Scattering Cross Section of Sound Waves by the Modal Element Method

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BY THE MODAL ELEMENT METHOD

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ABSTRACT

The modal element method has been employed to determine the scattered field from a plane acoustic wave impinging on a two dimensional body. In the modal element method, the scattering body is represented by finite elements, which are coupled to an eigenfunction expansion representing the acoustic pressure in the infinite computational domain surrounding the body. The present paper extends the previous work by developing the algorithm necessary to calculate the acoustic scattering cross section by the modal element method. The scattering cross section is the acoustical equivalent to the radar cross section (RCS) in electromagnetic theory. Since the scattering cross section is evaluated at infinite distance from the body, an asymptotic approximation is used in conjunction with the standard modal element method. For validation, the scattering cross section of the rigid circular cylinder is computed for the frequency range 0.1 \( \leq ka \leq 100 \). Results show excellent agreement with the analytic solution.

INTRODUCTION

The modal element method, which couples finite element algorithms to eigenfunction expansions, has been employed in calculating the scattered field from an acoustical plane wave impinging on a two dimensional body. The primary reasons for employing the modal element method are (1) to accurately describe the radiation boundary condition at the computational boundary and (2) to reduce the size of the numerical grid. In fact, for hard scatterers, the modal element method can effectively reduce a two dimensional scattering problem to a one dimensional problem by employing a single line of elements circumscribing the scattering body.

The modal element method has been given various titles, such as the unimoment method and the transfinite element method. In electromagnetics, the method has been applied to scattering from dielectric cylinders (Chang & Mei (1976), Lee & Cendes (1987), Baumeister & Kreider (1992)) and propagation in ducts (Baumeister (1991)). In acoustics, the method has also been applied to scattering from cylinders (Baumeister & Kreider (1993)) and propagation in ducts (Astley & Eversman (1981)). In all of these applications, an eigenfunction expansion is used to represent the acoustic pressure field in the far field. An asymptotic approximation of this expansion presents a simple means of determining the scattering cross section.

This paper presents the numerical algorithm for evaluating the acoustic scattering cross section by the modal element method. The scattering cross section is the acoustical equivalent to the radar cross section (RCS) in electromagnetic theory. Since the scattering cross section is evaluated at infinite distance from the body, asymptotic approximations are used in conjunction with the standard modal element method. For validation, the method is applied to scattering from rigid circular cylinders, for which the analytic solution is known.

NOMENCLATURE

- \( A_m^* \): modal amplitude of wave moving radially outwards
- \( a \): dimensionless circular cylinder radius
- \( H_m^{(1)} \): Hankel function of the first kind
- \( k \): wave number
- \( m \): mode number
- \( M_{\text{vec}} \): number of modal coefficients used in eigenfunction expansion
\( p \) dimensionless perturbation acoustic pressure  
\( r \) dimensionless radial coordinate  
\( \epsilon \) dimensionless property constant  
\( \Theta \) angle between radius vector and positive x axis  
\( \mu \) dimensionless property constant  
\( \sigma^2 \) acoustic scattering cross section  
\( \omega \) dimensionless frequency  

Superscript  
* complex conjugate  

**METHOD OF ANALYSIS**  

The present study is concerned with computing the scattering cross section of a two dimensional axisymmetrical rigid body due to an impinging plane wave traveling in the +x direction. In order to determine the acoustic scattering cross section, the scattered field must be computed. To accomplish this, the spatial domain is divided into two subdomains, the homogeneous domain and the finite element domain, as shown in Fig. 1. The grid system in Fig. 1 is designed to allow various structures to be imbedded into the grid, as shown by inserts A and B. For the special case of rigid bodies, the grid system can be reduced to a single ring of elements, which effectively reduces the two dimensional problem to a one dimensional problem.

In the finite element domain, an approximate solution for the total (incident + scattered) pressure at the element nodes is calculated by the Galerkin method. Linear triangular elements are used and the subdomain interface is approximated by piecewise linear segments. In the homogeneous domain, which extends to infinity, the pressure is represented by an eigenfunction expansion. The modal element method couples the two solution forms by imposing continuity on the pressure and velocity at the interface between the two subdomains. This coupling results in a single matrix equation in which the eigenfunction coefficients and the pressure at the finite element nodes are calculated simultaneously, yielding a global representation of the acoustic pressure field.

**GOVERNING EQUATION AND BOUNDARY CONDITIONS**

The scalar form of the acoustic wave equation can be written as (Baumeister & Kreider (1993))

\[
\frac{\partial}{\partial x} \left( \frac{1}{\epsilon} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} \frac{\partial p}{\partial y} \right) + \omega^2 \mu p = 0
\]

(1)

The harmonic time dependence \( e^{i\omega t} \) has been factored out. \( \epsilon \) and \( \mu \) are material property constants and \( \omega \) is the dimensionless frequency. The wave number is

\[
k = \sqrt{\omega^2 \mu \epsilon}
\]

(2)

In the homogeneous region surrounding the body, \( \epsilon = 1 \) and \( \mu = 1 \), while inside the body, \( \epsilon \) and \( \mu \) may assume other appropriate values depending on the material. Baumeister and Dahl (1989) present explicit expressions for \( \epsilon \) and \( \mu \) as a function of medium density, porosity, viscous loss coefficient and a heat transfer parameter.

At the interface between the finite element region and the analytic region, continuity is imposed on the pressure and velocity. The radiation boundary condition at infinity is automatically satisfied by the eigenfunction expansion introduced in the next section.

**ANALYTIC SOLUTION**

In the homogeneous domain, an exact expansion for the pressure can be derived (Morse and Ingard, 1968, p. 401) from eq. (1) by separation of variables:

\[
p = p^1 + p^s = e^{ikx} + \sum_{m=0}^{M_{ref}} A_m H_m^{(1)}(kr) \cos(m\theta).
\]

(3)

\( p^1 \) is the incident plane wave; \( p^s \) is the scattered wave. This expansion is valid for symmetric bodies. The modal coefficients \( A_m \) are unknowns to be determined. Formulas to estimate the number \( M_{ref} \) of modes needed for convergence can be found in Baumeister & Kreider (1993).
FINITE ELEMENT SOLUTION

The solution to eq. (1) will be obtained by linear finite element theory. In Fig. 1, the finite element region is divided into triangular elements defined by corner nodal points. It is assumed that all material properties are constant in each element. In the conventional weighted residual approach, the unknown pressure field is described in terms of all the unknown nodal values of the pressure.

The finite element aspects of converting eq. (1) and (3) and boundary conditions into an appropriate set of global difference equations can be found in Baumeister & Kreider (1993). The resulting set of global difference equations is solved for the pressure at the nodes and the modal coefficients $A_m^+$. With the modal coefficients determined, a relatively simple algorithm has been developed to determine the scattered cross section and is described in the next section.

BISTATIC ACOUSTIC SCATTERING CROSS SECTION

The bistatic acoustic scattering cross section per unit length for a two dimensional body can be defined as

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi \left| \frac{p^s \cdot p^s}{|p|^2} \right|^2 = \lim_{r \to \infty} 2\pi \left( \frac{p^s \cdot p^{s*}}{|p^s \cdot p^{s*}|} \right)$$

(4)

The standard superscript $c$ implies a two dimensional problem, where the cross section is often called the "echoing width" (Ruck, vol. 1, pp. 23). In three dimensions, the scattered field decays as $1/r$ while in two dimensions, considered herein, the scattered field decays as $1/r$. Thus, the $2\pi$ is employed in eq. (4) to negate the 2D scattered field decay and to yield a finite measure of the reflected power at infinity. The cross section defined in eq. (4) could be used in range equations specifically defined for 2D bodies.

For a unit incident plane wave, eq. (4) simplifies to

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi \left| \frac{p^s \cdot p^{s*}}{|p|^2} \right|^2 = \lim_{r \to \infty} 2\pi \left( \frac{p^s \cdot p^{s*}}{|p^s \cdot p^{s*}|} \right)$$

(5)

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi \left( \frac{p^s \cdot p^{s*}}{|p^s \cdot p^{s*}|} \right)$$

Using conventional numerical grids, it is often necessary to extrapolate the nodal values of the field on the outer boundary to infinity. For example, Noack and Anderson (1992) include a boundary element approximation to determine the scattering cross section at infinity. Fortunately, no such additional work is required with the modal element method. Since the scattering cross section is evaluated at infinity, the asymptotic expansion for the Hankel functions (Abramowitz & Stegun, 9.2.3)

$$H^{(1)}_m(z) \sim \sqrt{\frac{2}{\pi z}} e^{i \left( \frac{\pi}{4} - \frac{1}{4} \right)} e^{-\frac{1}{2} m \pi}$$

(6)

can be inserted into eq. (3), which is then substituted into eq. (4) to yield

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi e^{-i \left( \frac{\pi}{4} \right)} \sum_{m=0}^{M_{\text{coeff}}-1} A_m^+ e^{-i m \pi \cos(\theta)}$$

(7)

$$\sigma^c(\theta) = \lim_{r \to \infty} 2\pi e^{-i \left( \frac{\pi}{4} \right)} \sum_{m=0}^{M_{\text{coeff}}-1} A_m^+ e^{-i m \pi \cos(\theta)}$$

Once the $A_m^+$ have been calculated, the scattering cross section and the pressure field can be easily determined.

RESULTS AND COMPARISONS

Consider the following scattering problem: a unit plane wave, incident from the left, strikes a rigid circular cylinder of dimensionless radius $a = 1$ oriented with its axis normal to the propagation direction. The rigid body is simulated numerically through an impedance mismatch induced by setting $\varepsilon = 1 - 10^{-9} i$ and $\mu = 1$ for each internal finite element and $\varepsilon = 1$ and $\mu = 1$ in the homogeneous domain surrounding the body. This feature allows flexibility in the numerical implementation of the method. Rigid bodies with absorber coatings may be studied with only slight modification to the computer code by adding several finite element rings with the appropriate values of $\varepsilon$ and $\mu$.

This problem is solved for frequencies ranging from $ka = 0.1$ to $ka = 100$ in order to validate the method. This problem was chosen because it has an exact solution, given by eq. (3) with modal coefficients (Bowman et al., eq. (2.38))

$$A_m^+ = \left( \frac{(2 - 8 \delta_{m0}) i^m J_m(ka)}{H^{(1)}_m(ka)} \right)$$

(8)

and because it has been examined using other methods (Ling, 1986, Fig. 15).

In Fig. 2, the computed cross sections (square symbols) are compared to the corresponding exact solutions (solid line). The results agree very well with the exact solutions over a wide frequency range. Results for $ka = 50$ and $ka = 100$ show similar agreement but are omitted for brevity. In addition, the results agree well with the low frequency calculations $(k = 3.2)$ of Ling (1986, Fig. 15).

Additional validation of the algorithm is obtained by examining high frequency $(ka > 20)$ asymptotic approximations. In the backscattered direction $(\theta = 180)$, the optical approximation is valid (Ruck, et al. (1970), eq. (4.1) to (38)) and the backscattering cross section becomes

$$\sigma^c = \pi a = \pi$$

(9)
In the forward scattered direction ($\theta = 0$), commonly referred to as the diffracted field, the optical approximation yields (Ruck, et al. (1970), eq. (4.1) to (41)) the forward cross section

$$\sigma^c = 4ka^2 = 4k \quad (a = 1) \quad (10)$$

Figure 3 shows that the asymptotic limits are attained as frequency increases in both the forward and backscattered directions. Table I contains the numerical values obtained for various frequencies in these directions. Even at $ka = 100$, the relative error in the forward direction is only about 3 percent.

Some error at high frequencies can be attributed to the approximation involved with the asymptotic expansion given by eq. (7). However, the major source of error is in the calculated values of the mode coefficients $A_m^*$ in eq. (3). For example, in the $k = 100$ calculation, over 130 modal coefficients need to be determined. Consequently, some significant errors are involved in the calculation of the higher order modes.
CONCLUDING REMARKS

In the modal element method, the scattering body is imbedded in a finite element grid to calculate the acoustic pressure field. In the far field, the pressure is represented analytically by a Hankel function expansion. The asymptotic approximation of the Hankel functions are used to calculate the scattering cross section. The method is applicable to problems involving very high or low frequency scattering from bodies. In the validation examples, the numerical results are in good agreement with the corresponding exact solutions and asymptotic analytical limits.

References


## Abstract (Maximum 200 words)

The modal element method has been employed to determine the scattered field from a plane acoustic wave impinging on a two dimensional body. In the model element method, the scattering body is represented by finite elements, which are coupled to an eigenfunction expansion representing the acoustic pressure in the infinite computational domain surrounding the body. The present paper extends the previous work by developing the algorithm necessary to calculate the acoustics scattering cross section by the modal element method. The scattering cross section is the acoustical equivalent to the radar cross section (RCS) in electromagnetic theory. Since the scattering cross section is evaluated at infinite distance from the body, an asymptotic approximation is used in conjunction with the standard modal element method. For validation, the scattering cross section of the rigid circular cylinder is computed for the frequency range $0.1 < ka < 100$. Results show excellent agreement with the analytic solution.