TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE
FLOW PAST ELLIPTIC CYLINDERS
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TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW PAST ELLIPTIC CYLINDERS

By Carl Kaplan

SUMMARY

The method of Poggi is used to calculate, for perfect fluids, the effect of compressibility upon the flow on the surface of an elliptic cylinder at zero angle of attack and with no circulation. The result is expressed in a closed form and represents a rigorous determination of the velocity of the fluid at the surface of the obstacle insofar as the second approximation is concerned.

Comparison is made with Hooker's treatment of the same problem according to the method of Janzen and Rayleigh and it is found that, for thick elliptic cylinders, the two methods agree very well. The labor of computation is, moreover, considerably reduced by the present solution.

The third approximation to the compressible flow about circular cylinders, including the terms involving the factor \( r_o/c_o^4 \), is also obtained and compared with the result given by Poggi. It is found that the expression given by Poggi is incomplete with regard to the terms containing the factor \( r_o/c_o^4 \).

INTRODUCTION

The purpose of this paper is to employ the method of Poggi (reference 1) to determine the effect of compressibility on the flow about elliptic cylinders. This problem has already been considered by Hooker (reference 2) who made use of the method of Janzen and Rayleigh but, owing to the necessity for expanding a certain function in the analysis, the "thickness ratio" of the ellipse to which his result applies is limited. The thickness ratio of an ellipse is defined as the ratio \( b/a \), where \( a \) and \( b \) are the semimajor and semiminor axes, respectively. The method of Poggi, on the other hand, not only permits an unrestricted thickness ratio but also reduces the labor of computation.

Briefly, it may be said that Poggi considers compressible flow to be replaced by an incompressible flow due to a distribution of sinks and sources throughout the region of flow. The strength of the distribution in the plane of the profile is given by

\[
\frac{-1}{4\pi c} \left( \frac{\partial p}{\partial \xi} \frac{\partial r^2}{\partial \xi} + \frac{\partial \phi}{\partial n} \frac{\partial r}{\partial n} \right) d\xi d\eta
\]

and in the plane of the circle, into which the profile is mapped by a suitable conformal transformation, by

\[
\frac{1}{4\pi c} \left( r_o \frac{\partial r^2}{\partial \lambda} - \frac{\partial \phi}{\partial n} \frac{\partial r}{\partial n} \right) d\lambda d\theta
\]

where \( r, \theta \) are the polar coordinates of a point in the plane \( z = x + iy \) of the circle.

\( R, \delta \) the radius of the circle into which the profile is mapped and the angular coordinate on this circle, respectively.

\[
\lambda = \frac{R}{r}; \; \; \phi = -\frac{\partial \phi}{\partial \lambda}; \; \; \phi = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}; \; \; \phi \text{ is the velocity potential of the flow.}
\]

\( c, \) the magnitude of the velocity of the fluid in the plane of the profile.

\( c_s, \) the magnitude of the local velocity of sound.

Poggi then finds that the total velocity induced, at any point \( P(R, \delta) \) of the circular boundary by the foregoing system of sinks and sources, is:

\[
\Delta r = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \phi}{\partial \lambda} \frac{\partial r}{\partial \delta} \sin(\theta - \delta) d\lambda d\theta \tag{1}
\]

Poggi's method of approximating the compressible flow of a perfect fluid is based on the assumption that the incompressible flow is a suitable first approximation and that therefore the values pertaining to that flow may be substituted for \( r_o, \phi, \) and \( c^2 \) in equation (1). The value of \( \Delta r \) so obtained then represents the effect due to compressibility and is to be added to the already known value for the velocity of the incompressible flow. That is,

\[
v_{comp} = v_{incomp} + \Delta v \tag{2}
\]

It is to be noted that, in equation (1), the local velocity of sound \( c_s \) is not a constant but is related to the velocity \( c \) of the fluid in the plane of the profile by means of Bernoulli's equation and the equation of state of the fluid. Thus, if the adiabatic equation of state is adopted,

\[
c^2 = c_{o_s}^2 \left[ 1 + \frac{\gamma - 1}{2} \frac{r_o}{c_{o_s}} \left( 1 - \frac{c_s^2}{c_{o_s}^2} \right) \right] \tag{3}
\]

where \( c_o, r_o \) are the corresponding magnitudes in the undisturbed stream and \( \gamma = 1.408 \) for air.

In order to facilitate the solution of equation (1), it has been the custom to replace \( c \) by \( c_s \). This simpli-
The flow of a perfect compressible fluid past an elliptic cylinder

Let the $\xi$ plane be the plane of the ellipse and the $z$ plane be the plane of the corresponding circle. Then it is well known that the Joukowski transformation

$$\zeta = z + \frac{u}{z}$$

maps the circle of radius $a$ with its center at the origin of the $z$ plane into the line segment $(-2a, 0)$ to $(2a, 0)$ in the $\xi$ plane. Also, the circles concentric with the circle of radius $a$ are transformed into a family of confocal ellipses with common foci at $(-2a, 0)$ and $(2a, 0)$. If $R = a$ denotes the radius of one of these circles, then the semimajor and semiminor axes of the ellipse into which it is transformed are, respectively, $R + \frac{a^2}{R}$ and $R - \frac{a^2}{R}$. The thickness ratio $t$ then becomes:

$$t = \frac{R - \frac{a^2}{R}}{R + \frac{a^2}{R}} = \frac{1 - \frac{a^2}{R}}{1 + \frac{a^2}{R}}$$

or

$$\frac{a^2}{R} = \frac{1 - t}{1 + t}$$

where

$$\sigma = \frac{a}{R}$$

If $w$ denotes the complex potential of the incompressible flow in the $z$ plane when a stream of velocity $v_0$ impinges on a circle of radius $R$ in the direction of the negative $x$ axis, then

$$w = v_0 \left( z + \frac{R^2}{z} \right)$$

The complex velocity in the $\xi$ plane is then given by

$$\frac{dw}{d\xi} = \frac{d}{d\xi} \left( u + \frac{v_0^2}{z} \right)$$

or

$$\frac{d\xi}{d\zeta} = v_0^2 \frac{z^2 - R^2}{z^2 - a^2}$$

When $\lambda = \frac{R}{l}$ and $\sigma = \frac{a}{l}$ are introduced, it follows that

$$v_0^2 \left| \frac{d\zeta}{d\xi} \right| = v_0^2 \frac{1 - 2\lambda^2 \cos \theta + \lambda^4}{1 - 2\sigma\lambda^2 \cos \theta + \sigma^2\lambda^4}$$

Following Poggi's procedure, the Fourier development of $\frac{v^2}{v_0^2}$ will be obtained. Thus, by the use of the expansion

$$1 - 2\lambda\sin \theta \cos \theta + \lambda^2$$

(see appendix, sec. 1), it follows that

$$\frac{v_0^2}{v_0^2} = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_n \cos 2n\theta \right)$$

(11)

where

$$a_n = 2 \sum_{n=1}^{\infty} \frac{a_{2n} \cos 2n\theta}{1 - \sigma^2\lambda^2}$$

and for $n = 1, 2, \ldots$,

$$a_{2n} = \frac{2(1 - \sigma^2)(1 - \sigma^2\lambda^4)}{\sigma^2(1 - \sigma^2\lambda^4)} \left( \frac{\sigma\lambda^4}{1 - \sigma^2\lambda^4} \right)^n$$

Also from equation (8)

$$v_x = -v_0 (1 - \lambda^2) \cos \theta$$

$$v_y = v_0 (1 + \lambda^2) \sin \theta$$

(12)
Then, inserting the expressions for \( r^2, r_s, \) and \( r_0 \) given by equations (11) and (12) into equation (6) and making use of the integrals

\[
\int_0^{2\pi} \frac{\sin \gamma - \frac{1}{2} \sin 3\gamma}{1 - 2\lambda \cos \gamma - \lambda^2} \sin \gamma d\gamma = 0 \quad \text{if } n = 0
\]

see appendix, see. 11

\[
\int_0^{2\pi} \frac{\sin \gamma - \frac{1}{2} \sin 3\gamma}{1 - 2\lambda \cos \gamma - \lambda^2} \cos \mu d\gamma = -\pi \lambda^{n-1} \cos \mu \quad \text{if } n \geq 1
\]

it follows without difficulty that

\[
\frac{\Delta \rho}{\rho_0} = -\frac{\mu}{2} \sin \delta + \sum_{n=0}^{\infty} \frac{(2n+1)}{(2n+1)} \sin (2n+1) \delta
\]

where

\[
\mu = \frac{\sigma^2}{r_0^2}
\]

Substituting for the \( a_n \)'s from equation (11), equation (13) takes the form

\[
\frac{\Delta \rho}{\rho_0} = \frac{\mu}{2} \frac{1 - \sigma^2}{\sigma^2} \left[ \sin \delta - \left(1 - \sigma^2\right) \sum_{n=0}^{\infty} (2n+1) \sin (2n+1) \delta \right]
\]

Replacing \( \lambda^2 \) by \( \tau \), for purposes of integration only, it follows that

\[
I = \sum_{n=0}^{\infty} (2n+1) \sin (2n+1) \delta \int_0^{\pi} \frac{(1 - \sigma^2)^n}{(1 - \sigma^2)^n + \sin \delta} \sin (2n+1) \delta
\]

or

\[
I = R.P. \text{ of } \frac{1}{(1 - 2\sigma^2 \cos \delta + \sigma^2)^n} \int_0^{\pi} \frac{(1 - \sigma^2)^n}{(1 - \sigma^2)^n + \sin \delta} \sin (2n+1) \delta
\]

Replacing \( \lambda^2 \) by \( \tau \), for purposes of integration only, it follows that

\[
\frac{\Delta \rho}{\rho_0} = \frac{\mu}{2} \frac{1 - \sigma^2}{\sigma^2} \left[ \sin \delta - \left(1 - \sigma^2\right) \sum_{n=0}^{\infty} (2n+1) \sin (2n+1) \delta \right]
\]

or

\[
\frac{1}{(1 - 2\sigma^2 \cos \delta + \sigma^2)^n} \int_0^{\pi} \frac{(1 - \sigma^2)^n}{(1 - \sigma^2)^n + \sin \delta} \sin (2n+1) \delta
\]

Therefore

\[
\Delta \rho = \frac{\mu}{2} \frac{1 - \sigma^2}{\sigma^2} \left[ \sin \delta - \left(1 - \sigma^2\right) \sum_{n=0}^{\infty} (2n+1) \sin (2n+1) \delta \right]
\]

\[
\begin{align*}
\text{For } \delta = \frac{\pi}{2}, \text{ the position of maximum velocity on the surface of the elliptic cylinder,}
\end{align*}
\]

\[
\left(\frac{\Delta \rho}{\rho_0}\right)_{\delta = \frac{\pi}{2}} = \frac{\mu}{2} \frac{1 - \sigma^2}{\sigma^2} - \left(1 - \sigma^2\right) \sum_{n=0}^{\infty} (2n+1) \sin (2n+1) \delta
\]

\[
\begin{align*}
\text{It is interesting to note that the expression for } \Delta \rho/\rho_0 \text{ at the surface of a circular cylinder fixed in a stream of velocity } \rho_0 \text{ impinging on in the direction of the negative } x \text{ axis may be obtained from equation (14) by allowing } \sigma \left(\frac{a}{R}\right) \text{ to approach zero. Thus, making use of the expansions}
\end{align*}
\]

\[
\begin{align*}
\text{or}
\end{align*}
\]

\[
\begin{align*}
\text{This expression for } \Delta \rho/\rho_0 \text{ agrees with that obtained by the methods of Janzen, Rayleigh, and Poggi (reference 3).}
\end{align*}
\]

The effect of compressibility, i.e., \( \Delta \rho/\rho_0 \), having been found, it follows according to equation (2) that the total velocity at the circular boundary in the \( z \) plane is given by

\[
\left(\frac{\rho}{\rho_0}\right)_{\text{circuits}} = \frac{2}{\sigma} \sin \delta + \frac{\Delta \rho}{\rho_0}
\]

and on the elliptic profile in the \( \xi \) plane by

\[
\left(\frac{\rho}{\rho_0}\right)_{\text{eliptics}} = \frac{1}{(1 - 2\sigma^2 \cos \delta - \sigma^2)} \left(\frac{\rho}{\rho_0}\right)_{\text{circuits}}
\]
Table I shows the comparison between the values of \( (r'/r_0)_{\text{comp}} \) calculated according to equation (18) and those obtained by Hooker for an ellipse of thickness ratio \( t=\frac{1}{10} \) or \( \sigma^2=\frac{1}{10} \). The values for the corresponding incompressible flow are included. It is seen that the results of the two methods agree very well. This agreement is not unexpected since Hooker's method is particularly applicable to thick ellipses. Consider, however, a slender ellipse, say \( t=\frac{1}{100} \) or \( \sigma^2=\frac{1}{100} \). Table II shows the comparison between the exact calculations of the present method and the results obtained according to Hooker's method. The disagreement is more evident than that shown in Table I for the thicker ellipse.

Table II

<table>
<thead>
<tr>
<th>( \delta ) (deg.)</th>
<th>( \left(\frac{r'}{r_0}\right)_{\text{comp}} )</th>
<th>( \left(\frac{r'}{r_0}\right)_{\text{inc}} )</th>
<th>Incompressible</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.673</td>
<td>1.185</td>
<td>1.336</td>
</tr>
<tr>
<td>20</td>
<td>0.903</td>
<td>1.325</td>
<td>1.430</td>
</tr>
<tr>
<td>30</td>
<td>1.043</td>
<td>1.445</td>
<td>1.464</td>
</tr>
<tr>
<td>40</td>
<td>1.141</td>
<td>1.506</td>
<td>1.493</td>
</tr>
<tr>
<td>50</td>
<td>1.222</td>
<td>1.554</td>
<td>1.501</td>
</tr>
<tr>
<td>60</td>
<td>1.304</td>
<td>1.606</td>
<td>1.511</td>
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<td>1.362</td>
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</tr>
<tr>
<td>90</td>
<td>1.435</td>
<td>1.766</td>
<td>1.517</td>
</tr>
</tbody>
</table>

THE PRESSURE DISTRIBUTION

According to Bernoulli's theorem and the adiabatic equation of state, if \( p \) and \( \rho \) are the pressure and density of the fluid, then

\[
\frac{p}{\rho_0} \left( \rho_0 \right)^{\gamma} = 1 + \frac{\gamma-1}{2} \rho \left( 1 - \frac{v^2}{c_0^2} \right) \left[ 1 - \frac{v^2}{c_0^2} \right]^{\gamma-1}
\]

where \( \rho_0 \) and \( \rho_0 \) are the pressure and density, respectively, in the undisturbed stream. Expanding the right-hand side of the foregoing equation and neglecting

![Graph showing the pressure distribution](image)
THE ATTAINMENT OF THE LOCAL VELOCITY OF SOUND
AT THE SURFACE OF AN ELLIPTIC CYLINDER

According to equation (4) the critical velocity of the fluid is given by

\[
\left( \frac{v_{crit}}{c_0} \right)^2 = \frac{2}{\gamma + 1} \left( \frac{1 - \frac{\rho}{\rho_0}}{1} \right) \frac{1 - \frac{\rho}{\rho_0}}{1 + \frac{\rho}{\rho_0}} \log \left( \frac{1 + \frac{\rho}{\rho_0}}{1 - \frac{\rho}{\rho_0}} \right)
\]

For an elliptic cylinder, at zero angle of attack, the critical velocity occurs at \( \delta = \frac{\pi}{2} \), the position of maximum velocity on the cylinder and also in the region of flow. Hence substituting from equation (18) for \( (\rho/\rho_0)_{crit} \) at \( \delta = \frac{\pi}{2} \) yields a cubic equation in the variable \( \mu \).

Thus, from equation (15), if

\[
f(\sigma) = \frac{1 - \sigma^2}{2\sigma^2} \left[ 1 - \frac{1 - \sigma^2}{1 + \sigma^2} \left( \frac{1 - \sigma^2}{2\sigma^2} \log \frac{1 + \sigma^2}{1 - \sigma^2} \right) \right]
\]

then

\[
[f(\sigma)]^2 \mu^2 + 4f(\sigma) \mu^2 + \left[ 1 - \frac{\gamma - 1}{\gamma + 1} (1 + \sigma^2)^2 \right] \mu
\]

\[
-2 \left( 1 + \sigma^2 \right)^2 = 0
\]

where \( \gamma = 1.408 \) for air.

Table IV gives the critical values of \( v_0/c_0 \) for the entire range of thickness ratios including the limiting cases of the straight-line segment and the circular profile. Figure 3 shows the critical values of \( v_0/c_0 (= \sqrt{\mu}) \) plotted against the thickness ratio.

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**Table III**

<table>
<thead>
<tr>
<th>( \delta ) (deg.)</th>
<th>( \frac{v_{crit}}{c_0} )</th>
<th>( \frac{v_{crit}}{c_0} )</th>
</tr>
</thead>
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<td>0.959</td>
</tr>
<tr>
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<td>0.953</td>
</tr>
<tr>
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</tr>
<tr>
<td>45</td>
<td>0.940</td>
<td>0.941</td>
</tr>
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</table>

---

**Table IV**

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>Thickness ratio</th>
<th>( v_0/c_0 )</th>
</tr>
</thead>
<tbody>
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<td>1.000</td>
</tr>
<tr>
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<td>1.20</td>
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</tr>
<tr>
<td>1.3</td>
<td>1.30</td>
<td>1.300</td>
</tr>
<tr>
<td>1.4</td>
<td>1.40</td>
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</tr>
<tr>
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<td>1.50</td>
<td>1.500</td>
</tr>
<tr>
<td>1.6</td>
<td>1.60</td>
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</tr>
<tr>
<td>1.7</td>
<td>1.70</td>
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</tr>
<tr>
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<td>1.800</td>
</tr>
<tr>
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<td>1.900</td>
</tr>
<tr>
<td>2.0</td>
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</tr>
</tbody>
</table>

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**THE THIRD APPROXIMATION TO THE COMPRESSIBLE FLOW ABOUT CIRCULAR CYLINDERS**

In reference 2, the opinion is expressed by Hooker that the terms involving \( (v_0/c_0)^4 \), thus far neglected, may become of considerable importance as the local velocity of sound is approached on the ellipse. Hooker, however, did not investigate the matter any further. In reference 4, Poggi calculated these terms for the compressible flow about a circular cylinder, but a close examination of his work shows that not all such terms were taken into account. In what follows the terms neglected by Poggi will be obtained and compared with the already existing ones.
The fundamental integral equation (1) may be written as follows:

\[
\frac{\Delta r}{r_0} = \mu \int_0^{2\pi} \int_0^{2\pi} \frac{r_0^3}{r^3} \frac{\partial r}{\partial \lambda} \frac{\partial r}{\partial \theta} \left( 1 - \frac{\gamma - 1}{2} \frac{1}{r_0^2} \left( \frac{1}{r^2} \right) + \ldots \right) \sin (\theta - \delta) d\lambda d\theta
\]  

(22)

where \( \lambda = \frac{R}{r} \).

When the foregoing expressions are substituted into equation (22) and only the terms involving \( \mu \) and \( \mu^2 \) are evaluated, it is found that, besides the terms given by equation (23), the following ones involving \( \mu^2 \) must be included:

\[
\mu^2 \left( \frac{37}{40} \sin \delta - \frac{25}{24} \sin 3\delta + \frac{3}{8} \sin 5\delta \right) + \ldots
\]  

(25)

These terms seem to have been overlooked by both Poggi and Pistolesi (reference 3).

The third approximation to the compressible flow at the surface of the circular cylinder then becomes:

\[
\frac{\Delta r}{r_0} = \frac{v_{comp}}{r_0} - \frac{v_{incomp}}{r_0} + \frac{\Delta r}{r_0}
\]  

(24)

where \( \Delta r_0 \) is given by equation (23).

Equation (24) thus represents the third approximation to the compressible flow, the first approximation being the purely incompressible flow given by \( \frac{v_{incomp}}{r_0} \).

The third approximation may be obtained, at least in principle, by substituting for \( r_0 \), \( \theta_0 \), and \( \varphi \) in equation (22) expressions based on the second approximation. Such expressions, as far as the terms involving \( \mu \) are concerned, are given in reference 3 and are as follows:

\[
\frac{v_{comp}}{r_0} = (1 - \lambda^2) \cos \theta - \mu \left[ \left( \frac{13}{12} \lambda^2 - \frac{3}{2} \lambda^4 - \frac{5}{12} \lambda^6 \right) \cos \theta + \left( \frac{1}{4} \lambda^2 - \frac{1}{4} \lambda^4 \right) \sin 3\theta \right] + \ldots
\]

\[
(1 - \lambda^2) \sin \varphi \left[ \left( \frac{13}{12} \lambda^2 - \frac{3}{2} \lambda^4 + \frac{5}{12} \lambda^6 \right) \sin \theta + \left( \frac{3}{4} \lambda^2 - \frac{1}{4} \lambda^4 \right) \sin 3\theta \right] + \ldots
\]

\[
(1 - \lambda^2) \sin \lambda \varphi \left[ \left( \frac{13}{12} \lambda^2 - \frac{3}{2} \lambda^4 + \frac{5}{12} \lambda^6 \right) \sin \theta + \left( \frac{3}{4} \lambda^2 - \frac{1}{4} \lambda^4 \right) \sin 3\theta \right] + \ldots
\]

\[
\lambda^2 \gamma \cos \varphi \left[ \left( \frac{13}{12} \lambda^2 - \frac{3}{2} \lambda^4 + \frac{5}{12} \lambda^6 \right) \sin \theta + \left( \frac{3}{4} \lambda^2 - \frac{1}{4} \lambda^4 \right) \sin 3\theta \right] + \ldots
\]

It is interesting to compare the magnitudes of the various terms in equation (26) at the position of maximum velocity \( \delta = \pi/2 \) and for the critical value \( \mu = 0.1670 \) (obtained by means of equations (21) and (26)). Thus

\[
(2 \sin \delta)_{x=\frac{3}{2}} = 2
\]

\[
\mu \left( \frac{2}{3} \sin \delta - \frac{1}{2} \sin 3\delta \right)_{x=\frac{3}{2}} = 0.1948
\]

\[
\mu^2 \left( \frac{37}{40} \sin \delta - \frac{25}{24} \sin 3\delta + \frac{3}{8} \sin 5\delta \right)_{x=\frac{3}{2}} = 0.0653
\]

\[
\mu^2 (\gamma - 1) \left( \frac{53}{120} \sin \delta - \frac{11}{40} \sin 3\delta + \frac{3}{8} \sin 5\delta \right)_{x=\frac{3}{2}} = 0.0067
\]

Thus, it is seen that the terms involving \( \mu^2 \) do become of importance with regard to the \( \mu \) term as the local velocity of sound is approached on the circle and that the main contribution is made by expression (25).

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\[\text{LANGLEY MEMORIAL AERONAUTICAL LABORATORY,}
\text{NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,}
\text{LANGLEY FIELD, VA., FEBRUARY 11, 1938.}\]
APPENDIX

1. The Fourier Expansion of $1 - 2\sum \lambda'^{r} \cos 2\theta \lambda^{l}$

If $2\cos 2\theta = 1 - 2\sigma \lambda \cos 2\theta + \sum \lambda'^{r} \cos 2\theta \lambda^{l}$, then

$$H = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\sigma \lambda)^{n} \cos (n-2k)2\theta$$

Since, by the binomial theorem,

$$1 - \sigma^{2} \lambda^{2} \cos 2\theta = \sum_{r=0}^{\infty} (\sigma \lambda)^{r} \cos 2\theta \lambda^{l}$$

and

$$1 - \sigma^{2} \lambda^{2} \cos 2\theta = \sum_{r=0}^{\infty} (\sigma \lambda)^{r} \sin 2\theta \lambda^{l}$$

it follows that

$$H = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\sigma \lambda)^{n} \cos (n-2k)2\theta$$

Let

$$j+k=n$$

and therefore

$$j-k = n-2k; \quad j = n-k$$

The double series then becomes

$$H = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (\sigma \lambda)^{n} \cos (n-2k)2\theta \right)$$

The terms of this series can be grouped in pairs such that

$$H = 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\sigma \lambda)^{n} \cos (n-2k)2\theta$$

(1)

where $\frac{n}{2}$ or $\frac{n+1}{2}$ is the upper limit according as $n$ is even or odd and where the factor 2 is omitted from the term for which $n$ is even and $k = \frac{n}{2}$. This term is independent of $\theta$ and there is only one such term, not two.

The series (1) may be written as

$$H = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n} \cos (n-2k)2\theta$$

where

$$A_{n} = 2(\sigma \lambda)^{n}$$

Expanding this series and rearranging the terms in the form of a Fourier series,

$$H = \frac{1}{2} \sum_{n=0}^{\infty} A_{2n} + \sum_{n=1}^{\infty} \cos 2n\theta \sum_{k=0}^{n} A_{n+k}2k$$

But

$$\frac{1}{2} \sum_{n=0}^{\infty} A_{2n} = \sum_{n=1}^{\infty} (\sigma \lambda)^{n} = \frac{1}{1 - \sigma \lambda^{2}}$$

and

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n} (\sigma \lambda)^{n+k} = \frac{2(\sigma \lambda)^{n+1}}{1 - \sigma \lambda^{2}}$$

Therefore

$$H = \frac{1}{1 - \sigma \lambda^{2}} \left[ \sum_{n=1}^{\infty} (\sigma \lambda)^{n+1} \cos 2n\theta \right]$$

II. The Integrals

$$J_{1} = \int_{0}^{\pi/2} \sin (\theta - \delta) \frac{\sin (\theta - \delta)}{1 - 2\lambda \cos (\theta - \delta) - \lambda^{2}} \sin 2\theta d\theta$$

and

$$J_{2} = \int_{0}^{\pi/2} \sin (\theta - \delta) \frac{\sin (\theta - \delta)}{1 - 2\lambda \cos (\theta - \delta) - \lambda^{2}} \sin 2\theta d\theta$$

If $2\cos (\theta - \delta)$ is replaced by $e^{i\theta-\delta} + e^{-i\theta-\delta}$, then

$$\frac{1}{1 - 2\lambda \cos (\theta - \delta) - \lambda^{2}} \frac{\sin (\theta - \delta)}{1 - 2\lambda \cos (\theta - \delta) - \lambda^{2}} \sin 2\theta d\theta$$

$$= \frac{e^{i\theta-\delta}}{1 - 2\lambda \cos (\theta - \delta) - \lambda^{2}} - \frac{e^{-i\theta-\delta}}{1 - 2\lambda \cos (\theta - \delta) - \lambda^{2}}$$

$$= \frac{1}{2i} \left[ \sum_{n=0}^{\infty} \lambda^{n} \cos (n+1)\theta \cos \delta - \sum_{n=0}^{\infty} \lambda^{n} \cos (n+1)\theta \sin \delta \right]$$

Therefore

$$J_{1} + iJ_{2} = \frac{1}{2i} \sum_{n=0}^{\infty} \lambda^{n} \cos (n+1)\theta \cos \delta$$

Replacing $e^{-i\theta}$ by $\delta$,

$$J_{1} + iJ_{2} = \frac{1}{2} \sum_{n=0}^{\infty} \lambda^{n} \cos (n+1)\theta \delta$$

$$- \sum_{n=0}^{\infty} \lambda^{n} \cos (n+1)\theta \delta \frac{1}{j} dz$$

Since

$$\int_{\gamma} e^{z} = 0$$

it follows that $m = n - 1$ and therefore

$$J_{1} + iJ_{2} = \pi i \lambda^{n} e^{i\delta}$$

Hence, for $n \geq 1$,

$$J_{1} = \pi \lambda^{n-1} \sin n\delta$$

and

$$J_{2} = \pi \lambda^{n-1} \cos n\delta$$

(3)

III. The Fourier Expansion of

$$\frac{1}{(1 - 2\sigma \lambda \cos 2\theta + \sigma^{2} \lambda^{2})^{2}}$$

In analogy to section I, replace $2\cos 2\theta$ by $e^{i\theta} + e^{-i\theta}$.

Then

$$H = \frac{1}{(1 - 2\sigma \lambda \cos 2\theta + \sigma^{2} \lambda^{2})^{2}}$$

According to the binomial theorem

$$(1 - \sigma \lambda e^{i2\theta})^{-2} = \sum_{n=0}^{\infty} (j+1) (\sigma \lambda)^{j} e^{i2\theta}$$

and

$$(1 - \sigma \lambda e^{-i2\theta})^{-2} = \sum_{n=0}^{\infty} (j+1) (\sigma \lambda)^{j} e^{-i2\theta}$$

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Therefore
\[ H^2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (j+1)(k+1)(\sigma^2 \lambda^2)^{j+k} e^{2i(j-k)\theta} \]
Let
\[ j+k = n \]
and therefore
\[ j-k = n-2k, \quad j=n-k \]
Then
\[ H^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n-k+1)(k+1)(\sigma^2 \lambda^2)^n e^{2i(n-2k)\theta} \]
The exponent of \( e \) is \( 2i(n-k) - k \theta \). If \( k \) and \( n-k \) are interchanged, the exponent of \( e \) changes sign but the coefficient of \( e \) remains unaltered. The terms can therefore be grouped in pairs so that:
\[ H^2 = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n-k+1)(k+1)(\sigma^2 \lambda^2)^n \cos (n-2k)2\theta \]
where the factor 2 is omitted from the term for which \( n \) is even and \( k = \frac{n}{2} \).

The series (4) may be written as
\[ H^2 = \sum_{n,k=0}^{\infty} \sum_{k=0}^{n} A_{n,k} \cos (n-2k)2\theta \]
where
\[ A_{n,k} = 2(n-k+1)(k+1)(\sigma^2 \lambda^2)^n \]
Expanding this series and rearranging the terms in the form of a Fourier series,
\[ H^2 = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n,k} \cos 2n\theta \sum_{k=0}^{\infty} A_{n+2k,k} \]
But
\[ \frac{1}{2} \sum_{n=0}^{\infty} A_{n,k} \cos (n+1) = \sum_{k=0}^{\infty} (k+1)^2 (\sigma^2 \lambda^4)^k = \frac{1 + \sigma^2 \lambda^4}{(1 - \sigma^2 \lambda^4)^2} \]
and
\[ \sum_{k=0}^{\infty} A_{n+2k,k} = 2(\sigma^2 \lambda^2)^n \sum_{k=0}^{n} (n-k+1)(k+1)(\sigma^2 \lambda^4)^k \]
Therefore
\[ H^2 = \frac{1}{(1 - \sigma^2 \lambda^4)^2} \left[ (1 + \sigma^2 \lambda^4) + 2 \sum_{n=1}^{\infty} (n+1) - (n-1) \sigma^2 \lambda^4 \right] \cos 2n\theta \]

REFERENCES