematic errors. The first is a piecewise, nonlinear, periodic function which has a period that depends on the ratio
\[
m_{12} = \frac{N_2}{N_1} = \frac{b}{a}
\]
(60)

where \( N_i \) \((i = 1,2)\) is the number of gear teeth and \( b \) and \( a \) are the
minimum integral numbers with which the ratio $m_{12}$ can be expressed. The angle of rotation of gear 1 corresponding to the period of function $\Delta \phi_1(2)(\psi_1)$ is equal to $2\pi a$.

Such functions of kinematic errors are caused by (a) the eccentricity of gears and (b) the crossing of the theoretical axis with the axis of rotation of gears. (The shortest distance between these axes rotates in the process of motion.)

The second type of function $\Delta b_1(2)(b_1)$, shown in Fig. 6(b), has a period of $b_1 = \frac{2\pi}{N_1}$. This function is exerted by (a) errors in the generating process of gear teeth and (b) errors of gear axis location which do not change in the process of meshing, etc.

**APPLICATION OF THEORY TO ECCENTRICITY OF INVOLUTE SPUR GEARS**

Figure 7 shows base circles of radii $r_1(b_1)$ and $r_2(b_2)$ for two involute spur gears. The rotation centers of the gears are denoted $O_1(1)$ and $O_2(2)$. If the centers of base circles $O_i$ coincide with centers of rotation $O_i(1)$ ($i = 1,2$), and then vectors of gear eccentricity $\Delta e_i = O_i(1) - O_i$ are zero. The involute curves are in tangency at a point $M$ of the line of action KL.

To model the meshing of gears with eccentricity, gears 1 and 2 are translated from their theoretical positions by $O_i(1) - O_i(2) = \Delta e_i (i = 1,2)$. Now, the center $O_i$ will be offset from the center of rotation $O_i(1)$. Because of this displacement of the gears, the tangency of their involute curves is broken: the curves will either interfere with each other (intersect) or lose contact. To bring the involute curves into contact once again, it is sufficient to rotate gear 2 by a small angle $\Delta \phi_2(2)$. According to equation (58) the angle $\Delta \phi_2(2)$ may be determined with the equation

$$
(\Delta \phi_2(2) \times r_2(P)) \cdot n(M) = (\Delta e_1 - \Delta e_2) \cdot n(M)
$$

The triple product results in (Fig. 7)

$$
(\Delta \phi_2(2) \times r_2(P)) \cdot n(M) = \Delta \phi_2(2)(P) r_2 \cos \psi
$$
where \( b_0 \) is the pressure angle.

Vectors of eccentricity \( \Delta e_1 \) and \( \Delta e_2 \) form angles \( \beta_1 \) and \( \beta_2 \) with vector \( \overrightarrow{O_1(1)O_2(2)} \); these angles are measured in the direction of gear rotation (Fig. 7).

The dot products yield

\[
\Delta e_1 \cdot n^{(M)} = \Delta e_1 \sin(\beta_1 + \psi_0)
\]

\[
\Delta e_2 \cdot n^{(M)} = \Delta e_2 \sin(\psi_0 - \delta_e)
\]

It results from equations (61) to (62) that

\[
\Delta b^{(2)} = \frac{\Delta e_1 \sin(\beta_1 + \psi_0) + \Delta e_2 \sin(\beta - \psi_0)}{r_b^{(2)}}
\]

where

\[
r_b^{(2)} = r_2^{(P)} \cos \psi_0
\]

is the radius of the base circle of gear 2.

The center \( O_i \) (\( i = 1,2 \)) of the base circle rotates in the process of meshing; \( O_i^{(1)} \) and \( O_i^{(2)} \) are two instantaneous positions of this center (Fig. 8). Angles \( \beta_1 \) and \( \beta_2 \) can be represented as follows:

\[
\beta_1 = \beta_{10} + \varphi_1, \quad \beta_2 = \beta_{20} + \varphi_2
\]

where \( \beta_{10} \) and \( \beta_{20} \) correspond to the initial positions of centers \( O_1 \) and \( O_2 \), with \( \varphi_1 = \varphi_2 = 0 \).

Equations (64) and (66) yield

\[
\Delta \phi^{(2)} = \frac{\Delta e_1 \sin(\varphi_1 + \gamma_1) + \Delta e_2 \sin(\varphi_2 + \gamma_2)}{r_b^{(2)}}
\]

where

\[
\gamma_1 = (\beta_{10} + \psi_0) ; \quad \gamma_2 = (\beta_{20} - \psi_0)
\]

For convenience, consider the kinematic error function to have zero magnitude at \( \varphi^{(1)} = \varphi^{(2)} = 0 \). Then, the kinematic error becomes
\[ \Delta \phi^{(2)} = \Phi^{(2)}(\psi_1) - \Phi^{(2)}(0) \]

\[ \frac{\Delta e_1 [\sin(b_1 + \gamma_1) - \sin \gamma_1]}{r_b^{(1)}} m_{21} + \frac{\Delta e_2 [\sin(b_2 + \gamma_2) - \sin \gamma_2]}{r_b^{(1)}} m_{22} \]  \hspace{1cm} (68)

where

\[ m_{21} = \frac{\omega_2}{\omega_1} = \frac{r_b^{(1)}}{r_b^{(2)}} = \frac{N_1}{N_2} \]

Equation (68) represents the kinematic error of a gear train with two gears as the sum of two harmonics. The periods of these harmonics are equal to the periods of complete revolutions of the gears.

Equation (68) may be made symmetric as follows:

\[ \Delta \phi^{(2)} = \frac{\Delta e_1 [\sin(b_1 + \gamma_1) - \sin \gamma_1]}{r_b^{(1)}} m_{21} + \frac{\Delta e_2 [\sin(b_2 + \gamma_2) - \sin \gamma_2]}{r_b^{(2)}} m_{22} \]

\[ = \sum_{i=1}^{2} \frac{\Delta e_i [\sin(b_i + \gamma_i) - \sin \gamma_i]}{r_b^{(i)}} m_{2i} \]  \hspace{1cm} (69)

Here, \( m_{22} = 1 \) and \( \varphi_2 = \varphi_1 m_{21} \)

Equation (69) can be generalized for a train with \( n \) gears as follows:

\[ \Delta \phi^{(n)} = \sum_{i=1}^{n} \frac{\Delta e_i [\sin(b_i + \gamma_i) - \sin \gamma_i]}{r_b^{(i)}} m_{ni} \]  \hspace{1cm} (70)

where \( \Delta \phi^{(n)} \) is the resulting kinematic error of the gear train represented as the angle of rotation of gear \( n \) (the output gear).

A complicated gear train is a combination of pairs of gears. The parameter \( \gamma_i \) may be represented as
\[ \gamma_i = \beta_{i0} + \psi_0 \]

for the driving gear of the pair, and as

\[ \gamma_i = \beta_{i0} - \psi_0 \]

for the driven gear of the pair. For instance, for computational purposes, a train of three gears must be replaced by two pairs of gears. The idler (intermediate gear) is considered as the driven gear in the first pair, and as the driving gear in the second pair.

Designate the kinematic error exerted by the eccentricity \( \Delta e_i \) of gear number \( i \) as

\[
\Delta \phi_i = \frac{\Delta e_i [\sin(\varphi_i + \gamma_i) - \sin \gamma_i]}{r_b(i)}
\]

(71)

where \( \Delta \phi_i \) is the error of the rotation angle \( \varphi_i \). The maximum possible value of this error is

\[
\Delta \phi_{i,\text{max}} - \Delta \phi_{i,\text{min}} = \frac{2 \Delta e_i}{r_b(i)}
\]

(72)

The kinematic error of the train may be represented as

\[
\Delta \phi(n) = \sum_{i=1}^{n} \Delta \phi_i m_{ni}
\]

(73)

Usually gear trains are applied for the reduction of angular velocities and thus \( m_{ni} \) is less than 1. It results from equation (73) that the last gears of a train (numbers \( n, n-1, n-2 \)) induce the largest part of the resulting kinematic error \( \Delta \phi(n) \). Therefore, the precision of these gears must be higher than the others.
The largest value of the kinematic error function $\Delta e(n)$ and its distribution above and below the abscissa depend on the combination of parameters $\gamma_i (i = 1, 2, \ldots, n)$. Figure 9 shows the distribution of a function $\Delta e_1 (\gamma_i)$ exerted by eccentricity of gear $i$ of the train.

The resulting errors of a gear train may be compensated for in part, by definite rules of assembly of gears with eccentricity. For instance, for gears with tooth numbers $N_1 = N_2$ and equal eccentricities $\Delta e_1 = \Delta e_2$ the resulting kinematic error will be approximately zero if eccentricity vectors $\Delta e_1$ and $\Delta e_2$ (Fig. 7) are directed opposite each other.

**APPLICATION OF THEORY TO ECCENTRICITY OF SPIRAL BEVEL GEARS**

For spatial gears the word "eccentricity" is used to describe that the geometric axis of a gear is parallel to, but does not coincide with, its axis of rotation (Fig. 10). As the eccentric gear rotates its geometric axis generates a cylindrical surface of radius $\Delta e$. The eccentricity vector $\Delta e$ is a vector which rotates about the gear axis. The initial position of vector $\Delta e$ (its position at the beginning of motion) is given by angle $\alpha$ (Fig. 10).

Figure 11 shows coordinate systems $S_1(X_1, Y_1, Z_1)$ and $S_f(X_f, Y_f, Z_f)$, which are rigidly connected to gear 1 and the frame, respectively. System $S_h$ is an auxiliary coordinate system, which is also rigidly connected to the frame. Driving gear 1 rotates about axis $Z_h$. The position of $\Delta e_1$ in coordinate system $S_1$ is given by the angle $\alpha_1$, which is made by $\Delta e_1$ and axis $X_1$. The current position of $\Delta e_1$ in coordinate system $S_f$ (or $S_h$) is defined by the angle $(\varphi_1 + \alpha_1)$ and the matrix equality

$$[\Delta e_f^{(1)}] = [L_{fh}] [\Delta e_h^{(1)}]$$

$$= \begin{bmatrix} \cos \gamma_1 & 0 & \sin \gamma_1 \\ 0 & 1 & 0 \\ -\sin \gamma_1 & 0 & \cos \gamma_1 \end{bmatrix} \begin{bmatrix} \Delta e_1 \cos(\varphi_1 + \alpha_1) \\ -\Delta e_1 \sin(\varphi_1 + \alpha_1) \\ 0 \end{bmatrix}$$ (74)
Here \( [\Delta e_h^{(1)}] \) is the matrix of vector \( \Delta e_1 \) in terms of its projections on the axes of coordinate system \( S_h \). The 3 by 3 matrix \( [L_f] \) transforms elements of the column matrix \( \Delta e_h^{(1)} \) to coordinate system \( S_f \) from coordinate system \( S_h \).

Matrix equality (74) yields

\[
[\Delta e_f^{(1)}] = \begin{bmatrix}
\Delta e_1 \cos(\phi_1 + a_1) \cos \gamma_1 \\
-\Delta e_1 \sin(\phi_1 + a_1) \\
-\Delta e_1 \cos(\phi_1 + a_1) \sin \gamma_1
\end{bmatrix}
\]  

(75)

The vector of eccentricity of the driven gear can be defined in a similar way. Figure 12 shows coordinate systems \( S_2 \) and \( S_f \) rigidly connected to gear 2 and the frame, respectively. The auxiliary coordinate system \( S_p \) is also rigidly connected to the frame.

Vector \( \Delta e_f^{(2)} \) is represented by matrix equalities

\[
[\Delta e_f^{(2)}] = [L_{fp}] [\Delta e_p^{(2)}]
\]

\[
= \begin{bmatrix}
\cos \gamma_2 & 0 & -\sin \gamma_2 \\
0 & 1 & 0 \\
\sin \gamma_2 & 0 & \cos \gamma_2
\end{bmatrix}
\begin{bmatrix}
\Delta e_2 \cos(\phi_2 + a_2) \\
\Delta e_2 \sin(\phi_2 + a_2) \\
0
\end{bmatrix}
\]

(76)

which after matrix multiplication gives

\[
[\Delta e_f^{(2)}] = \begin{bmatrix}
\Delta e_2 \cos(\phi_2 + a_2) \cos \gamma_2 \\
\Delta e_2 \sin(\phi_2 + a_2) \\
\Delta e_2 \cos(\phi_2 + a_2) \sin \gamma_2
\end{bmatrix}
\]

(77)

Kinematic errors induced by gear eccentricities may now be found by applying Equation (58) as follows:
Here $\Delta e_f^{(1)}$ and $\Delta e_f^{(2)}$ are given by matrices (75) and (77); vector $\Delta e_f^{(2)}$ (Fig. 12) is represented by the matrix

$$
\begin{bmatrix}
\cos \gamma_2 & 0 & -\sin \gamma_2 \\
0 & 1 & 0 \\
\sin \gamma_2 & 0 & \cos \gamma_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \Delta e_2 \end{bmatrix}
$$

Equations (78) and (79) yield

$$
\Delta e_2 = -\frac{n_x \Sigma \Delta e_x + n_y \Sigma \Delta e_y + n_z \Sigma \Delta e_z}{-y \cos \gamma_2 + (x \cos \gamma_2 + z \sin \gamma_2)n_y - y \sin \gamma_2 n_z}
$$

where $\Sigma \Delta e_x = \Delta e_x^{(1)} - \Delta e_x^{(2)}$, $\Sigma \Delta e_y = \Delta e_y^{(1)} - \Delta e_y^{(2)}$, $\Sigma \Delta e_z = \Delta e_z^{(1)} - \Delta e_z^{(2)}$ and the subscript $f$ was dropped.

Projections $n_y$ and $n_z$ of the surface unit normal $n$, and coordinates $x$, $y$, and $z$ of the contact point change in the process of motion. But since the changes in these variables are relatively small, they may be neglected (Fig. 13).

$$
n_f = \sin c_c \ j_f + \cos c_c \ (\cos \beta \ k_f + \sin \beta \ \ell_f)
$$

$$
x_f = 0, \quad y_f = 0, \quad z_f = L
$$
Equations (80) and (81) yield

\[ \Delta \varphi_2(\varphi_1) = \frac{A(\varphi_1)}{L \sin \gamma_2 \cos \psi \cos \beta} \]  

(82)

where

\[ A(\varphi_1) = a_1 \sin (\varphi_1 + a_1) + b_1 \cos (\varphi_1 + a_1) + a_2 \sin (\varphi_2 + a_2) + b_2 \cos (\varphi_2 + a) \]

\[ a_1 = -\Delta e_1 \cos \psi \cos \beta \]
\[ a_2 = -\Delta e_2 \cos \psi \cos \beta \]
\[ b_1 = \Delta e_1 (\cos \gamma_1 \sin \psi \cos \beta - \sin \gamma_1 \cos \psi \sin \beta) \]
\[ b_2 = -\Delta e_2 (\cos \gamma_2 \sin \psi \cos \beta + \sin \gamma_2 \cos \psi \sin \beta) \]

\[ \varphi_2 = \frac{N_1}{N_2} \varphi_1 \]

It is concluded from the form of equation (82) that kinematic errors induced by the eccentricity of spiral-bevel gears may be represented as the sum of four harmonics: the period of two harmonics coincides with the period of revolution of gear 1; the period of the other two coincides with the period of revolution of gear 2.

The function \( \Delta \varphi_2(\varphi_1) \) represented by equation (82) is a smoothed, continuous function which serves as a first approximation. In reality the true function \( \varphi_2(\varphi_1) \) breaks as different sets of teeth come into mesh. This break can be discovered if \( \Delta \varphi_2(\varphi_1) \) is determined by equation (80).

GEAR TRAIN VIBRATION AND NOISE MEASUREMENT

To illustrate the principles discussed on the subject of gear train precision, Figs. 14 and 15 are used. These figures show some frequency spectrum measurements made on a helicopter transmission running in a test stand [7]. The transmission had a spiral-bevel input stage with 19 teeth on the pinion...
and 71 teeth on the gear. The pinion was turning at 6200 rpm and the output shaft at 355.5 rpm. The output stage was a spur planetary arrangement with a 27-tooth sun, 3 planet gears, each with 35 teeth and a 99-tooth ring gear which was splined to the transmission housing. An accelerometer was mounted on the case immediately outside the spline.

Figure 14 shows a broadband frequency spectrum measurement of the vibration signal. The spur mesh frequency was 583 Hz and the spiral bevel mesh frequency was 1963 Hz. The spiral-bevel vibration signature was much stronger than the spur signature. This indicates that the meshing accuracy was better for the spur mesh than for the spiral-bevel mesh. There are also other peaks in the spectrum at multiples of the fundamental frequencies of 1963 Hz and 583 Hz. These other peaks are the higher harmonics due to the noise and vibration pulsations as the teeth mesh being different from the pure sinusoidal shape as shown in Fig. 9.

Figure 15, an expanded region of the autospectrum plot given in Fig. 14, shows many peaks that are symmetrically located about the spur gear mesh fundamental frequency peak at 583 Hz. These peaks locate the sideband frequencies which are due to sources of modulation in the time dependent vibration waveform. Each source of modulation may produce one pair of sidebands if it is a harmonic modulator. If nonharmonic, the side bands will repeat many times, as in the case in Fig. 15.

SUMMARY OF RESULTS

Kinematic precision is affected by errors that are the result of either intentional adjustments or accidental defects in the manufacturing and assembly of gear trains. A general method for the determination of kinematic precision of gear trains has been explained. The general method is based on the exact kinematic relations for the contact point motions of the gear tooth sur-
faces under the influence of errors. An approximate method was also explained.

Example applications of the general and approximate methods were demonstrated for gear trains consisting of involute (spur and helical) gears, circular arc (Wildhaber-Novikov) gears, and spiral-bevel gears. Gear noise measurements from a helicopter transmission were presented and discussed with relation to the kinematic precision theory. The following results were obtained:

1. The exact numerical iterative procedure for finding kinematic errors, $\Delta b_2$, is as follows: From equation (20) find

$$\Delta \varphi_2 = \varphi_2(\varphi_1, \Delta Q) - \varphi_2^0(\varphi_1)$$

where the angles $\varphi_2$ and $\varphi_2^0$ have been determined from an iterative solution of the nonlinear algebraic system of five equations, which are separable into two systems of two and three equations each as follows:

$$F_i(\varphi_1, \varphi_2) = 0, i = 1, 2$$

$$\begin{align*}
A - K_1(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2, \Delta Q) &= 0 \\
H_1 - K_2(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2, \Delta Q) &= 0 \\
H_2 - K_3(u_1, \theta_1, \varphi_1, u_2, \theta_2, \varphi_2, \Delta Q) &= 0
\end{align*}$$

where $A, H_1, H_2$, and $\Delta Q$ are given values and $\varphi_2^0$ is determined by solving with $\Delta Q = 0$, whereas $\varphi_2$ is determined by the full solution of the five equations.

2. The approximate equation for kinematic error $\Delta \varphi(2)$ is

$$(\Delta b(2) \times r_2^{(p)}(p) \times S\Delta q) \cdot n(M) = 0$$

where $r_2^{(p)}$ is the radius vector to the pitch point, $n(M)$ is the surface normal vector at the contact point $M$, and $S\Delta q$ is the sum of known error vectors.
3. Application of the formulas showed that Wildhaber-Novikov gears are sensitive to any errors which cause changes in the center distance. A formula for the location of the gear-tooth contact point was given.

4. It was found that for a pair of spur gears the kinematic error function due to eccentricities is a sum of two simple harmonics. For a multistage speed reducer, it was concluded that accuracy in the final stages has the most impact on kinematic error. For gears with approximately equal known eccentricities, the kinematic error may be compensated for by directing the eccentricities opposite one another.

5. For a pair of spiral-bevel gears, the kinematic error function due to eccentricities is a sum of four harmonics.
REFERENCES


Figure 3. - Generation of circular arc gears.

Figure 4. - Gear teeth not in contact.
Figure 7. Base circles with eccentricity of rotation.

Figure 8. Two positions of base circle during rotation.
Figure 12. Coordinate frames used to describe eccentricity for gear 2.

Figure 11. Coordinate frames used to describe eccentricity for gear 1.
Figure 13. - Spiral bevel gear showing surface unit normal vector $\eta_j$.

Figure 14. - Baseband frequency spectrum showing spiral bevel amplitude compared with spur.
Figure 15. - Narrow band frequency spectrum showing sidebands around the spur mesh frequency.