STABLE BOUNDARY CONDITIONS AND DIFFERENCE SCHEMES FOR NAVIER-STOKES EQUATIONS

Pravir Dutt

NASA Contract No. NAS1-17070
August 1985

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665
The Navier-Stokes equations can be viewed as an incompletely elliptic perturbation of the Euler equations. By using the entropy function for the Euler equations as a measure of 'energy' for the Navier-Stokes equations, we are able to obtain nonlinear 'energy' estimates for the mixed initial boundary value problem. These estimates are used to derive boundary conditions which guarantee $L^2$ boundedness even when the Reynolds number tends to infinity. Finally, we propose a new difference scheme for modelling the Navier-Stokes equations in multidimensions for which we are able to obtain discrete energy estimates exactly analogous to those we obtained for the differential equation.
INTRODUCTION

For computational problems involving the Navier-Stokes equations, it is necessary to limit the domain of computation and introduce artificial boundary conditions. Naturally, we would like these boundary conditions to be stable, compatible with weak boundary layers, and to remain valid even when the Reynolds number tends to infinity. Such a set of boundary conditions were proposed by Gustaffson and Sundström in [4]. They used energy estimates on the linearized Navier-Stokes equations to obtain boundary conditions of maximal dissipative type. In this report we define an 'energy' in terms of the entropy function for the Euler equations and obtain fully nonlinear 'energy' estimates from which we are able to extract a family of boundary conditions with the above properties. An attractive feature of these boundary conditions is that they are easy to implement and can be expressed in terms of the physics of the problem.

The Navier-Stokes equations are an incompletely elliptic perturbation of the Euler equations — which are themselves a hyperbolic system of conservation laws with entropy functions. It was observed by Mock [5] that by introducing the gradient of the entropy as a new variable a system of hyperbolic conservation laws can be reduced to a symmetric, hyperbolic system in terms of this new variable. Further, Harten [5] showed that if the dissipative terms in the Navier-Stokes equations are rewritten in terms of this new variable then the matrix coefficients of the dissipative terms have certain symmetry properties. We are able to show that the augmented matrix formed from these matrix coefficients is, in fact, negative semidefinite. This observation is crucial to the energy estimates we obtain for the Navier-Stokes equations.
This leads us to propose a new difference scheme for modelling Navier-Stokes equations in multidimensions. We are able to obtain discrete "energy" estimates — which are exact analogs of the "energy" estimates we obtained for the differential equation — at the semidiscrete level, even for meshes with unequal mesh widths. Thus we are able to propose boundary conditions and a difference scheme for the Navier-Stokes equations which give a priori boundedness of 'energy' for all time.

This report is organized as follows: In Section 2 we define the Navier-Stokes equations and obtain the necessary results to derive the "energy" estimates of Section 3. In Section 4 we propose a family of "stable" boundary conditions and relate them to the physics of the problem. In Section 5 we propose a new method for differencing the Navier-Stokes equations in multidimensions and obtain discrete "energy" estimates for our difference scheme. Finally in Section 6 we obtain stable boundary conditions for the difference scheme and conclude by displaying some numerical simulations in Section 7.

2. PRELIMINARIES

We consider systems of hyperbolic conservation laws of the form:

\[
q_t + \sum_{i=1}^{d} f_i(q) x_i = 0.
\]

Here \(q(x,t)\) is an \(n\) column vector of unknowns, \(f_i(q)\) is a vector valued function of \(n\) components, \(x = (x_1, \ldots, x_d)\) and \(f = (f_1, \ldots, f_d)\).
We can rewrite (2.1) in matrix form:

\[ q_t + \sum_{i=1}^{d} A^i(q) q_{x_1} = 0 \]

where \( A^i(q) = f_i^i \). The system (2.1) is called hyperbolic if the matrix

\[ \sum_{i=1}^{d} \omega_i A^i(q) \]

has real eigenvalues and a complete set of eigenvectors for all real \( \omega_i \).

Following Mock, a scalar function \( V(q) \) is an entropy function for (2.1) if:

i) \( V \) satisfies

\[ V \frac{f_i}{q} = f_i^i \]

where \( f_i^i(q) \) is some scalar function called entropy flux in the \( x_i \) direction.

ii) \( V \) is a convex function of \( q \).

It follows from (2.4) upon multiplying (2.1) by \( V^T q \) that every smooth solution of (2.1) also satisfies:

\[ V_t + \sum_{i=1}^{d} F^i_{x_1} = 0 \]

where \( F = (F^1, \ldots, F^d) \).
The Euler Equations of Gas Dynamics

Description of variables:

\( p \) denotes density,
\( u \) denotes velocity in the \( x \) direction,
\( v \) denotes velocity in the \( y \) direction,
\( w \) denotes velocity in the \( z \) direction,

\( m_u, m_v, \) and \( m_w \) are the components of momentum in the \( x, y \) and \( z \) directions respectively,

\( T \) is the temperature,
\( p \) is the pressure,
\( U \) is the thermodynamic entropy,
\( E \) is the energy,
\( R \) is the universal gas constant,
\( \gamma \) is the ratio of specific heats.

Note that we shall use \( (x, y, z) \) and \( (x_1, x_2, x_3) \) interchangeably to denote the spatial vector \( \mathbf{x} \).

We shall also need the following thermodynamic relations:

\[
T = \frac{(\gamma - 1)}{R} \left( \frac{E}{p} - \frac{(m_u^2 + m_v^2 + m_w^2)}{2p^2} \right)
\]

\[
p = (\gamma - 1) \left( E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2p} \right)
\]

\[
U = \log \left( E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2p} \right) - \gamma \log p = \log p - \gamma \log \rho,
\]

up to an additive constant.
T, p and \( \rho \) will always be restricted to be positive because of obvious physical considerations. \( q \) will always denote the vector:

\[
q = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
E
\end{bmatrix} = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
E
\end{bmatrix}.
\]

The Euler equations are of the form:

\[
q_t + f_1 x + f_2 y + f_3 z = 0
\]

where

\[
f_1 = \begin{bmatrix}
\rho u \\
\rho u^2 + p \\
\rho uv \\
\rho uw \\
(E + p)u
\end{bmatrix}, \quad f_2 = \begin{bmatrix}
\rho v \\
\rho vu \\
\rho v^2 + p \\
\rho vw \\
(E + p)v
\end{bmatrix}, \quad f_3 = \begin{bmatrix}
\rho w \\
\rho uw \\
\rho vw \\
\rho w^2 + p \\
(E + p)w
\end{bmatrix}.
\]

We shall write the Euler equations in operator notation as:

\[
(2.6) \quad \text{Eq} = 0.
\]

The Euler equations have a family of strictly convex entropy functions defined by

\[V(q) = -\rho h(U).\]

The preferred entropy function in most physical applications is:
\[ V(q) = -\rho U = -\rho \log \left( E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho} \right) + \gamma \rho \log \rho. \]

The entropy flux functions turn out to be:

\[ F^1 = -m_u U = -m_u \log \left( E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho} \right) + \gamma m_u \log \rho \]

\[ F^2 = -m_v U \]

\[ F^3 = -m_w U. \]

It should be noted that the entropy function \( V(q) \) is strictly convex but may be nonpositive in general.

**Navier-Stokes Equations**

We shall denote the Navier-Stokes equations in operator notation as:

\[ (2.7) \quad Nq = 0 \]

where

\[ Nq = Eq + (-D)q, \]

where \((-D)q = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} A^{ij}(q)q_{x_j} \right) x_i.\]

We can represent (2.7) in the alternative form:

\[ Nq = Eq + \sum_{i=1}^{3} (h^i)x_i.\]
where
\[ h^1 = \sum_{j=1}^{3} A^{ij}(q)q_{x_j} \]

and \( h^1, h^2, h^3 \) are as follows:
\[
\begin{align*}
  h^1 &= \begin{bmatrix}
    0 \\
    \theta_{xx} \\
    \theta_{yx} \\
    \theta_{zx} \\
    \theta_{xx} u + \theta_{yz} v + \theta_{zx} w - k \frac{\partial T}{\partial x}
  \end{bmatrix}, \\
  h^2 &= \begin{bmatrix}
    0 \\
    \theta_{xy} \\
    \theta_{yy} \\
    \theta_{zy} \\
    \theta_{xy} u + \theta_{yy} v + \theta_{zy} w - k \frac{\partial T}{\partial y}
  \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
  h^3 &= \begin{bmatrix}
    0 \\
    \theta_{xz} \\
    \theta_{yz} \\
    \theta_{zz} \\
    \theta_{xz} u + \theta_{yz} v + \theta_{zz} w - k \frac{\partial T}{\partial z}
  \end{bmatrix}
\end{align*}
\]

where:
\[
\begin{align*}
  \theta_{xx} &= -2\mu \frac{\partial u}{\partial x} - \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\
  \theta_{yy} &= -2\mu \frac{\partial v}{\partial y} - \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\
  \theta_{zz} &= -2\mu \frac{\partial w}{\partial z} - \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\
  \theta_{xy} &= \theta_{yx} = -\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
  \theta_{yz} &= \theta_{zy} = -\mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
  \theta_{zx} &= \theta_{xz} = -\mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)
\end{align*}
\]
Here the parameters $\mu, \lambda$ and $k$ are as follows:

$\mu$ is the shear coefficient of viscosity,

$\lambda$ is the second coefficient of viscosity,

$k$ is the coefficient of thermal conductivity.

Clearly

$$(-D)q = h^1_x + h^2_y + h^3_z.$$ 

We write the Navier-Stokes equations as

$$(2.8) \quad Nq = Eq + (-D)q$$

where $E$ is the differential operator corresponding to the Euler equations and $D$ is the elliptic perturbation (depending on the parameters $\mu, \lambda$ and $k$) due to the dissipative terms in the Navier-Stokes equations.

The entropy function $V(q) = -\rho U$ for the Euler equations is not positive valued in general. Since we wish to interpret the entropy as a measure of "energy" of the system, we can normalize $V(q)$ and define a new entropy $\tilde{V}(q)$ which has the properties:

(i) $\tilde{V}(q) > 0 \forall q$

(ii) $\tilde{V}(q) = 0 \Leftrightarrow q = q$ for some fixed $q$.

For the entropy $V$ is not altered by adding to it an arbitrary inhomogeneous linear function. Hence we define $\tilde{V}(q)$ as follows:
The associated entropy flux functions are given by

\[ \tilde{F}^i(q) = F^i(q) - F^i(\bar{q}) - \sum_{j=1}^5 \frac{\partial V}{\partial q_j} \tilde{q}(f^i(q) - f^i(\bar{q}))_j. \]  

We choose as our rest state \( \bar{q} = (\bar{\rho}, \bar{m}_u, \bar{m}_v, \bar{m}_w, \bar{E}) \) such that:

\begin{align*}
\bar{\rho} &> 0 \\
\bar{u} &= \bar{v} = \bar{w} = 0 \\
\bar{T} &> 0.
\end{align*}

In a later section we shall put further restrictions on \( \bar{\rho} \) and \( \bar{T} \). With this choice of \( \bar{q} \) we obtain:

\begin{align*}
\tilde{V}(q) &= \rho \left[ \bar{U} - U + \frac{(\gamma - 1)}{2RT} (u^2 + v^2 + w^2) - \gamma + \frac{T}{T} \right] + (\gamma - 1)\bar{\rho} \\
\tilde{F}^1(q) &= \rho u \left[ \bar{U} - U + \frac{(\gamma - 1)}{2RT} (u^2 + v^2 + w^2) - \gamma + \frac{T}{T} \right].
\end{align*}

Or more compactly:

\begin{align*}
\tilde{F}^1(q) &= u(\tilde{V}(q) - (\gamma - 1)\bar{\rho}) \\
\tilde{F}^2(q) &= v(\tilde{V}(q) - (\gamma - 1)\bar{\rho}) \\
\tilde{F}^3(q) &= w(\tilde{V}(q) - (\gamma - 1)\bar{\rho}).
\end{align*}
We make the change of variables \( v = \tilde{V} \) and rewrite the operator \( Dq \) in terms of \( v \). Here

\[
v = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}.
\]

Substituting these relations in

\[
(-D)q = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} A_{ij}^q q_{x_j} \right) x_i
\]

yields

\[
(-D)q = (-\tilde{D})v = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} C_{ij}^q(v) q_{x_j} \right) x_i
\]

where \( C_{ij}^q(v) = A_{ij}^q q_{x_j} \). Harten [5] observed that the matrix coefficients \( C_{ij}^q(v) \) satisfy the symmetry relation:

\[
C_{ij}^q = (C_{ji}^q)^T.
\]

For the energy estimates we shall derive in Section 3 we need to show that

\[
(2.11) \quad \sum_{i=1}^{3} \sum_{j=1}^{3} (\xi_i^1)^T C_{ij}^q(v)(\xi_j^1) \leq 0 \quad \xi_i^1, \xi_j^1 \in \mathbb{R}^5.
\]

Clearly this is equivalent to proving that the augmented matrix \( \tilde{C} \) defined by
\[
\hat{C} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\]

is negative semidefinite.

Observe that \( C_{ij} = (C_{ji})^T \hat{C} \) is symmetric. Hence to check that \( \hat{C} \) is negative semidefinite it is enough to perform an LU decomposition of \( \hat{C} \) and then check that the diagonal elements of the upper triangular matrix \( U \) are negative. Thus we obtain:

\[(2.12) \quad (2.11) \text{ holds } \iff \lambda + \mu \geq 0.\]

Typically, for physical fluids \( \lambda \leq 0 \) and \( \mu \geq 0 \). Further for most fluids under nonextreme flow conditions the following relation holds:

\[
\frac{\gamma \mu}{Pr} \geq \lambda + 2\mu \geq \mu.
\]

Here \( Pr \) denotes the Prandtl number. So (2.11) holds under these conditions.

3. NONLINEAR ENERGY ESTIMATES

Let \( \Omega \) denote a bounded domain in \( \mathbb{R}^3 \) and let \( \partial \Omega \) denote the boundary of \( \Omega \). Consider the mixed initial boundary value problem in \( \Omega \):

\[(3.1) \quad \partial_q = 0 \quad \psi \rightarrow \in \Omega, \quad t \geq 0\]

where \( \partial \) is the Navier-Stokes differential operator with initial condition:
\[ (3.2) \quad q(\vec{x}, t)|_{t=0} = q_0(\vec{x}) \quad \forall \vec{x} \in \Omega \]

and boundary condition:

\[ (3.3) \quad Bj = 0 \quad \forall \vec{x} \in \partial \Omega, \quad t \geq 0 \]

where \( B \) is a boundary operator.

Define:

\[ (3.4) \quad S(t) = \int_{\Omega} V(q(\vec{x}, t)) d\vec{x}. \]

We claim that \( S(t) \) gives us an estimate of the energy of the system at time \( t \). Note that \( \tilde{V}(q) \) is a strictly convex, non-negative function of \( q \), i.e.,

i) \( \tilde{V}(q) \geq 0 \)

ii) \( \tilde{V}(q) = 0 \iff q = \bar{q} \)

iii) \( \tilde{V}_{qq} > cI \) where \( c > 0 \), at least in some appropriate physical domain. In particular, (iii) implies:

\[ (3.5) \quad \tilde{V}(q) \geq \alpha \|q - \bar{q}\|^2 \]

for some \( \alpha > 0 \). Hence we conclude that \( S(t) \) has the following properties:

i) \( S(t) \geq 0 \)

ii) \( S(t) = 0 \iff q(\vec{x}, t) \equiv \bar{q} \quad \forall \vec{x} \in \Omega \)

iii) \( S(t) \leq k \), where \( k \) is a constant \( \Rightarrow \)
if \( q(x, t) \) lies in a domain where an inequality of the form (3.5) holds.

We wish to study the time evolution of \( S(t) \). Recollect that the Navier-Stokes equations are:

\[
Nq = Eq + (-D)q = 0
\]

where

\[
(-D)q = \sum_{i=1}^{3} \sum_{j=1}^{3} (A^{ij}(q)q_{x_j})_{x_i}
\]

Making the change of variables \( v = V_q \) gives:

\[
(-D)q = (-\tilde{D})v = \sum_{i=1}^{3} \sum_{j=1}^{3} (C^{ij}(v)v_{x_j})_{x_i}
\]

where the matrix coefficients \( C^{ij}(v) \) satisfy the condition:

\[
(3.6) \quad \sum_{i=1}^{3} \sum_{j=1}^{3} \xi^i T C^{ij}(u)\xi^j \leq 0 \quad \forall \xi^i, \xi^j \in \mathbb{R}^3.
\]

This leads us to the following theorem.

**Theorem 3.1:** Consider the mixed initial boundary value problem (3.1)-(3.3). Let \( \eta_i, \cdots, \eta_d \) denote the outward unit normal to the boundary \( \partial \Omega \) and let \( \varphi^1, \cdots, \varphi^d \) denote the entropy flux functions as in (2.10). Then any piecewise smooth solution of (3.1)-(3.3) satisfies the energy estimate:

\[
(3.7) \quad \frac{dS}{dt} \leq \int_{\partial \Omega} \left( \sum_{i=1}^{3} \varphi^i \xi^i q_{x_i} + \sum_{i=1}^{3} \sum_{j=1}^{3} \nabla q \cdot A^{ij}(q)q_{x_j} \right) d\sigma.
\]
Here \( d\sigma \) is an element of surface area.

\textbf{Proof:} We prove it here for the case \( q(x, t) \) is 'smooth enough.' We have

\begin{equation}
Nq = q_t + \sum_{i=1}^{3} f^i(q)x_i + \sum_{i=1}^{3} \left( \sum_{j=1}^{3} A^i_{ij}(q)x_j \right)x_i = 0.
\end{equation}

Premultiplying (3.8) by \( \bar{V}_q \) we get

\begin{equation}
\bar{V}_q q_t + \sum_{i=1}^{3} \bar{V}_q (f^i(q))x_i + \sum_{i=1}^{3} \bar{V}_q \left( \sum_{j=1}^{3} A^i_{ij}(q)x_j \right)x_i = 0.
\end{equation}

By (2.4), this reduces to

\begin{equation}
\bar{V}_t + \sum_{i=1}^{3} \bar{V}_x \left( \sum_{j=1}^{3} A^i_{ij}(q)x_j \right)x_i = 0.
\end{equation}

From (3.9) we obtain

\begin{equation}
\frac{dS}{dt} = - \int_{\Omega} \left\{ \sum_{i=1}^{3} \bar{V}_x^i + \sum_{i=1}^{3} \bar{V}_q \left( \sum_{j=1}^{3} A^i_{ij}(q)x_j \right)x_i \right\} \bar{d}x,
\end{equation}

or

\begin{equation}
\frac{dS}{dt} = - \int_{\Omega} \left\{ \sum_{i=1}^{3} \bar{V}_x^i + \sum_{i=1}^{3} v^T \left( \sum_{j=1}^{3} c^i_{ij}(v)x_j \right)x_i \right\} \bar{d}x.
\end{equation}

By the divergence theorem

\begin{equation}
\frac{dS}{dt} = - \int_{\partial \Omega} \left\{ \sum_{i=1}^{3} \bar{V}_x^i \zeta^i + \sum_{i=1}^{3} v^T \zeta^i c^i_{ij}(v)x_j \right\} d\sigma
\end{equation}
\[ + \int_{\Omega} \left( \sum_{i=1}^{3} \sum_{j=1}^{3} v_{i}^{T} \mathbf{C}_{ij}^{(v)} v_{j} \right) dx. \]

But by (3.6)
\[ \int_{\Omega} \left( \sum_{i=1}^{3} \sum_{j=1}^{3} v_{i}^{T} \mathbf{C}_{ij}^{(v)} v_{j} \right) dx \leq 0. \]

Hence we obtain
\[ \frac{dS}{dt} \leq - \int_{\partial \Omega} \left( \sum_{i=1}^{3} \tilde{F}_{i} \zeta_{i} + \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{v}_{i}^{T} \zeta_{i} \mathbf{C}_{ij}^{(v)} v_{j} \right) d\sigma. \]

This finally yields
\[ (3.7) \quad \frac{dS}{dt} \leq - \int_{\partial \Omega} \left( \sum_{i=1}^{3} \tilde{F}_{i} \zeta_{i} + \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{v}_{i} q_{i} A_{ij}^{(q)} q_{j} \right) d\sigma. \]

Remark: Estimate (3.7) is a fully nonlinear 'energy' estimate which holds for a broad class of solutions of the mixed initial boundary value problem (3.1)-(3.3). In fact (3.7) is simpler to obtain and broader in scope than the linearized energy estimates in current use.
4. STABLE BOUNDARY CONDITIONS FOR THE NAVIER-STOKES EQUATIONS

Since the Navier-Stokes equations are rotationally invariant, we can consider a moving coordinate frame \((x, y, z)\) where \(x\) points in the direction of the inward normal and \(y\) and \(z\) are tangential to \(\partial \Omega\). Of course, we reorient \(u, v,\) and \(w\) so that \(u\) is the component of velocity in the \(x\) direction, \(v\) in the \(y\) direction, and \(w\) in the \(z\) direction. Let \(\xi\) denote the outward unit normal to \(\partial \Omega\). Clearly \(\xi = (-1, 0, 0)\).

Then the nonlinear energy estimate (3.7) takes the simpler form:

\[
\frac{dS}{dt} \leq \int_{\partial \Omega} \left( \nabla^1 + \sum_{j=1}^{3} \nabla q_{1j} A_{1j} q_{xj} \right) d\sigma
\]

or
(4.2) \[ \frac{dS}{dt} \leq \int_{\Omega} (F^1 + \mathcal{V} h^1) d\sigma \]

where \[ h^1 = \sum_{j=1}^{3} A^1 j q_{x_j}. \]

Substituting the relations obtained in Section 2 we obtain

\[ \tilde{\mathcal{V}} q h^1 = -\frac{(\gamma - 1)}{R} \left[ 0, \frac{u}{T}, \frac{v}{T}, \frac{w}{T}, -(\frac{1}{T} - \frac{1}{\Gamma}) \right] \begin{bmatrix} 0 \\ 2\mu \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \\ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \left( 2\mu u \frac{\partial u}{\partial x} + \lambda u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + \mu v \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right. \\
\left. + \mu u \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] \cdot \\
\frac{\partial T}{\partial x}, \]

or,

(4.3) \[ \tilde{\mathcal{V}} q h^1 = \frac{k(\gamma - 1)}{R} \left( \frac{1}{T} - \frac{1}{\Gamma} \right) \frac{\partial T}{\partial x} - \frac{(\gamma - 1)}{RT} \left\{ (2\mu + \lambda) u \frac{\partial u}{\partial x} + \lambda u \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right. \\
\left. + \mu v \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \mu u \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right\}. \]

So finally we obtain the needed 'energy' estimate:
\[
\frac{dS}{dt} \leq \int_{\partial \Omega} u(\bar{V} - (\gamma - 1)p)\,d\sigma + \int_{\partial \Omega} \frac{k(\gamma - 1)}{RT} \frac{T - T}{\bar{T}} \frac{\partial T}{\partial x} \,d\sigma
\]

\[
= \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} u \left( 2u + \lambda \right) \frac{3u}{\partial x} + \lambda \left( \frac{3v}{\partial y} + \frac{3w}{\partial z} \right) \,d\sigma
\]

\[
- \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} v \mu \frac{3v}{\partial x} + \frac{3u}{\partial y} \,d\sigma - \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} \omega \mu \left( \frac{3w}{\partial x} + \frac{3u}{\partial z} \right) \,d\sigma
\]

where

\[
(4.5) \quad u(\bar{V} - (\gamma - 1)p) = \rho u \left[ U - U + \frac{(\gamma - 1)}{2RT} \left( u^2 + v^2 + w^2 \right) - \gamma + \frac{T}{\bar{T}} \right]
\]

and

\[
(4.6) \quad \bar{V} = \rho \left[ U - U + \frac{(\gamma - 1)}{2RT} \left( u^2 + v^2 + w^2 \right) - \gamma + \frac{T}{\bar{T}} \right] + (\gamma - 1)p.
\]

So far we have left \( \bar{T} \) and \( \bar{p} \) vaguely defined. We are now going to specify \( \bar{T} \) and \( \bar{p} \), but first we have to make a few assumptions about the solution \( q(x, t) \) to the mixed initial boundary value problem (3.1)-(3.3).

Assumptions:

\[
\begin{align*}
\{(4.7) \quad & i) \exists \rho_{\min} > 0 \exists \rho(x, t) \geq \rho_{\min}, \forall x \in \Omega, t \geq 0 \\
& \quad \{(4.7) \quad ii) \exists \rho_{\max} > \rho_{\min} \exists \rho(x, t) < \rho_{\max}, \forall x \in \Omega, t \geq 0 \\
& \quad \{(4.7) \quad iii) \exists T_{\min} > 0 \exists T(x, t) \geq T_{\min}, \forall x \in \Omega, t \geq 0. \}
\end{align*}
\]
Assumptions (i)-(iii) exclude cavitation, freezing of the fluid, and such esoteric phenomena as the formation of black holes. Further, it should be noted that if (4.7) is valid $\tilde{V}(q)$ is strictly convex and hence, then we are able to obtain boundary conditions which ensure that $S(t)$ remains bounded, i.e.,

$$ S(t) \leq k \quad \forall \, t \geq 0 \text{ where } k \text{ is a constant}; \text{ then by the discussion following (3.5)} $$

$$ \Rightarrow \left( \int_{\Omega} \| q(x, t) - q \|^{2} \, dx \right)^{1/2} \leq C \quad \text{(another constant)} \quad \forall \, t \geq 0. $$

Hence we obtain $L^2$ boundedness of the solution for all time.

Choose $0 < \tilde{\rho} < \rho_{\text{min}}$ such that

$$ (4.8) \quad \tilde{V} - (\gamma - 1)\tilde{\rho} > 0 $$

and $0 < \tilde{T} < T_{\text{min}}$ such that

$$ (4.9) \quad \bar{U} - \bar{U} < 0 $$

$q$ satisfying (4.7).

It is easy to see that this can always be done. For by (4.6)

$$ \tilde{V}(q) \geq \tilde{\rho} \left[ \bar{U} - \bar{U} - \gamma + \frac{T}{\tilde{T}} \right] + (\gamma - 1)\tilde{\rho} $$

or

$$ \tilde{V}(q) \geq \tilde{\rho} \left[ (\gamma - 1) \log \left( \frac{\rho}{\tilde{\rho}} \right) - \gamma + \frac{T}{\tilde{T}} - \log \left( \frac{\tilde{T}}{T} \right) \right] + (\gamma - 1)\tilde{\rho}. $$
Clearly
\[ \frac{T}{T} - \log \left( \frac{T}{T} \right) \geq 1 \quad \forall \, T \geq 0. \]

Hence we get
\[ \tilde{V}(q) \geq \rho \left[ (\gamma - 1) \log \left( \frac{\rho}{\rho} \right) - \gamma + 1 \right] + (\gamma - 1)\rho. \]

Thus if \( \rho \geq \rho_{\text{min}} > 0 \) we can choose \( \overline{\rho} < \rho_{\text{min}} \) such that
\[ \tilde{V}(q) - (\gamma - 1)\overline{\rho} > 0. \]

Hence \((4.8)\) holds.

Now
\[ \overline{U} - U = (\gamma - 1) \log \left( \frac{\rho}{\rho} \right) - \log \left( \frac{T}{T} \right). \]

If \((4.7)\) is valid then
\[ \overline{U} - U < (\gamma - 1) \log \left( \frac{\rho_{\text{max}}}{\rho} \right) - \log \left( \frac{T_{\text{min}}}{T} \right). \]

We can choose \( 0 < \overline{T} < T_{\text{min}} \) such that
\[ (\gamma - 1) \log \left( \frac{\rho_{\text{max}}}{\rho} \right) - \log \left( \frac{T_{\text{min}}}{T} \right) < 0. \]

Hence \((4.9)\) holds.

The Navier-Stokes equations are:

\[ (4.10) \quad Nq = Eq + (-D)q = 0 \]
where $E$ is the hyperbolic differential operator corresponding to the Euler equations and $D$ is the incompletely elliptic perturbation, depending on the parameters, $\mu$, $\lambda$, and $k$, due to the dissipative terms in the Navier-Stokes equations. It is assumed that $\mu$, $\lambda$, and $k$ are proportional to a small parameter $\varepsilon > 0$. Thus the Navier-Stokes equations may be viewed as an incompletely elliptic perturbation of a system of hyperbolic equations.

The question we are concerned with is: for which boundary conditions are the solutions of the perturbed problem (i.e., $\varepsilon > 0$) well defined in some time interval $0 \leq t \leq \tau_0$ and are bounded in an appropriate norm as $\varepsilon \to 0$.

Michelson [8] suggests boundary conditions of the type given below for the singular perturbation problem:

\begin{equation}
\text{(4.11a)} \quad Bq = S(\{(\varepsilon D)^{\alpha}\}_{\alpha \in A}; \frac{\alpha}{\varepsilon} \in \Omega, \ t \geq 0, \ \varepsilon > 0) = 0
\end{equation}

where

$$
(\varepsilon D)^{\alpha} = \varepsilon |\alpha| D_{x_1}^{a_1} D_{x_2}^{a_2} D_{x_3}^{a_3} D_{t}^{a_4}
$$

and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a multi-index in some finite set $A \subset \mathbb{Z}^4$.

It is well known that a singular perturbation problem of the type we are considering exhibits a boundary layer phenomenon. We wish to choose boundary conditions in such a way that a strong boundary layer does not develop. For the boundary conditions to be compatible with a weak boundary layer, we need a condition of the sort given by:

\begin{equation}
\text{(4.11b)} \quad B|_{\varepsilon=0} q = S(\{(\varepsilon D)^{\alpha}\}_{\alpha \in A}; \frac{\alpha}{\varepsilon} \in \Omega, \ t \geq 0, \ \varepsilon = 0) = 0
\end{equation}
gives a well-posed boundary value problem for the hyperbolic part of the Navier-Stokes equation:

\[ N|_{\varepsilon=0} q = Eq = 0 \]

(see [8]).

Further, from Strikwerda's work on initial boundary value problems for incompletely parabolic systems, we know that the number of boundary conditions which should be imposed for the Navier-Stokes equations to obtain a well-posed problem are:

- 5 boundary conditions for inflow boundary and
- 4 boundary conditions for outflow boundary.

At the same time the unperturbed hyperbolic system requires:

- 5 boundary conditions for supersonic inflow,
- 4 boundary conditions for subsonic inflow,
- 1 boundary condition for subsonic outflow, and
- none for supersonic outflow.

The boundary conditions we are going to impose will be of the form:

\[
(4.12) \quad \varepsilon R \frac{\partial g}{\partial x} + Sq = g
\]

where \( R \) is a matrix of rank at most 4.

To get a set of boundary conditions that also works for \( \varepsilon = 0 \) \( S \) must be chosen in such a way that \( Sq = g \) is a proper set of boundary conditions for the unperturbed hyperbolic problem. If \( Sq = g \) gives too many boundary conditions for the unperturbed hyperbolic problem, the solution will contain boundary layer components of the form \( \exp(-x/\varepsilon) \) for \( \varepsilon + 0 \); (see [4]).
We now proceed to suggest a set of maximal dissipative boundary conditions for the Navier-Stokes equations which are compatible with weak boundary layers.

**Outflow Boundary Conditions**

We need to specify 4 boundary conditions for the perturbed problem. For supersonic outflow (-u > c > 0) we need not specify any boundary conditions for the unperturbed hyperbolic problem, while for subsonic outflow (c > -u > 0) we need to specify only one. From (4.4) we have

\[
\frac{dS}{dt} \leq \int_{\partial \Omega} u(\bar{V} - (\gamma - 1)\bar{p}) d\sigma + \int_{\partial \Omega} \frac{k(\gamma - 1)}{RT} \left( \frac{T}{T} \right) \frac{\partial T}{\partial x} d\sigma \\
- \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} u \left( 2 \mu + \lambda \right) \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) d\sigma \\
- \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} v u \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) d\sigma - \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} w u \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) d\sigma.
\]

By (4.8)

\[
\int_{\partial \Omega} u(\bar{V} - (\gamma - 1)\bar{p}) d\sigma \leq 0.
\]

The boundary conditions we suggest which give a decay of energy are:
\[
\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - \alpha_1 \frac{\partial T}{\partial x} = g_1, \quad \alpha_1 \geq 0
\]

\[
(2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \alpha_2 u = g_2, \quad \alpha_2 \geq 0
\]

(4.13)

\[
\mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) - \alpha_3 v = g_3, \quad \alpha_3 \geq 0
\]

\[
\mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) - \alpha_4 w = g_4, \quad \alpha_4 \geq 0.
\]

It should be noted that if we put \( \alpha_1 = 0 \) we must also put \( g_1 = 0 \) for \( i = 1, \ldots, 4 \). Since we want (4.13) to be compatible with the preceding discussion, we end up with the following set of maximal dissipative boundary conditions:

**Supersonic outflow**

\[
\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} = 0
\]

\[
(2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]

(4.14)

\[
\mu \left( \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \right) = 0
\]

\[
\mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = 0
\]
Subsonic Outflow

\[ \frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} = 0 \]

\[ (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \alpha_2 u = \xi_2 \quad \alpha_2 \geq 0 \]

\[ (4.15) \]

\[ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = 0 \]

\[ \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = 0 \]

Inflow Boundary Conditions

By (4.9)

\[ \int_{\partial \Omega} \rho u [U - \bar{U}] d\sigma \leq 0 \]

Subsonic Inflow

Since \( c > u \) we have \( u^2 \leq \gamma RT \). Hence

\[ \int_{\partial \Omega} \frac{(\gamma - 1)}{2RT} (\rho u)^2 d\sigma \leq \int_{\partial \Omega} \frac{(\gamma - 1)}{2RT} (\rho u) \gamma RT d\sigma \]

\[ = \frac{dS}{dt} \leq \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} u \left( (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) d\sigma \]

\[ + \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} v \left( \frac{(\rho u)}{2} v - \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) d\sigma \]

\[ + \int_{\partial \Omega} \frac{(\gamma - 1)}{RT} w \left( \frac{(\rho u)}{2} w - \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right) d\sigma \]

\[ + \int_{\partial \Omega} \frac{1}{T} \left[ \frac{C_f}{T} \frac{T}{\bar{T}} \frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} + \frac{(\gamma^2 - \gamma + 2)}{2} (\rho u) T \right] d\sigma. \]
The boundary conditions we could specify are:

\[ \rho u = g_1 \]

\[ (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0 \]

\[ (4.16) \]

\[ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) - \alpha_3 v = g_3 \]

\[ \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) - \alpha_4 w = g_4 \]

\[ \frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - \alpha_5 T = g_5. \]

Here we must impose the conditions

\[ \alpha_3 > g_1/2 \]

\[ \alpha_4 > g_1/2 \]

\[ \alpha_5 > \frac{(\gamma^2 - \gamma + 2)}{2} g_1 + \epsilon \text{ for some suitable } \epsilon > 0 \]

depending on \( \bar{T} \).
Supersonic Inflow

\[
\frac{dS}{dt} < \int_{\Omega} \rho u (\vec{U} - \vec{u}) \, d\sigma + \int_{\Omega} \frac{(\gamma - 1)}{\rho T} \left( \frac{\partial u}{\partial x} u - \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \, d\sigma
\]

\[
+ \int_{\Omega} \frac{(\gamma - 1)}{\rho T} \left( \frac{\partial v}{\partial y} v - \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \, d\sigma + \int_{\Omega} \frac{(\gamma - 1)}{\rho T} \left( \frac{\partial w}{\partial z} w - \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right) \, d\sigma
\]

\[
+ \int_{\Omega} \frac{1}{\rho T} \left( \frac{\partial T}{\partial t} k (\gamma - 1) \frac{\partial T}{\partial x} + \rho u \frac{\partial T}{\partial x} \right) \, d\sigma.
\]

The boundary conditions we could specify are:

\[
\rho u = g_1
\]

\[
(2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda (\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) - \alpha_2 u = g_2
\]

\[
\mu (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) - \alpha_3 v = g_3
\]

\[
\mu (\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) - \alpha_4 w = g_4
\]

\[
\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - \alpha_5 T = g_5.
\]

Here the following conditions must hold:

\[
\alpha_2, \alpha_3, \alpha_4 > \frac{g_1}{2}
\]

and \( \alpha_5 > g_1 + \varepsilon \) for some suitable \( \varepsilon > 0 \).
Discussion of the Boundary Conditions

The outflow boundary conditions we have proposed are of the following type:

(4.18) \( \frac{\partial T}{\partial x} = 0. \)

(4.18) says that the computational boundary corresponds to an insulated wall and there is no conduction of heat across it.

(4.19) \( \theta_{xx} = \theta_{yx} = \theta_{zx} = 0 \)

(4.19) asserts that there is no shearing of the fluid either in the normal or tangential direction against the computational boundary.

For an inflow boundary one of the boundary conditions we have specified is:

(4.20) \( \rho u = g. \)

In other words, we specify the momentum influx across the computational boundary.

For the temperature component of the state vector, we have the boundary condition:

(4.21) \( \frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - a_5 T = g_5. \)
(4.21) has an interesting physical interpretation. For injection through a porous wall into the main stream (sometimes called transpiration) the injected fluid may be, say, a coolant at a temperature considerably different from the wall, and one needs to consider an energy balance at the wall. A good approximation for coolant injection is to use the boundary condition proposed by Roberts [in Truit (1960, chapter 11)].

\[ (4.22) \text{Injection: } k \frac{\partial T}{\partial x} = \rho_w u_w c_p (T_w - T_{\text{coolant}}) \]

where \( \rho_w u_w \) is the momentum flow of coolant per unit area through the wall, \( T_w \) is the temperature of the wall, \( T_{\text{coolant}} \) is the temperature of the coolant and \( \frac{\partial T}{\partial x} \bigg|_w \) is the temperature gradient at the wall. (For more information refer to [12, chapter 1.4].)

By choosing \( \alpha_5 \) and \( g_5 \) appropriately (4.21) can be geared to satisfy (4.22). The physical effect of such a boundary condition would be to make the temperature of the fluid within the computational boundary stabilize at \( T_{\text{coolant}} \) over time. When we discretize the boundary conditions, however, we can choose \( g_5 \) so that the discretized version of (4.21) is of extrapolation type. This has the effect of allowing the system to evolve as if there were no boundary.

For the velocity component of the state vector, the boundary conditions we have specified are:
For one-dimensional fluid flow the boundary conditions (4.23) would take the simple form:

\[(2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \alpha_2 u = g_2 \]

\[\mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) - \alpha_3 v = g_3 \]

\[\mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) - \alpha_4 w = g_4. \]

For perfectly diffuse reflection of a gas or fluid against a solid wall we have the boundary condition (after Maxwell in 1879):

\[u_w = \ell \left( \frac{\partial u}{\partial x} \right)_w \]

where \(u_w\) is the velocity of the fluid at the wall, \(\ell\) is the mean free path of the gas or fluid at the wall and \(\left( \frac{\partial u}{\partial x} \right)_w\) is the velocity gradient at the wall (see [12]).

By choosing \(g_2\) and \(\alpha_2\) appropriately we can choose (4.24) to satisfy (4.25). Since we want the boundary conditions to be nonreflecting, however, this would be a bad choice indeed. If we let the computational boundary move with the same velocity as that of the fluid at the boundary interface, we should not have any diffuse reflection. And, in fact, we can discretize (4.24) to achieve this exactly. The discrete boundary conditions so obtained are of extrapolation type and the effect of the computational boundary is minimized.
A last remark is that the boundary conditions proposed are of maximal dissipative type and such boundary conditions are intrinsically radiative.

5. STABLE SEMI-DISCRETE DIFFERENCE SCHEMES FOR THE DIFFERENTIAL EQUATION

The stable difference scheme we are going to propose is valid for the Navier-Stokes equations in multidimensions. However, to avoid getting drowned in notation we restrict the discussion that follows to Navier-Stokes equations in one space dimension.

Consider the incompletely parabolic system obtained from a hyperbolic system of conservation laws with entropy:

\[(5.1) \quad q_t + f(q)_x + (A(q)q_x)_x = 0.\]

Let \( \tilde{V}(q) \) be a normalized entropy function as in (2.9) for the hyperbolic system of conservation laws:

\[(5.2) \quad q_t + f(q)_x = 0 \]

and let \( v^T = \tilde{V}_q \).

We can rewrite (5.1) as:

\[(5.3) \quad q_t + f(q)_x + (C(v)v_x)_x = 0 \]

where we assume \( C(v) \leq 0 \).
Divide the x axis into a mesh of points \( \{ x_i \}_{i \in \mathbb{N}} \), where \( x_i = i\Delta x \) and \( \Delta x \) is the mesh width.

Let \( q_j(t) = q(x_j, t) \). Consider the semidiscrete difference approximation to the hyperbolic system of conservation laws (5.2):

\[
\frac{dq_j}{dt} + \frac{\Delta-}{\Delta x} (h(q_{j-r}, \ldots, q_{j+s})) = 0
\]

where \( r \) and \( s \) are integers \( \geq 0 \) and \( h(q_{j-r}, \ldots, q_{j+s}) \) is the numerical flux function corresponding to the flux function \( f(q) \), i.e., \( h(q_{j-r}, \ldots, q_{j+s}) \) satisfies the consistency condition:

\[
h(q, \ldots, q) = f(q).
\]

Let the order of accuracy of the semidiscrete difference approximation be \( \alpha \) where we can take \( \alpha \geq 1 \).

Further, suppose (5.4) satisfies a semidiscrete entropy inequality:

\[
\frac{d\tilde{\Psi}(q_j)}{dt} + \frac{\Delta-}{\Delta x} [H(q_{j-r}, \ldots, q_{j+s})] \leq 0
\]

where \( H(q_{j-r}, \ldots, q_{j+s}) \) is the numerical entropy flux function corresponding to the entropy flux function \( \tilde{F}(q) \), i.e., \( H(q_{j-r}, \ldots, q_{j+s}) \) satisfies the consistency relation:

\[
H(q, \ldots, q) = \tilde{F}(q).
\]
We propose the following semidiscrete approximation for the Navier-Stokes equation (5.1):

\[
\frac{d q_j}{d t} + \frac{\Delta}{\Delta x} \left( h(q_{j-r}, \cdots, q_{j+s}) \right) + \frac{\Delta}{\Delta x} \left( C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x} \right) = 0
\]

where

\[
v_{j+1/2} = \frac{v_j + v_{j+1}}{2}.
\]

It is easy to see that the order of accuracy \( \beta \) of (5.8) is given by \( \beta = \min(\alpha, 2) \). Hence if (5.4) is a second-order accurate approximation to (5.2) then (5.8) is a second-order accurate approximation to (5.1). In any case, (5.7) is at least a first-order accurate approximation to (5.1).

**Remark:** The only second-order accurate semidiscrete difference scheme for a hyperbolic system of conservation laws which satisfies an entropy condition of the type (5.6) of which we are aware of is one proposed by Osher in [10].

**Lemma 5.1:** The semidiscrete difference approximation (5.8) to the Navier-Stokes equations (5.1) satisfies the entropy inequality:

\[
\frac{d}{d t} (q_j) + \frac{\Delta}{\Delta x} \left[ H(q_{j-r}, \cdots, q_{j+s}) \right] + \frac{v_j^T}{\Delta x} \Delta - \left( C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x} \right) \leq 0.
\]

Fix \( \Delta t > 0 \). Let

\[
q_j^*(\Delta t) = q_j - \frac{\Delta t}{\Delta x} \Delta - \left( h(q_{j-r}, \cdots, q_{j+s}) \right).
\]

Let \( q^E(t) \) denote the solution to the semidiscrete difference scheme
(5.10) \[ \frac{dq^E}{dt} + \frac{\Delta t}{\Delta x} h(q^E_{j-r}, \ldots, q^E_{j+s}) = 0 \]

with \( q^E_j(0) = q_j \). Then \( q^*_j(\Delta t) = q^E_j(\Delta t) + O(\Delta t^2) \). Since (5.10) satisfies the entropy inequality:

\[ \frac{d\tilde{V}}{dt}(q^E_j) + \frac{\Delta t}{\Delta x} [H(q^E_{j-r}, \ldots, q^E_{j+s})] \leq 0 \]

\( \Psi \) sequence \( \{\Delta t_k\} \ni \Delta t_k \to 0 \) \( \exists \) a sequence \( \{\epsilon_k\} \ni \epsilon_k \to 0 \) and the discrete entropy inequality:

\[ \frac{\Psi(q^*_j(\Delta t_k)) - \Psi(q_j)}{\Delta t_k} + \frac{\Delta t}{\Delta x} [H(q^E_{j-r}, \ldots, q^E_{j+s})] \leq \epsilon_k \]

holds.

We now define a discrete difference approximation for the Navier-Stokes equation:

\[ \bar{q}_j(\Delta t) = q^*_j(\Delta t) - \frac{\Delta t}{\Delta x} \Delta - (C(v_{j+1/2}) \frac{\Delta^+ v_j}{\Delta x}) \cdot \]

Let \( q_j(t) \) denote the solution to the semidiscrete difference scheme:

\[ \frac{dq_j}{dt} + \frac{\Delta t}{\Delta x} (h(q^E_{j-r}, \ldots, q^E_{j+s})) + \frac{\Delta t}{\Delta x} (C(v_{j+1/2}) \frac{\Delta^+ v_j}{\Delta x}) = 0. \]

Then it is obvious that \( \bar{q}_j(\Delta t) = q_j(\Delta t) + O(\Delta t^2) \). Define an approximate entropy function

\[ W(\bar{q}_j(\Delta t)) = \tilde{V}(q^*_j(\Delta t)) - v_j^T \frac{\Delta t}{\Delta x} \Delta - (C(v_{j+1/2}) \frac{\Delta^+ v_j}{\Delta x}). \]
By (5.11),
\[ W(q_j(\Delta t_k)) < \tilde{V}(q_j) - \frac{\Delta t_k}{\Delta x} \Delta - [H(q_{j-r}, \ldots, q_{j+s})] \]
\[ - \frac{v_j^T}{\Delta x} \left( C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x} \right) + \epsilon_k \Delta t_k, \]
or
\[ \frac{W(q_j(\Delta t_k))}{\Delta t_k} - \tilde{V}(q_j) \leq - \frac{\Delta -}{\Delta x} [H(q_{j-r}, \ldots, q_{j+s})] \]
\[ - \frac{v_j^T}{\Delta x} \left( C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x} \right) + \epsilon_k. \]

By a Taylor series argument:
\[ W(q_j(\Delta t)) = \tilde{V}(q_j(\Delta t)) + O(\Delta t^2). \]

And since \( \tilde{q}_j(\Delta t) = q_j(\Delta t) + O(\Delta t^2) \),
\[ \frac{d\tilde{V}}{dt} (q_j) \leq - \frac{\Delta -}{\Delta x} [H(q_{j-r}, \ldots, q_{j+s})] - \frac{v_j^T}{\Delta x} \Delta - (C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x}) \]
by letting \( \Delta t_k \to 0 \) in (5.12).

Define a discrete version of the energy \( S(t) \) by:
\[ (5.13) \quad S(t) = \sum_{j=1}^{N} \Delta x \tilde{V}(q_j). \]

We can now easily obtain a discrete version of the energy estimates of Section 2. (Henceforth we shall denote \( h(q_{j-r}, \ldots, q_{j+s}) \) by \( h_j \) and \( H(q_{j-r}, \ldots, q_{j+s}) \) by \( H_j \).)
Theorem 5.1: Consider the semidiscrete difference approximation:

\[ \frac{dq_j}{dt} = -\frac{\Delta}{\Delta x} [h_j] - \frac{\Delta}{\Delta x} [C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x}] \]

and let

\[ S(t) = \sum_{j=1}^{N} \Delta x \tilde{V}(q_j). \]

Then the following 'energy' estimate holds:

\[ \frac{dS}{dt} \leq \left[ H_0 + v_1^T C(v_{1/2}) \frac{\Delta v_0}{\Delta x} \right] - \left[ H_N + v_{N+1}^T C(v_{N+1/2}) \frac{\Delta v_N}{\Delta x} \right]. \]

Proof: By Lemma 5.1

\[ \frac{d\tilde{V}(q_j)}{dt} \leq -\frac{\Delta}{\Delta x} [H_j] - v_j^T \frac{\Delta}{\Delta x} [C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x}]. \]

Hence

\[ \frac{dS}{dt} = \sum_{j=1}^{N} \Delta x \frac{d\tilde{V}(q_j)}{dt} \leq \sum_{j=1}^{N} \Delta - [H_j] - \sum_{j=1}^{N} v_j^T \Delta - [C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x}] \]

\[ \Rightarrow \frac{dS}{dt} \leq H_0 - H_N + \sum_{j=1}^{N} \Delta + v_1^T C(v_{1/2}) \frac{\Delta v_0}{\Delta x} \]

\[ + v_{N+1}^T C(v_{N+1/2}) \frac{\Delta v_N}{\Delta x} - v_{N+1}^T C(v_{N+1/2}) \frac{\Delta v_N}{\Delta x} \]

using summation by parts. Since

\[ C \leq 0 \Rightarrow \sum_{j=1}^{N} \Delta + v_j^T C(v_{j+1/2}) \frac{\Delta v_j}{\Delta x} \leq 0. \]
Hence

\[ \frac{dS}{dt} < \left[ H_0 + v_j^T C(v_{1/2}) \frac{A_+ v_0}{\Delta x} \right] - \left[ H_N + v_{N+1}^T C(v_{N+1/2}) \frac{A_+ v_N}{\Delta x} \right].\]

Remark: The generalization of this theorem to multidimensions is obvious.

We now extend these results to uneven meshes, though restricting (5.4) to the Godunov scheme.

The Godunov Scheme

Consider a grid of points \( \{x_j\}_{j \in \mathbb{N}} \) and define \( \Delta x_j = x_{j+1} - x_j \). Define

\[ x_{j-1/2} = \frac{x_{j-1} + x_j}{2} \quad \text{and} \quad \delta x_j = \frac{\Delta x_{j-1} + \Delta x_j}{2}. \]

Figure 5.1
Consider the hyperbolic system of conservation laws:

\[ q_t + f(q)_x = 0 \]  \hspace{1cm} (5.15)

with Riemann initial data:

\[ q(x, t)|_{t=0} = \begin{cases} q_j & \text{for } x < 0 \\ q_{j+1} & \text{for } x > 0 \end{cases} \]

Then the Godunov flux \( f^G(q_j, q_{j+1}) = f_j \) is defined as:

\[ f^G(q_j, q_{j+1}) = f(q(x = 0, t = 0^+)) \]  \hspace{1cm} (5.16)

where \( q(x, t) \) is the solution to the Riemann problem (5.15).

The semidiscrete version of the Godunov scheme is:

\[ \frac{dq_j}{dt} + \frac{A_-}{\Delta x} [f^G(q_j, q_{j+1})] = 0 \]  \hspace{1cm} (5.17)

where \( f^G(q_j, q_{j+1}) \) is defined in (5.16). The semidiscrete Godunov scheme satisfies the entropy inequality:

\[ \frac{d\tilde{V}(q_j)}{dt} + \frac{A_-}{\Delta x} [F^G(q_j, q_{j+1})] \leq 0 \]  \hspace{1cm} (5.18)

where \( F^G(q_j, q_{j+1}) = \tilde{F}_j \) is the numerical entropy flux for the Godunov scheme and is defined as:

\[ F^G(q_j, q_{j+1}) = \tilde{F}(q(x = 0, t = 0^+)) \]
where \( q(x, t) \) is the solution to the Riemann problem (5.15).

A natural modification of (5.8) for uneven meshes is:

\[
(5.19) \quad \frac{dq_j}{dt} + \frac{\Delta^-}{\delta x_j} \left(f^G(q_j, q_{j+1})\right) + \frac{\Delta^-}{\delta x_j} \left(C(v_j + \frac{1}{2}) \frac{\Delta^+}{\Delta x_j} v_j\right) = 0.
\]

Clearly (5.19) is a first-order accurate semidiscrete difference approximation to (5.1). We claim that (5.19) satisfies the entropy inequality:

\[
(5.20) \quad \frac{d\mathcal{V}(q_j)}{dt} + \frac{\Delta^-}{\delta x_j} \left(f^G(q_j, q_{j+1})\right) + v_j^T \frac{\Delta^-}{\delta x_j} \left(C(v_j + \frac{1}{2}) \frac{\Delta^+}{\Delta x_j} v_j\right) < 0.
\]

The proof is exactly the same as in Lemma 5.1.

Define a discrete version of the 'energy' \( S(t) \) for the uneven mesh \( \{x_j\} \) by:

\[
(5.21) \quad S(t) = \sum_{j=1}^{N} \delta x_j \mathcal{V}(q_j).
\]

Then we obtain the following theorem.

**Theorem 5.2:** Consider the semidiscrete difference approximation:

\[
(5.19) \quad \frac{dq_j}{dt} = -\frac{\Delta^-}{\delta x_j} \left(f^G_j\right) - \frac{\Delta^-}{\delta x_j} \left(C(v_j + \frac{1}{2}) \frac{\Delta^+}{\Delta x_j} v_j\right)
\]

and let

\[
S(t) = \sum_{j=1}^{N} \delta x_j \mathcal{V}(q_j).
\]
Then the following `energy' estimate holds:

\[
(5.22) \quad \frac{dS}{dt} \leq \left[ F_{0}^{G} + v_{1}^{T} C(v_{1/2}) \frac{\Delta + v_{0}}{\Delta x_{0}} \right] - \left[ F_{N}^{G} + v_{N+1}^{T} C(v_{N+1/2}) \frac{\Delta + v_{N}}{\Delta x_{N}} \right].
\]

**Proof:** The proof is exactly as in Theorem 5.1.

**Remark:** Theorem 5.2, though a simple extension of Theorem 5.1, will prove very useful in proposing stable boundary conditions for semidiscrete difference approximations to the Navier-Stokes equations.
6. STABLE BOUNDARY CONDITIONS FOR THE DIFFERENCE SCHEMES

Figure 6.1

Suppose we have the boundary condition at \( x = 0 \):

\[ \beta \frac{\partial T}{\partial x} - T = g \]

where \( \beta \) can be arbitrarily small.

Then a discrete version of this boundary condition would be:

\[ \beta \frac{(T_1 - T_0)}{\Delta x_0} - T_0 = g \]

\[ \Rightarrow T_0 = \frac{\beta T_1 - g \Delta x_0}{(\beta + \Delta x_0)} . \]

So if \( \Delta x_0 \gg \beta \Rightarrow T_0 = -g \) and hence could give rise to a steep 'numerical boundary layer.' To avoid this we should have \( \Delta x_0 \) of the same order of magnitude as \( \beta \) or smaller. If, however, we choose an even mesh this becomes computationally infeasible since \( \beta \) can be arbitrarily small.

To overcome this difficulty we propose the following: Divide the interval \( [x_0, x_{N+1}] \) into \( N + 2 \) points \( x_0, x_1, \ldots, x_N, x_{N+1} \). As before let \( \Delta x_J = x_{J+1} - x_J \). Choose
where we may choose $\Delta x^- \ll \Delta x$. (See Figure 6.1.) So the mesh points are evenly spaced in the interior, but $x_0$ is close to $x_1$ and $x_{N+1}$ is close to $x_N$.

Define $\delta x = \frac{\Delta x + \Delta x^-}{2}$. Henceforth we shall restrict $q$ to be $q = (\rho, m, E)^T$.

The semidiscrete difference approximation for the mixed initial boundary value problem is tailored according to (4.19):

\[
\begin{align*}
\frac{dq_1}{dt} + \frac{\Delta x}{\delta x} \left[ f^G(q_1, q_2) \right] + \frac{1}{\delta x} \left[ C(v_{3/2}) \frac{\Delta + v_1}{\Delta x} - C(v_{1/2}) \frac{\Delta + v_0}{\Delta x} \right] = 0 \\
\frac{dq_j}{dt} + \frac{\Delta x}{\delta x} \left[ f^G(q_j, q_{j+1}) \right] + \frac{\Delta x}{\delta x} \left[ C(v_{j+1/2}) \frac{\Delta + v_j}{\Delta x} \right] = 0 \quad \text{for } j=1, \ldots, N-1 \\
\frac{dq_N}{dt} + \frac{\Delta x}{\delta x} \left[ f^G(q_N, q_{N+1}) \right] + \frac{1}{\delta x} \left[ C(v_{N+1/2}) \frac{\Delta + v_N}{\Delta x} - C(v_{N-1/2}) \frac{\Delta + v_{N-1}}{\Delta x} \right] = 0 \\
\end{align*}
\]

(6.1) with initial condition:

\[ q_j(0) = Q_j \]

and boundary conditions:

\[ B(q_0, q_1, q_2) = 0 \ \forall t \]

\[ \tilde{B}(q_{N-1}, q_N, q_{N+1}) = 0 \ \forall t \]

where $B$ and $\tilde{B}$ are boundary operators.
In general, B and \( \tilde{B} \) may be an underdetermined set of boundary conditions. For example, \( B(q_0, q_1, q_2) \) may be of the form:

\[
\begin{align*}
  u_0 &= u_1 \\
  T_0 &= T_1 \\
  \text{and } \rho_0 \text{ is unspecified.}
\end{align*}
\]

If \( B \) is a "legitimate" boundary operator we can connect \( q_1 \) to \( q_0 \) by just two waves, say a 2- contact and a 3- shock or rarefaction, such that \( B(q_0, q_1, q_2) = 0 \) holds. We make this statement more precise by examining the boundary Riemann problem.

Suppose we have a boundary (a wall) given by \( x = st \). We want to give a number, \( m \), of nonlinear boundary conditions depending on \( q \) at the boundary:

\[
\begin{align*}
  h(q)|_{x=st} &= 0 \\
  h: \mathbb{R}^n \to \mathbb{R}^m, h \in C^\infty
\end{align*}
\]

(6.2)

and initial condition:

\[
q(x, t)|_{t=0} = q_r \text{ (a constant state) for } x > 0.
\]

The underlying differential equation is

(6.3) \( q_t + f(q)_x = q_t + A(q)q_x = 0 \)

where
A(q) = \frac{\partial f}{\partial q}.

We want to construct similarity solutions for this initial boundary problem. Clearly, for the problem to be well-posed we have to choose \( q_r \) so that

\[ \lambda_1(q_r) < \cdots < \lambda_k(q_r) < s < \lambda_{k+1}(q_r) < \cdots < \lambda_n(q_r) \]

where \( \lambda_i(q) \) are the eigenvalues of the matrix \( A(q) \) and \( m = n - k \).

We can connect \( q_r \) to the boundary by \( n - k \) waves. The rarefunction and shock curves of the \( j^{th} \) family determine a wave curve:

\[
W^j(q_r, \epsilon_j) = \begin{cases} 
R^j(q_r, \epsilon_j), & \epsilon_j > 0 \\
S^j(q_r, \epsilon_j), & \epsilon_j < 0 
\end{cases}
\]

where \( W^j \) is twice continuously differentiable in both its arguments (see [2]).

Given \( q_r \) and sufficiently small parameters \( \epsilon_{k+1}, \cdots, \epsilon_n \) we define states \( q^n, \cdots, q^{k+1} \) inductively by:

\[
q^n = W^n(q_r, \epsilon_n)
\]

\[
q^{j-1} = W^j(q^j, \epsilon_j) \text{ for } j = n, \cdots, k + 2.
\]

Set:

\[
q^{k+1} = q^{k+1}(q_r, \epsilon_{k+1}, \cdots, \epsilon_n).
\]
Then the boundary condition \( h(q) = 0 \) becomes \( m \) equations in the \( n - k \) unknowns \( \varepsilon_{k+1}, \ldots, \varepsilon_n \):

\[
h(q, (q_{r}, \varepsilon_{k+1}, \ldots, \varepsilon_n)) = 0.
\]

By the implicit function theorem we will be able to solve for \( \varepsilon_{k+1}, \ldots, \varepsilon_n \) in terms of \( q_r \) in a \( C^2 \) manner if \( m = n - k \) and the Jacobian \( \frac{\partial h}{\partial \varepsilon} \) has rank \( m \). Differentiating we get the local solvability condition:

\[
(6.4) \quad \frac{\partial h}{\partial q} \left( \begin{array}{c|c|c}
1 & \cdots & 1 \\
\vdots & & \vdots \\
r_{k+1} & \cdots & r_n
\end{array} \right)_{q=q_r}
\]

has rank \( m = n - k \)

where \( r_i \) in the \( i^{th} \) eigenvector of the matrix \( A(q) \) corresponding to the eigenvalue \( \lambda_i \). The construction is described in Figure (6.2).
(6.4) is also a necessary condition for the linearized initial boundary value problem to be well-posed.

We relate this now to the boundary operator \( B(q_0, q_1, q_2) = 0 \). Let \( q_2 \) denote \( q_0 \) and \( q_r \) denote \( q_1 \). Then \( B(q_0, q_1, q_2) = 0 \) corresponds to the \( m \) boundary conditions:

\[
(6.5) \quad h(q)|_{x=0} = 0
\]

and the initial condition for the initial boundary Riemann problem (6.2) is:

\[
q(x, t)|_{t=0} = q_r = q_1.
\]

So \( B(q_0, q_1, q_2) \) is a 'legitimate' operator if (6.4) holds. Assuming that it is, we can connect \( q_1 \) to a state \( q_0 \) such that \( B(q_0, q_1, q_2) = 0 \) holds.

Returning to the difference scheme (6.1) we can now define the Godunov flux \( t^G(q_0, q_1) \) using this construction. Suppose at \( t = 0 \) \( q_1(x, t) = q_1 \) and we have the boundary operator \( B(q_0, q_1, q_2) = 0 \).

Consider the Riemann initial boundary value problem:

\[
q(x, t)|_{t=0} = q_1 \text{ for } x_{1/2} < x < x_{3/2}
\]

and boundary condition:

\[
h(q) = 0 \text{ for } x = x_0 \text{ corresponding to } B(q_0, q_1, q_2) = 0
\]

as in (6.5).
Then if (6.4) holds we can join \( q_1 \) to a state \( q_0 \) in a small enough neighborhood of \( q_1 \) so that \( B(q_0, q_1, q_2) = 0 \).

In general, the number of boundary conditions, \( m \), specified by the boundary operator \( B \) will be more than the number of boundary conditions needed for the hyperbolic Euler equations since we are solving the Navier-Stokes equations, i.e., \( m > n - k \). Hence, when we connect \( q_1 \) to a state \( q_0 \) lying to the left of the line \( x = x_{1/2} \) some of the waves will lie in the region \( x < x_{1/2} \). This corresponds to letting waves be radiated at the boundary.

We know that we can connect \( q_1 \) to a state \( q_0 \) such that \( B(q_0, q_1, q_2) = 0 \) if \( q_0 \) and \( q_1 \) are sufficiently close. We chose an uneven mesh precisely for this reason. We set \( \Delta x_0 = \Delta x_N = \Delta x^- \) and \( \Delta x_i = \Delta x_1 \) for \( i = 1, \ldots, N - 1 \) where \( \Delta x^- \) could be arbitrarily small. In fact, by choosing \( \Delta x^- \) small enough we can make \( q_0 \) lie as close to \( q_1 \) as we want. This has the effect of making the waves in the cell bounded by \( x = x_0 \) and \( x = x_1 \) very weak and hence we conclude that

\[
F^G(q_0, q_1) = F^G(q_1, q_1) = \tilde{F}(q_1).
\]
Recollect that the energy estimate we obtained for an uneven mesh was of the form:

\[
\frac{dS}{dt} \leq \left[ F^G(q_0, q_1) + v^T_1 C(v_{1/2}) \frac{\Delta + v_0}{\Delta x} \right] - \left[ F^G(q_N, q_{N+1}) + v^T_{N+1} C(v_{N+1/2}) \frac{\Delta + v_N}{\Delta x} \right].
\]

(6.6)

Since \( q_0 + q_1 \) as \( \Delta x^* \to 0 \) the estimate (6.6) reduces to:

\[
\frac{dS}{dt} \leq \left[ \hat{F}(q_1) + \hat{v}_{q}(q_1)A(q_1) \frac{\Delta + q_0}{\Delta x^*} \right] - \left[ \hat{F}(q_N) + \hat{v}_{q}(q_N)A(q_N) \frac{\Delta + q_N}{\Delta x^*} \right]
\]

(6.7)

for sufficiently small \( \Delta x^* \).

One further point that should be noted is that when we take a fully discrete version of our semidiscrete scheme the CFL condition we have to impose to prevent wave interactions is:

\[
\Delta t \lambda(A) \leq \frac{\Delta x + \Delta x^*}{2}
\]

(6.8)

where \( \lambda(A) \) is the spectral radius of the matrix \( A(q) \). Clearly (6.8) is not an unduly restrictive condition.

For the numerical simulation presented in the next and last section, we nondimensionalize the Navier-Stokes equations.

Let \( L \) be a reference length, \( \rho_f \) a reference density, and \( u_f \) a reference velocity. Define:

\[
x^* = x/L,
\]

\[
t^* = tu_f/L,
\]

\[
u^* = u/u_f,
\]
\[
p^* = \frac{p}{\rho_f u_f^2},
\]
\[
T^* = \frac{T}{R u_f^2},
\]
\[
\rho^* = \frac{\rho}{\rho_f},
\]
\[
E^* = \frac{E}{\rho_f u_f^2}.
\]

We also need the following parameters, which occur extensively in fluid dynamics:

\[
Pr = \frac{u_R y}{k(\gamma - 1)} = \frac{\mu c_p}{k},
\]
\[
Re = \frac{\rho_f u_f L}{\mu}.
\]

Here \( Pr \) and \( Re \) are abbreviations for the Prandtl number and Reynolds number respectively. Henceforth we drop the "*" for notational convenience. For our numerical simulation we use Stokes' assumption:

\[
\lambda = -2/3u.
\]

The nondimensionalized Navier-Stokes equations then take the form:

\[
\rho_t + (\rho u)_x = 0
\]
\[
(\rho u)_t + (\rho u^2 + p)_x = \frac{4}{3Re} \frac{\partial^2 u}{\partial x^2}
\]
\[
(E)_t + [(E + p)u]_x = \frac{4}{3Re} \frac{\partial}{\partial x} (u \frac{\partial u}{\partial x}) + \frac{\gamma}{(\gamma - 1)Pr Re} \frac{\partial^2 T}{\partial x^2}.
\]
Rewriting the dissipative term \((-D)q\) in terms of our new variable \(v = \frac{\nabla}{q}\), we obtain:

\[
(C(v)v_x)_x = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{4uT}{3Re} & -\frac{4uT}{3Re} \\
0 & -\frac{4uT}{3Re} & -\frac{4u^2 T}{3Re} - \frac{\gamma T^2}{(\gamma - 1)Re Pr} \\
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} \\
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{u}{T} \\
\frac{1}{T} \\
\end{pmatrix}
\]

Our nondimensionalized boundary conditions take the form:

**Supersonic Outflow**

\[\frac{\partial u}{\partial x} = 0\]

\[\frac{\partial T}{\partial x} = 0\]

**Subsonic Outflow**

\[\frac{4}{3Re} \frac{\partial u}{\partial x} - a_2 u = g_2 \text{ where } a_2 > 0\]

\[\frac{\partial T}{\partial x} = 0\]

**Subsonic Inflow**

\[
\rho u = g_1
\]

\[\frac{\partial u}{\partial x} = 0\]

\[\frac{\gamma}{Re Pr} \frac{\partial T}{\partial x} - a_3 T = g_3 \text{ where } a_3 > \frac{(\gamma^2 - \gamma + 2) g_1}{2}\]
Supersonic Inflow

\[ \rho u = g_1 \]

\[ \frac{4}{3\text{Re}} \frac{\partial u}{\partial x} - a_2 u = g_2 \]

\[ \frac{\gamma}{\text{Re} \text{Pr}} \frac{\partial T}{\partial x} - a_3 T = g_3 \text{ where } a_2 > \frac{g_1}{2}, a_3 > g_1. \]

We now propose a discrete version of these boundary conditions which are of extrapolation type and hence minimize the effect of the computational boundary.

Supersonic Outflow

\[ u_0 = u_1 \]

\[ T_0 = T_1 \]

We can connect \( q_0 \) to \( q_1 \) without any waves at all by putting

\[ \rho_0 = \rho_1. \]

Subsonic Outflow

The boundary conditions proposed for supersonic outflow would work equally well for subsonic outflow. This corresponds to choosing \( a_2 = 0 \) in the subsonic outflow boundary conditions. If we choose \( a_2 > 0 \) the discretized boundary conditions would be:

\[ \frac{4}{3\text{Re}} \frac{(u_1 - u_0)}{\Delta x} - a_2 u_1 = g_1 \]
where we could put

\[ g_1 = \frac{4}{3\text{Re}} \frac{(u_2 - u_1)}{\Delta x} - a_2 u_2. \]

Then

\[ u_0 = \frac{(4/3\text{Re}) - a_2 \Delta x' u_1 - g_1 \Delta x'}{(4/3\text{Re})}. \]

Clearly \( u_0 + u_1 \) as \( \Delta x' \to 0 \). We choose \( \rho_0 \) by joining \( q_1 \) to \( q_0 \) by two wave interactions.

**Subsonic Inflow**

\[ (\rho u)_0 = g_1 = (\rho u)_1 - \frac{\Delta x'}{\Delta x} [(\rho u)_2 - (\rho u)_1] \]

\[ u_0 = u_1 \]

\[ \frac{\gamma}{\text{Re} \: \text{Pr}} \frac{(T_1 - T_0)}{\Delta x'} - a_3 T_1 = g_3 = \frac{\gamma}{\text{Re} \: \text{Pr}} \frac{(T_2 - T_1)}{\Delta x} - a_3 T_2. \]

Here \( a_3 > (\gamma^2 - \gamma + 2)g_1/2 \).

**Supersonic Inflow**

\[ (\rho u)_0 = g_1 = (\rho u)_1 - \frac{\Delta x'}{\Delta x} [(\rho u)_2 - (\rho u)_1] \]

\[ \frac{4}{3\text{Re}} \frac{(u_1 - u_0)}{\Delta x} - a_2 u_1 = g_2 = \frac{4}{3\text{Re}} \frac{(u_2 - u_1)}{\Delta x} - a_2 u_2 \]

\[ \frac{\gamma}{\text{Re} \: \text{Pr}} \frac{(T_1 - T_0)}{\Delta x'} - a_3 T_1 = g_3 = \frac{\gamma}{\text{Re} \: \text{Pr}} \frac{(T_2 - T_1)}{\Delta x} - a_3 T_2. \]
Here $a_2 > (g_1)/2$, $a_3 > g_1$.

As mentioned earlier all the boundary conditions are such that $q_0 + q_1$ as $\Delta x^\gamma \to 0$ and they are a discretized version of the boundary conditions for the differential equations, i.e.,

$$\varepsilon R \frac{\partial q}{\partial x} + Sq = g.$$

It is easy to verify from (6.7) that for $\Delta x^\gamma$ small enough we get bounded growth of the discrete version of the energy $S(t)$ in time by choosing $a_2$ and $a_3$ appropriately.

7. NUMERICAL RESULTS

Throughout the simulations we use the following parameter values:

\[ Pr = 0.7 \]
\[ \gamma = 1.4 \]
\[ \Delta x = 0.1 \]
\[ \Delta x^\gamma = 0.000001 \]
\[ \Delta t = 0.01. \]

**Simulation 1:** For the first simulation we use a very large value of the Reynolds number

\[ \text{Re} = 10^6. \]

We run the program with Riemann initial data:
\[ \rho_L = 1.0 \]
\[ u_L = 3.0 \]
\[ p_L = 1.0 \]

and

\[ \rho_R = 0.4734821 \]
\[ u_R = 2.1393370 \]
\[ p_R = 0.3333333. \]

At the left boundary our boundary conditions correspond to supersonic inflow. At the right boundary we have supersonic outflow boundary conditions.

The solution to the Euler equations with initial data corresponding to the above Riemann problem is a 3 shock moving to the right with a speed \[ = 3.774567. \]

The numerical simulations bear this out very well. The boundary conditions are radiative and allow the shock to pass through. Further for long time the solution stabilizes to a constant state.

**ACKNOWLEDGEMENTS**

The author is grateful to Professor S. Osher for his valuable help and encouragement and would like to thank Professor Ralston and Professor Bube for helpful discussions.
Figure 7.1
Figure 7.2

\[ T = 0.05 \]
Figure 7.3
Figure 7.4
Figure 7.5
Figure 7.6
Figure 7.7
Figure 7.8
Figure 7.9

$T = 0.2$

The graph shows a plot of $\rho$ against $x$, with $\rho$ ranging from 3.4 to 0.0 and $x$ ranging from 0.0 to 1.0.
T = 1.5

Figure 7.10
Figure 7.11
Figure 7.12
Figure 7.13

$T = 0.1$
Figure 7.14

$T = 0.2$

![Graph showing $T = 0.2$ with values along the y-axis from 0.0 to 3.4 and x-axis from 0.0 to 1.0.](image-url)
Simulation 2: We use a low Reynolds number:

\[ \text{Re} = 500. \]

Once more we run the program with Riemann initial data:

\[ \rho_L = 1.0 \]
\[ u_L = 1.0 \]
\[ p_L = 1.0 \]

and

\[ \rho_R = 1.625000 \]
\[ u_R = 0.3798263 \]
\[ p_R = 2.000000. \]

At the left boundary we have subsonic inflow boundary conditions and at the right boundary the boundary conditions correspond to subsonic outflow.

The solution to the Euler equations with the above Riemann initial data is a 1 shock moving to the left with a speed \( = 0.6124516. \)

The numerical results, once again, have all the desirable properties we observed in Simulation 1.
Figure 7.16

T = 0.0

X

RHO

2.2

2.0

1.8

1.6

1.4

1.2

1.0

0.8

0.6

0.4

0.2
Figure 7.17
Figure 7.18

$T=0.4$
Figure 7.19
$T = 1.5$

Figure 7.20
\textbf{Figure 7.21}
Figure 7.22
$T = 0.4$

**Figure 7.23**
Figure 7.24
Figure 7.25
Figure 7.26
Figure 7.27
Figure 7.28

$T = 0.4$

![Graph showing the relationship between $P$ and $X$ at $T = 0.4$.]
Figure 7.29

$T=0.8$
Figure 7.30
REFERENCES


STABLE BOUNDARY CONDITIONS AND DIFFERENCE SCHEMES FOR NAVIER-STOKES EQUATIONS

Pravir Dutt

Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665

National Aeronautics and Space Administration
Washington, D.C. 20546

J. C. South Jr.
Additional Support: NSF Grant No. 82-00788 and ARO Grant No. DAAG 29-82-0090 and NASA Grant NAG1-506.

The Navier-Stokes equations can be viewed as an incompletely elliptic perturbation of the Euler equations. By using the entropy function for the Euler equations as a measure of energy, we are able to obtain nonlinear energy estimates for the mixed initial boundary value problem. These estimates are used to derive boundary conditions which guarantee $L^2$ boundedness even when the Reynolds number tends to infinity. Finally, we propose a new difference scheme for modelling the Navier-Stokes equations in multidimensions for which we are able to obtain discrete energy estimates exactly analogous to those we obtained for the differential equation.
End of Document